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Discrete singular convolution for beam analysis

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Abstract

This paper explores the utility of a discrete singular convolution (DSC) algorithm for beam analysis. Regularized Shannon and Dirichlet kernels are selected to illustrate the present algorithm. Three classes of benchmark beam problems, including bending, vibration and buckling, are utilized to test numerical accuracy and speed of convergence of the present approach. Numerical experiments indicate that the DSC is a simple and reliable algorithm for beam analysis. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Conventionally, the design and optimization of engineering structures are based on theory and experimentation. However, engineering experiments can be extremely expensive. Most practical engineering designs are either extremely difficult or impossible to be accomplished by analytical methods. Therefore, numerical simulations play a more and more important role in structural analysis. The quality of the design and optimization depends crucially on both the basic theory and computational methods. The availability of inexpensive high-performance computers has given tremendous impetus to a general area entitled *computer-aided design*. In fact, there has been an ongoing interest in computational methodology by numerous researchers in every field of science and engineering. Most effort has been centered in developing either global methods [1–5] or local methods [6–15] for solving a variety of time-dependent and time-independent computational problems. In general, global methods are much more accurate than local methods, while the major advantages of local methods are their flexibility in handling complex geometries and boundary conditions. In ordinary applications, it is relatively safe and efficient to use either a global method or a local one for numerically solving an ordinary differential equation or a partial differential equation. However, when a differential equation has singularities and/or homoclinic orbits, neither the global methods nor the local methods can be applied without numerical instabilities. It is desirable to have a method that has both global methods' accuracy and local methods' flexibility.

Recently, we have developed a discrete singular convolution (DSC) algorithm [16,46] as a potential approach for computer realization of Hilbert transform, Abel transform, Radon transform and delta transform. For solving differential equations, the DSC algorithm has global methods' accuracy and local methods' flexibility. Many DSC kernels, such as regularized Shannon's kernel (RSK), regularized Dirichlet kernel (RDK) and regularized Lagrange kernel, have been constructed for various applications [16,17,46]. The DSC algorithm has been shown to be very accurate and efficient for stochastic analysis [16,46], quantum eigenvalue estimation [18] and solving Maxwell's equations [17]. It is found to play a special role in integrating the sine-Gordon equation with initial values close to homoclinic orbits [19], for which previous methods encounter great difficulty [20]. The first four invariant functionals of a Hamiltonian system described by the Korteweg–de Vries equation are conserved to the machine precision [21]. The DSC method was utilized to resolve the non-linear pattern dynamics described by the Cahn–Hilliard equation in a circular domain [22], which, due to artificial singularities at the origin and complex phase space structures, is a

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very challenging problem for conventional numerical methods. Recently, we have shown that DSC algorithm provides up to machine precision for solving the Navier–Stokes equation with periodic boundary conditions [23]. Most recently, we have demonstrated the use of the DSC algorithm for solving incompressible fluid flow with complex boundaries [24].

The purpose of the present paper is to explore the utility of the DSC algorithm for beam analysis, which is of great practical significance to civil, mechanical, and aerospace engineering. Apart from a few analytically solvable cases, there is no general solution for static analysis of structures. Numerical simulation is one of the major approaches. Various numerical methods have been used with great success for structural computations. These include the series expansion [25] for orthotropic plates, integral equation approach [26–28], methods of finite strips and finite elements [29–31], domain decomposition approaches [32,33], Galerkin methods [34], differential quadrature methods [35–37], and Rayleigh–Ritz variational methods [38–40], to name only a few. As our first attempt, we consider only three typical classes of beam problems.

This paper is organized as follows. Section 2 is devoted to a brief review of the philosophy and implementation of the discrete singular convolution algorithm. The reader is referred to Refs. [16,46] for more details. Applications to beam analysis are presented in Section 3. Three types of important beam problems, including bending under transverse loads, vibrations subjected to tensile or compressive axial forces, and bucking failure critic value estimations, are selected for the present illustration. The conclusion is given in Section 4.

2. Discrete singular convolution

Singular convolutions appear in many science and engineering problems, such as electromagnetics, Hilbert transform, Abel and Radon transforms. DSC is a general approach for numerically solving singular convolution problems. By appropriate realizations of a singular convolution kernel, the DSC can be an extremely efficient, accurate and reliable algorithm for scientific computations [16,46].

The theory of *singular convolution* can be discussed in the context of distributions. We denote T a distribution and $\eta(t)$ an element of the space of test functions. A singular convolution can be expressed as:

$$F(t)=(T*\eta)(t)=\int_{-\infty}^{\infty} T(t-x)\eta(x)dx. \tag{1}$$

Here $T(t-x)$ is a singular kernel. Depending on the form of the kernel T, the singular convolution is the central issue for a wide range of science and engineering problems. For example, singular kernels of the *Hilbert type* have a general form of:

$$T(x)=\frac{1}{x^n},\ (n=1, 2, \ldots). \tag{2}$$

Here, kernel $T(x)=\frac{1}{x}$ commonly occurs in electrodynamics, theory of linear response, signal processing, theory of analytic functions, and the Hilbert transform; $T(x)=\frac{1}{x^2}$ is the kernel used in tomography. Singular kernels of the *Abel type*

$$T(x)=\frac{1}{x^{\beta}},\ (0<\beta<1) \tag{3}$$

was introduced in the tautochrone problem. It has applications in the area of holography and interferometry with phase objects and is of practical importance in aerodynamics, heat and mass transfer, and plasma diagnostics. Other interesting examples are singular kernels of the *delta type*

$$T(x)=\delta^{(n)}(x),\ (n=0, 1, 2,\ldots). \tag{4}$$

Here, kernel $T(x)=\delta(x)$ is important for the interpolation of surfaces and curves (including atomic, molecular and biological potential energy surfaces); and $T(x)=\delta^{(n)}(x)$, $(n=1, 2, \ldots)$ are essential for numerically solving partial differential equations. However, since these kernels are singular, they cannot be directly digitized in computer. Hence, the singular convolution, Eq. (1), is of little numerical merit. To avoid the difficulty of using singular expressions directly in computer, sequences of approximations (T_α) of the distribution T can be constructed

$$\lim_{\alpha\to\alpha_0} T_\alpha(x)\to T(x), \tag{5}$$

where α_0 is a generalized limit. Obviously, in the case of $T(x)=\delta(x)$, the sequence, $T_\alpha(x)$, is a delta sequence. Moreover, with a sufficiently smooth approximation, it is useful to consider a *discrete singular convolution* (DSC)

$$F_\alpha(t)=\sum_k T_\alpha(t-x_k)f(x_k), \tag{6}$$

where $F_\alpha(t)$ is an approximation to $F(t)$ and $\{x_k\}$ is an appropriate set of discrete points on which the DSC (6) is well defined. Note that, the original test function $\eta(x)$ has been replaced by $f(x)$. The mathematical property or requirement of $f(x)$ is determined by the approximate kernel T_α. In general, the convolution is required being Lebesgue integrable.

A sequence of approximation can be improved by a regularizer

$$\lim_{\sigma\to\infty} R_\sigma(x)=1. \tag{7}$$

The regularizer is designed to increase the regularity of convolution kernels. For the delta sequence, it follows from Eq. (5) that

$$\int \lim_{\alpha\to\alpha_0} T_\alpha(x)R_\sigma(x)dx=R_\sigma(0)=1, \tag{8}$$

where $R_\sigma(0)=1$ is the special requirement for a *delta regularizer*. A typical delta regularizer used in this work and elsewhere [41] is $\exp(-x^2/2\sigma^2)$.

As a special example, Shannon's kernel $\sin\alpha x/\pi x$ is a delta sequence kernel

$$\lim_{\alpha\to\infty} <\frac{\sin\alpha x}{\pi x}, \eta(x)>=\eta(0). \tag{9}$$

Other important examples include the Dirichlet kernel

$$\frac{\sin\left[\left(l+\frac{1}{2}\right)(x-x')\right]}{2\pi\sin\left[\frac{1}{2}(x-x')\right]},$$

the modified Dirichlet kernel

$$\frac{\sin\left[\left(l+\frac{1}{2}\right)(x-x')\right]}{2\pi\tan\left[\frac{1}{2}(x-x')\right]},$$

and the de la Vallée Poussin kernel

$$\frac{1}{\pi\alpha}\frac{\cos[\alpha(x-x')]-\cos[2\alpha(x-x')]}{(x-x')^2}.$$

For sequences of both the delta type and the Hilbert type, an interpolating (or quasi interpolating) algorithm sampling at *Nyquist frequency*, $\alpha=\frac{\pi}{\Delta}$, has great advantage over a non-interpolating discretization. Hence the Shannon's kernel is discretized and regularized [41] as:

$$\frac{\sin[\alpha(x-x')]}{\pi(x-x')}\to\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right). \tag{10}$$

Not only the interpolating (or quasi interpolating) nature guarantees the highest accuracy on the set of grid points, but also it provides the highest possible computational efficiency off a grid. This is because the *Nyquist interval* given by $[-\pi/\Delta, \pi/\Delta]$, is the largest possible sampling interval that is free of alias whenever the L^2 function $f(x)$ under study satisfies the *Nyquist condition*

$$\operatorname{supp}\hat{f}(k)\subset\left\{-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right\}. \tag{11}$$

This fact can be formally given by Shannon's sampling theorem

$$f(x)=\sum_{k=-\infty}^{\infty} f(x_k)\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)}. \tag{12}$$

The significance of Shannon's sampling theorem is that, by a discrete, but infinite set of sampling data $\{f(x_k)\}$ one can actually recover a bandlimited L^2 function on a real line. Shannon's sampling theorem has great impact on information theory, signal and image processing because the Fourier transform of Shannon's kernel is an ideal low-pass filter for signals bandlimited to $[-\pi/\Delta, \pi/\Delta]$.

The uniform, Nyquist rate, interpolating discretization and the regularization are used for the Dirichlet kernel:

$$\frac{\sin\left[\left(l+\frac{1}{2}\right)(x-x')\right]}{2\pi\sin\left[\frac{1}{2}(x-x')\right]}\to \tag{13}$$

$$\frac{\sin(\pi/\Delta(x-x_k))}{(2M+1)\sin(\pi/\Delta(x-x_k/2M+1))}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right).$$

In comparison to Shannon's kernel, the Dirichlet kernel has one more parameter M which can be optimized to achieve better results in computations. Usually, we set a sufficiently large M for various numerical applications. Obviously, the Dirichlet kernel converts to Shannon's kernel at the limit of $M\to\infty$. These uniform interpolating discretization and the regularization will also be used for the modified Dirichlet kernel

$$\frac{\sin\left[\left(l+\frac{1}{2}\right)(x-x')\right]}{2\pi\tan\left[\frac{1}{2}(x-x')\right]}\to \tag{14}$$

$$\frac{\sin(\pi/\Delta(x-x_k))}{(2M+1)\tan(\pi/\Delta(x-x_k/2M+1))}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right),$$

and for the de la Vallée Poussin kernel

$$\frac{1}{\pi\alpha}\frac{\cos[\alpha(x-x')]-\cos[2\alpha(x-x')]}{(x-x')^2}\to \tag{15}$$

$$\frac{2\cos\,(\pi/\bar{\Delta})(x-x_k)-\cos(2\pi/\bar{\Delta})(x-x_k)}{3\,[\pi(x-x_k)/\bar{\Delta}]^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right),$$

where $\bar{\Delta}=3\Delta/2$. Since π/Δ is proportional to the highest frequency which can be reached in the Fourier representation, the Δ should be very small for a given problem involving very oscillatory functions or very high frequency components.

We use a *symmetrically* (or *antisymmetrically*) *truncated*, *translation invariant* singular kernel

$$f^{(n)}(x)\approx\sum_{k=-W}^{W}\delta^{(n)}_{\alpha,\sigma}(x-x_k)f(x_k),\ (n=0,1,2,\ldots), \tag{16}$$

where $2W+1$ is the computational bandwidth, or effective kernel support, which is usually smaller than the whole computational domain, $[a, b]$. Here $\delta^{(n)}_{\alpha,\sigma}(x-x_k)$ is a collective symbol for the nth derivative of any of the right-hand side of Eqs. (10), (13), (14), and (15).

In the present study we limit our attention to the RSK and RDK. Nevertheless, various other delta sequence kernels can be similarly employed [16,46]. It is noted that the RSK is parameter free, which is convenient for applications. The $2M+1$ parameter for the RDK is chosen as 188 for all calculations. We note that as long as the $2M+1$ value is chosen sufficiently large the numerical results are not sensitive to the specific values used.

3. Application

To illustrate the use of the DSC algorithm and test its accuracy for beam analysis, we consider three typical problems, including bending, vibration and buckling of Kirchhoff beams. These computations require solving differential equations of fourth order and the involved DSC algorithm is presented in the first subsection. The DSC results are described in the last three subsections.

3.1. Formulation

Consider an operator $\mathcal{O}$ having a differential part $\mathcal{D}$ and a function part F

$$\mathcal{O}=\mathcal{D}+F. \tag{17}$$

In the DSC approach, it is convenient to choose a grid representation for the coordinate so that the function part F of the operator is diagonal. Hence, its discretization is simply given by a direct interpolation on the grid

$$F(x)\rightarrow F(x_k)\delta_{m,k}. \tag{18}$$

The differential part of the operator on the coordinate grid is then represented by functional derivatives

$$\mathcal{D}=\sum_n d_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\rightarrow\sum_n d_n(x_m)\delta^{(n)}_{\alpha,\sigma}(x_m-x_k), \tag{19}$$

where $d_n(x)$ is a coefficient. Thus, the full DSC-matrix representation for the operator, $\mathcal{O}$, is given by:

$$\mathcal{O}(x_m,x_k)=\sum_n d_n(x_m)\delta^{(n)}_{\alpha,\sigma}(x_m-x_k)+F(x_m)\delta_{m,k}. \tag{20}$$

Here the derivatives are *analytically* given by:

$$\delta^{(n)}_{\alpha,\sigma}(x_m-x_k)=\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\delta_{\alpha,\sigma}(x-x_k)\right]_{x=x_m}. \tag{21}$$

The differentiations in Eq. (21) can be easily carried out for a given $\delta_{\alpha,\sigma}(x-x_k)$. For example, if $\delta_{\pi/\Delta,\sigma}(x-x_k)=[\sin(\pi/\Delta)(x-x_k)]/[\pi(x-x_k)/\Delta]\exp[-(x-x_k)^2/2\sigma^2]$, we have for $x\neq x_k$ [41]:

$$\begin{aligned}\delta^{(1)}_{\pi/\Delta,\sigma}(x-x_k)=&\frac{\cos(\pi/\Delta)(x-x_k)}{(x-x_k)}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&-\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&-\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right),\end{aligned} \tag{22}$$

$$\begin{aligned}\delta^{(2)}_{\pi/\Delta,\sigma}(x-x_k)=&-\frac{(\pi/\Delta)\sin(\pi/\Delta)(x-x_k)}{(x-x_k)}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)-2\frac{\cos(\pi/\Delta)(x-x_k)}{(x-x_k)^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&-2\frac{\cos(\pi/\Delta)(x-x_k)}{\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&+2\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)^3}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&+\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)+\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta\sigma^4}(x-x_k)\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right),\end{aligned} \tag{23}$$

$$\begin{aligned}\delta^{(3)}_{\pi/\Delta,\sigma}(x-x_k)=&-\frac{(\pi^2/\Delta^2)\cos(\pi/\Delta)(x-x_k)}{(x-x_k)}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)+3\frac{(\pi/\Delta)\sin(\pi/\Delta)(x-x_k)}{(x-x_k)^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&+3\frac{(\pi/\Delta)\sin(\pi/\Delta)(x-x_k)}{\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&+6\frac{\cos(\pi/\Delta)(x-x_k)}{(x-x_k)^3}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&+3\frac{\cos(\pi/\Delta)(x-x_k)}{(x-x_k)\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&+3\frac{(x-x_k)\cos(\pi/\Delta)(x-x_k)}{\sigma^4}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\\&-6\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)^4}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)\end{aligned} \tag{24}$$

$$-3\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)^2\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$-\frac{(x-x_k)^2\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta\sigma^6}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right),$$

and

$$\delta^{(4)}_{\pi/\Delta,\,\sigma}(x-x_k)=4\frac{(\pi^2/\Delta^2)\cos(\pi/\Delta)(x-x_k)}{(x-x_k)^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)+\frac{(\pi^3/\Delta^3)\sin(\pi/\Delta)(x-x_k)}{(x-x_k)}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$+4\frac{(\pi^2/\Delta^2)\cos(\pi/\Delta)(x-x_k)}{\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$-12\frac{(\pi/\Delta)\sin(\pi/\Delta)(x-x_k)}{(x-x_k)^3}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$-6\frac{(\pi/\Delta)\sin(\pi/\Delta)(x-x_k)}{(x-x_k)\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$-6\frac{(\pi/\Delta)(x-x_k)\sin(\pi/\Delta)(x-x_k)}{\sigma^4}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$-24\frac{\cos(\pi/\Delta)(x-x_k)}{(x-x_k)^4}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right) \quad (25)$$
$$-12\frac{\cos(\pi/\Delta)(x-x_k)}{(x-x_k)^2\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$-4\frac{(x-x_k)^2\cos(\pi/\Delta)(x-x_k)}{\sigma^6}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$+24\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)^5}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$+12\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)^3\sigma^2}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$+3\frac{\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta(x-x_k)\sigma^4}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$-2\frac{(x-x_k)\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta\sigma^6}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right)$$
$$+\frac{(x-x_k)^3\sin(\pi/\Delta)(x-x_k)}{\pi/\Delta\sigma^8}\exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right).$$

At $x=x_k$, it is convenient to evaluate these derivatives separately

$$\delta^{(1)}_{\pi/\Delta,\,\sigma}(0)=0 \quad (26)$$

$$\delta^{(2)}_{\pi/\Delta,\,\sigma}(0)=-\frac{1}{\sigma^2}-\frac{\pi^2}{3\Delta^2} \quad (27)$$

$$\delta^{(3)}_{\pi/\Delta,\,\sigma}(0)=0 \quad (28)$$

and

$$\delta^{(4)}_{\pi/\Delta,\,\sigma}(0)=\frac{1}{5}\,\frac{15+10(\pi^2/\Delta^2)\sigma^2+(\pi^4/\Delta^4)\sigma^4}{\Delta^4}. \quad (29)$$

Derivative operations of other kernels can be computed in a similar way.

Note that the differentiation matrix in Eq. (20) is banded when the bandwidth parameter $2W+1$ of Eq. (16) is smaller than N, the number of grid points in the computational domain. The kernel support $2W+1$ is the same on every grid point x_k of the computational domain [16,46]. This gives rise to great advantage over global methods in large scale computations. As the generalized functions can be easily extended to higher dimensions, the present treatment can be easily extended to higher dimensions, for example, by tensorial products. We refer expression (20) and its higher dimensional generalizations as DSC matrices.

3.2. Bending under transverse loads

Within the scope of the classical Kirchhoff theory, the bending of a beam can be treated as a boundary value problem [42], governed by the Euler–Bernoulli equation

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w(x)}{dx^2}\right)=q(x),\ \ x\in[0,\,a], \quad (30)$$

where $w(x)$ represents the deflection of the beam of length a, $q(x)$ is the transverse load applied to the beam, $E(x)$ is the elastic modulus of the material of which the beam is composed, and $I(x)$ is the moment of inertia of the cross-section. The product $EI(x)$ is usually referred to as the flexural rigidity of the beam. The edge of the beam can be free, clamped, simply supported, elastically supported, and sliding and these can be appropriately combined to give the completed boundary conditions of the beam.

For simplicity, we consider a beam of constant flexural rigidity and length a, subject to the transverse load $q(x)=q_0\sin(\pi x/a)$ distributed over the entire beam (q_0 is a constant)

$$\frac{d^4w(x)}{dx^4}=\frac{q_0}{EI}\sin\left(\frac{\pi x}{a}\right),\ \ x\in[0,\,a], \quad (31)$$

and clamped boundary condition

$$w(0)=\frac{dw(0)}{dx}=0,\ \ w(a)=\frac{dw(a)}{dx}=0. \quad (32)$$

This problem admits an exact solution which provides an objective measure for potential numerical methods of beam analysis

$$w(x)=\frac{a^2q_0}{\pi^4EI}\left[\pi x^2-a\pi x+a^2\sin\left(\frac{\pi x}{a}\right)\right]. \quad (33)$$

In the present computation, we set $q_0/EI=1$. This problem is treated as a boundary value one in which the boundary values are specified in a DSC-collocation matrix. The resulting linear equation is solved by a standard LU decomposition code. Totals of 5, 11 and 21 grid points are employed in the closed interval [0, 1] in association with $\sigma/\Delta=2.2$, 2.55 and 3.9, respectively. The L_2 and L_∞ errors are listed in Table 1. It is seen that in the case of five grid points (just three inner points), the results are already accurate to five significant figures. The accuracy increases more than 10 times when the grid is refined by a factor of 2. The RSK and the RDK show similar behaviour in this boundary value problem.

3.3. Vibration under axial forces

The natural vibration of a beam of uniform flexural rigidity, subjected to either tensile or compressive axial forces whose magnitude is below the value of the so called critical Euler elastic buckling, can be formulated as an eigenvalue problem [42]

$$\frac{d^4w(x)}{dx^4}+\frac{S}{EI}\frac{d^2w(x)}{dx^2}=k^2\frac{mA}{EI}w(x),\ \ x\in[0,a], \tag{34}$$

with the simply supported boundary condition

$$w(0)=\frac{d^2w(0)}{dx^2}=0,\ \ w(a)=\frac{d^2w(a)}{dx^2}=0, \tag{35}$$

where a positive S represents the magnitude of a compressive force and a negative S represents the magnitude of a tensile force, m and A are the mass density of the material and the cross-sectional area of the beam, respectively. This problem is also analytically solvable and its exact solutions have the form of:

$$w_n(x)=B_n\sin\left(\frac{n\pi x}{a}\right),\ n=1,2,\ldots, \tag{36}$$

where B_n is an arbitrary constant, which satisfies all of the boundary conditions in Eq. (35). A set of eigenvalues are given by:

$$k_n^2\frac{mA}{EI}=\left(\frac{n\pi}{a}\right)^4-\frac{S}{EI}\left(\frac{n\pi}{a}\right)^2. \tag{37}$$

To ensure the statement of an eigenvalue problem, the compressive force is required to be bounded from above

$$S<EI\left(\frac{n\pi}{a}\right)^2. \tag{38}$$

To demonstrate the high accuracy of the DSC algorithm for beam eigenvalue problems, we discretize the grid by using RSK and RDK. All eigenvalues are obtained via direct diagonalization by using a standard eigenvalue solver. In the closed interval of $[0, 10\pi]$, three sets of grids, N=11, 21 and 31, are employed in association with $\sigma/\Delta=2.1$, 3.1 and 4.0, respectively. Table 2 lists the absolute errors when no load is imposed ($S/EI=0$). The errors in the cases of compressive force $S/EI=0.01$ and the tensile force $S/EI=-1$ are given in Tables 3 and 4, respectively. We note that errors for the first three eigenvalues are extremely small even if only 11 grid points are deployed in the large interval of 10π. When N=31, all of the first 10 eigenvalues are of machine precision.

3.4. Buckling failure analysis

In the framework of elastic Kirchhoff's beam theory, the buckling phenomenon associated with compressive force $\mathcal{N}$ can be modeled [42] by the following boundary value problem

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w(x)}{dx^2}\right)+\mathcal{N}\frac{d^2w(x)}{dx^2}+k_0w(x)=0,\quad x\in[0,a]. \tag{39}$$

Here, for a given flexural rigidity distribution $EI(x)$ and constant k_0, there is a critical value $\mathcal{N}_{cr}$ of the compressive force $\mathcal{N}$, which leads a sudden change of the undeformed stable equilibrium state to a bent unstable one. This problem does not admit a general solution for mixed boundary conditions. However, for simply supported boundary conditions

$$w(0)=\frac{d^2w(0)}{dx^2}=0,\ \ w(a)=\frac{d^2w(a)}{dx^2}=0, \tag{40}$$

and a constant flexural rigidity EI_0, the buckling failure critical value is analytically given by:

$$\mathcal{N}_{cr}=\frac{EI_0\pi^4+a^4k_0}{a^2\pi^2}. \tag{41}$$

Table 1
Computational errors for bending under load

N	RSK			RDK		
	5	11	21	5	11	21
L_2	2.68 (−05)	1.93 (−06)	1.80 (−07)	2.68 (−05)	1.93 (−06)	1.79 (−07)
L_∞	3.63 (−05)	4.03 (−06)	4.21 (−07)	3.65 (−05)	4.02 (−06)	4.21 (−07)

Table 2
Errors for vibration eigenvalues (*S*/*EI*=0)

n	$k_n^2[mA/EI]$	N=11		N=21		N=31	
		RSK	RDK	RSK	RDK	RSK	RDK
1	0.0001	2.3 (−08)	2.2 (−08)	6.8 (−12)	6.9 (−12)	1.3 (−14)	2.2 (−14)
2	0.0016	6.8 (−07)	6.8 (−07)	6.0 (−12)	6.2 (−12)	3.9 (−14)	3.8 (−14)
3	0.0081	1.6 (−05)	1.6 (−05)	4.9 (−12)	5.0 (−12)	1.7 (−14)	1.6 (−14)
4	0.0256	2.5 (−04)	2.5 (−04)	2.9 (−12)	3.0 (−12)	3.0 (−14)	2.8 (−14)
5	0.0625	2.4 (−03)	2.4 (−03)	2.3 (−11)	2.2 (−11)	7.3 (−15)	1.1 (−14)
6	0.1296	1.5 (−02)	1.5 (−02)	6.2 (−10)	6.1 (−10)	2.1 (−14)	1.9 (−14)
7	0.2405	7.4 (−03)	7.4 (−03)	1.4 (−08)	1.4 (−08)	3.3 (−15)	4.4 (−15)
8	0.4096	1.6 (−01)	1.6 (−01)	2.7 (−07)	2.6 (−07)	8.4 (−15)	7.8 (−15)
9	0.6561	3.5 (−01)	3.5 (−01)	3.9 (−06)	3.8 (−06)	3.8 (−15)	3.8 (−15)
10	1.0000	4.3 (−01)	4.3 (−01)	4.5 (−05)	4.4 (−05)	2.6 (−14)	2.4 (−14)

Table 3
Errors for vibration eigenvalues (*S*/*EI*=0.01)

n	$k_n^2[mA/EI]$	N=11		N=21		N=31	
		RSK	RDK	RSK	RDK	RSK	RDK
1	0.0000	2.2 (−08)	2.2 (−08)	6.7 (−12)	6.9 (−12)	2.3 (−14)	2.2 (−14)
2	0.0012	6.8 (−07)	6.8 (−07)	6.0 (−12)	6.2 (−12)	3.8 (−14)	3.9 (−14)
3	0.0072	1.6 (−05)	1.6 (−05)	4.9 (−12)	5.0 (−12)	1.7 (−14)	1.4 (−14)
4	0.0240	2.5 (−04)	2.4 (−04)	2.9 (−12)	3.0 (−12)	2.8 (−14)	2.9 (−14)
5	0.0600	2.4 (−03)	2.4 (−03)	2.3 (−11)	2.2 (−11)	8.0 (−15)	1.2 (−14)
6	0.1260	1.5 (−02)	1.5 (−02)	6.2 (−09)	6.1 (−10)	2.1 (−14)	1.8 (−14)
7	0.2352	8.8 (−03)	8.8 (−03)	1.4 (−08)	1.4 (−08)	3.5 (−15)	5.3 (−15)
8	0.4032	1.6 (−01)	1.6 (−01)	2.7 (−07)	2.6 (−07)	9.7 (−15)	5.6 (−15)
9	0.6480	3.5 (−01)	3.5 (−01)	3.9 (−06)	3.8 (−06)	1.0 (−14)	4.7 (−15)
10	0.9900	4.3 (−01)	4.3 (−01)	4.5 (−05)	4.4 (−05)	1.8 (−14)	2.2 (−14)

Table 4
Errors for vibration eigenvalues (*S*/*EI*=−1.00)

n	$k_n^2[mA/EI]$	N=11		N=21		N=31	
		RSK	RDK	RSK	RDK	RSK	RDK
1	0.0101	3.5 (−08)	3.5 (−08)	8.5 (−12)	8.8 (−12)	1.9 (−14)	2.0 (−14)
2	0.0416	8.9 (−07)	8.8 (−07)	7.6 (−12)	7.8 (−12)	4.1 (−14)	4.0 (−14)
3	0.0981	2.2 (−05)	2.1 (−05)	6.2 (−12)	6.4 (−12)	1.8 (−14)	2.0 (−14)
4	0.1856	3.4 (−04)	3.4 (−04)	3.8 (−12)	4.0 (−12)	2.8 (−14)	3.6 (−14)
5	0.3125	3.4 (−03)	3.4 (−03)	2.5 (−11)	2.5 (−11)	6.7 (−15)	9.0 (−15)
6	0.4896	2.2 (−02)	2.2 (−02)	6.7 (−10)	6.6 (−10)	2.3 (−14)	2.0 (−14)
7	0.7301	1.3 (−01)	1.3 (−01)	1.6 (−08)	1.5 (−08)	1.0 (−14)	6.9 (−15)
8	1.0496	4.5 (−01)	4.5 (−01)	2.9 (−07)	2.9 (−07)	7.5 (−15)	5.8 (−15)
9	1.4661	6.4 (−01)	6.4 (−01)	4.3 (−06)	4.2 (−06)	4.7 (−15)	3.6 (−15)
10	2.0000	7.2 (−01)	7.2 (−01)	4.9 (−05)	4.9 (−05)	2.0 (−14)	1.9 (−14)

In Table 5 one finds relative errors of the results computed by using the DSC algorithm. The calculation is carried out on a domain of [0, 1] for simply supported beams. The critical values are estimated for six different elastic coefficients k_0 with a constant flexural rigidity EI_0=1. Two sets of grids, N=11 and N=21, are chosen in association with σ/Δ=2.15 and 2.8, respectively. Relative error slightly decreases as the elastic coefficient

Table 5
Relative errors for buckling failure critic values

k_0	Exact	RSK		RDK	
		N=11	N=21	N=11	N=21
50	14.9356635832	1.13 (−04)	1.03 (−10)	1.13 (−04)	1.31 (−10)
100	20.0017227653	8.41 (−05)	7.76 (−11)	8.42 (−05)	1.00 (−10)
200	30.1338411296	5.58 (−05)	4.96 (−11)	5.58 (−05)	6.57 (−11)
400	50.3980778580	3.34 (−05)	2.96 (−11)	3.34 (−05)	4.00 (−11)
800	90.9265513149	1.85 (−05)	1.69 (−11)	1.85 (−05)	2.14 (−11)
1600	171.983498228	9.78 (−06)	9.09 (−12)	9.78 (−06)	1.16 (−11)

increases. All results are very accurate, considering that the grids are very small. In fact, when grid is refined by a factor of 2, the accuracy is improved by 10^6 times.

4. Conclusion

This paper explores the utility of a DSC algorithm [16,46] for beam analysis. RSKs and RDKs are selected for the present computations. However, similar results can be obtained by other DSC kernels as prescribed elsewhere [16,17,41,46]. Three types of beam problems, including beam bending under loads, vibration under forces and buckling critical values, are considered to illustrate the present approach. The Kirchhoff beam theory is used as the framework for all of the abovementioned problems.

In the first example, we consider the bending of a Kirchhoff beam in statics. Since this problem can be simulated with a linear ordinary differential equation which admits an analytical solution, the results of the DSC approach can be objectively evaluated. By using a small number of grid points (N=5, 11 and 21), the DSC algorithm can achieve very high precisions. Both the RSK and RDK perform very well for this problem.

The second example is a class of beam eigenvalue problems rendered from vibrations under axial forces. Both compressive and tensile forces are considered in the present treatment. This example is also very valuable because it is analytically solvable. The performances of the RSK and RDK are excellent; only 11 grid point in a large interval of 10π is required to achieve the accuracy of five significant figures or better for the first three eigenvalues. Machine precision is reached for all the first 10 eigenvalues when the mesh is slightly refined (N=31). The last problem tackled is the buckling phenomenon for elastic Kirchhoff beams. It is also known as the classical Euler problem and can be modeled by a linear ordinary differential equation. It in fact admits an analytical solution for the uniform flexural rigidity and simply supported boundary condition. We estimate the buckling failure critical values for a number of elastic coefficients k_0. Very high precisions are attained by using small sets of grid points.

Our results from these three classes of test problems indicate that the DSC algorithm is a reliable and robust approach for the numerical analysis of Kirchhoff beams. Numerical results for more complicated beam and frame problems, which usually do not have analytical solutions, will be presented elsewhere.

The feature of the DSC algorithm includes the following. It is much more accurate than the standard local methods, such as the finite element methods, for numerical approximations. It requires very few computational grid points to achieve high accuracy and thus, is suitable for large scale computations. In particular, the DSC approach is found to be extremely accurate and stable for the treatment of high frequency vibrations [43]. It provides reliable predictions for plate vibration modes up to the order of a few thousands [44], which is unattainable by any other existing method at present. The DSC algorithm also provides a reliable framework for the implementation of a new shock capture scheme [45]. A small number of grid points is required for the DSC-based treatment of non-smooth shock fronts.

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