THE TWENTY-SEVENTH HERZOG PRIZE EXAMINATION

November 13, 1999

1. Determine which of the numbers 1999^{1999} or 2000^{1998} is the larger.

2. Prove or disprove: If a sequence of integers in arithmetic progression contains a square, then it must contain infinitely many squares.

3. A ladybug (beetle of the family *Coccinellidae*) walks along the real line. She starts at point a and walks toward b. Halfway there, she realizes that she forgot to eat an aphid at a and turns back. But halfway back to a she decides to go to b anyway and turns around again, only to change her mind again, halfway to b. She continues in this manner. Find all cluster points (*i.e.* accumulation points) of the resulting sequence of turning points.

4. Given segment AB in the plane, determine the locus of all points P such that $\angle PAB + \angle PBA$ is constant. (Give a detailed description in terms of the constant.)

5. Let a, b, c be positive real numbers. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$$

6. Let $p(x)$ be a polynomial with real coefficients, and with all zeros real. Suppose also that $p(x) \geq 0$ for all real numbers x. Prove that $p(x) = \{q(x)\}^2$ for some polynomial $q(x)$ with real coefficients.