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Wavelets and distributed approximating functionals

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Abstract

A general procedure is proposed for constructing father and mother wavelets that have excellent time-frequency localization and can be used to generate entire wavelet families for use as wavelet transforms. One interesting feature of our father wavelets (scaling functions) is that they belong to a class of generalized delta sequences, which we refer to as distributed approximating functionals (DAFs). We indicate this by the notation wavelet-DAFs. Correspondingly, the mother wavelets generated from these wavelet-DAFs are appropriately called DAF-wavelets. Wavelet-DAFs can be regarded as providing a pointwise (localized) spectral method, which furnishes a bridge between the traditional global methods and local methods for solving partial differential equations. They are shown to provide extremely accurate numerical solutions for a number of nonlinear partial differential equations, including the Korteweg–de Vries (KdV) equation, for which a previous method has encountered difficulties (J. Comput. Phys. 132 (1997) 233). © 1998 Elsevier Science B.V.

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The theory and application of wavelets has been one of the most rapidly developing, multi-disciplinary subjects of the last decade [1–3]. The pervasive nature of the subject is creating scientific and technical linkages among mathematicians, engineers and scientists. There have been spectacular wavelet successes in telecommunications and electrical engineering, including a host of applications such as image and signal processing, data compression and pattern recognition. Since wavelets are intimately and significantly related to spline theory and the theory of approximations, it would seem that wavelet theory ought to lead to entirely new approaches for scientific computations, where traditional methods are either global [4–7] or local [8–11]; it is well known that global spectral methods are accurate and efficient for linear partial differential equations, whereas local methods are simple and convenient for nonlinear partial differential equations. It is extremely desirable to develop an approach that delivers global method accuracy, while also providing local method flexibility and simplicity, particularly for nonlinear partial differential equations involving singularities. Singularities are crucial for describing a variety of physical phenomena [12], ranging from black holes in astronomy, shock waves in compressible gas flow, vortex sheets in high Reynolds number incompressible fluid flow, to critical points of Bose–Einstein condensates. Obtaining accurate numerical solutions for such problems is still a major challenge [12]. Wavelet theory has been expected to fulfill this task and has been extensively studied recently for this purpose [13–15]. However, these efforts have been hindered either by the technical difficulties of incorporating multiresolution analysis into

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the treatment of boundary conditions or by the lack of accurate and efficient wavelets and/or wavelet packets for solving partial differential equations (in the sense of computational physics, where an accuracy on the order of one part in 10^6 or better is often required). In this work we propose a general scheme for constructing wavelets by using generalized delta sequences. These wavelets are found to be both extremely accurate and robust for solving linear and nonlinear partial differential equations.

We shall consider the Gabor functions [16–18] defined by

$$G_{a_k,\sigma}(x) = e^{-x^2/2\sigma^2}\cos(a_k x)\,, \tag{1}$$

as a basic constituent in our construction of scaling functions (called father wavelets in some literature). Here, $a_k = ka$ and a is the central frequency. Although Gabor functions are *not* orthogonal, and they do *not* satisfy the wavelet admissibility condition, they are known to provide excellent time-frequency representation of a signal due to their Gaussian decay [1–3]. Thus, they are very useful for general time-frequency analyses. In such applications, the best results are usually obtained if the window size σ varies as a function of the central frequency a, and $\sigma = r\pi/a$, where r is a parameter chosen during computations [17]. However, it should be remarked that the direct use of Gabor functions for solving partial differential equations has not been very successful.

We next consider the Mth order Dirichlet kernel, $D_M(x)$, which is given by [2]

$$D_M(x) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{M}\cos(kx)\right] = \frac{\sin(M+\frac{1}{2})x}{2\pi\sin\frac{x}{2}}\,. \tag{2}$$

The Dirichlet kernel is useful for the reconstruction of a function $f(x)$ by its Fourier series. That is, the Mth partial sum f_M of a Fourier series f can be obtained by the convolution of f with the Dirichlet kernel of degree M,

$$f(x) \approx f_M(x) = \int_0^{2\pi} f(y)D_M(x-y)\,dy\,. \tag{3}$$

As M approaches infinity, any well-behaved periodic function is exactly recovered by this procedure. That is,

$$f(x) = \lim_{M\to\infty} f_M(x)\,, \tag{4}$$

which implies that the Dirichlet kernel is a type of delta sequence. The Dirichlet kernel has been widely used as a statistical estimator in the mathematical literature [2]. Rutkowski [19] demonstrated its use for extraction of nonlinear regression from noisy data.

Chui and Mhaskar [20] have discussed the use of Dirichlet kernels as wavelets in $L^2[0,2\pi]$. It is well known in the wavelet community that smoothness and decay rate are two of the most important criteria for evaluating a wavelet basis. This suggests that an appropriate combination of the Gabor functions and the Dirichlet kernel should produce father wavelets which are both *smooth* and *rapidly decaying*. To this end, we combine Gabor functions, Eq. (1), with the Dirichlet kernel, Eq. (2), to postulate a scaling function (father wavelet) of the form

$$\begin{aligned}\phi(x) &= C_{M,\sigma}e^{-x^2/2\sigma^2}\left[\sum_{k=0}^{M}\cos\left(\frac{2\pi}{L}kx\right) - \frac{1}{2}\right]\\ &= C_{M,\sigma}e^{-x^2/2\sigma^2}\left[\frac{\sin(M+\frac{1}{2})\frac{2\pi}{L}x}{2\sin\frac{\pi}{L}x}\right],\end{aligned} \tag{5}$$

where the domain of x is taken to be the real line. Here, σ and L are parameters with the dimension of length. The present procedure is based on mathematical regularization [21] and its computational utility must be tested numerically. It is noted that the introduction of the Gaussian factor destroys the periodicity of the Dirichlet kernel. Operationally, one can think of smoothing the ideal low pass filter implicit in the definition of the Dirichlet kernel with a Gaussian envelope. According to wavelet convention [2], the constant $C_{M\sigma}$ is determined by the zero frequency Fourier transform $\hat{\phi}$ of ϕ,

$$\begin{aligned}\hat{\phi}(0) &= \int_{-\infty}^{\infty}\phi(x)\,dx\\ &= C_{M\sigma}\sqrt{2\pi\sigma^2}\left[\frac{1}{2} + \sum_{k=1}^{M}\exp\left(-\frac{2\pi^2\sigma^2k^2}{L^2}\right)\right]\\ &= 1\,.\end{aligned} \tag{6}$$

We shall call these Dirichlet–Gabor scaling functions or father wavelets and they can be used to generate *both* continuous and discrete wavelet families for performing wavelet transforms by the standard translation and dilation procedures. These procedures give

$$\phi_{m,n}(x) = a^{-m/2}\phi\left(\frac{x-nb}{a^m}\right)$$
$$(a > 1, b > 0;\ m, n \in \mathbb{Z}), \qquad (7)$$

where the factor $a^{-m/2}$ is introduced to ensure the usual $L^2(R)$ normalization of a wavelet expansion. We introduce a corresponding mother wavelet as

$$\psi(x) = C_{M,\sigma}\left\{ e^{-x^2/2\sigma^2}\left[\frac{\sin(M+\frac{1}{2})\frac{2\pi}{L}x}{2\sin\frac{\pi}{L}x}\right] - \frac{1}{a}e^{-x^2/2a^2\sigma^2}\left[\frac{\sin(M+\frac{1}{2})\frac{2\pi}{aL}x}{2\sin\frac{\pi x}{aL}}\right]\right\}. \qquad (8)$$

It follows from Eqs. (6)–(8) that the Fourier transform of the mother wavelet $\hat{\psi}$ must vanish at the origin, that is,

$$\hat{\psi}(0) = \int \psi(x)\,dx = 0. \qquad (9)$$

Therefore, ψ is indeed a "small wave" and is admissible for wavelet transforms [2]. A multiresolution analysis based on the Dirichlet–Gabor wavelets can be constructed. However, since in a multiresolution analysis it is difficult to handle complex boundary conditions of the kinds that arise in computational physics and engineering applications [15], we do not discuss this matter here but rather will consider it elsewhere [21].

The basic idea underlying the present work is the construction of smooth, decaying scaling functions or father wavelets which are generalized delta sequences parameterized by the quantities σ and M, which in general can themselves be functions of x. As such, they are also distributed approximating functionals (DAFs) [22,23]. We see that for a given finite σ,

$$\lim_{M\to\infty}(C_{M,\sigma}\pi)^{-1}\phi(x) = e^{-x^2/2\sigma^2}\delta_{2\pi}\left(\tfrac{\pi}{L}x\right), \qquad (10)$$

is effectively a delta function (Here $\delta_{2\pi}(\frac{\pi}{L}x)$ is a 2π periodic Dirac delta function). Furthermore, when σ approaches zero,

$$\lim_{\sigma\to 0^+}\{C_{M,\sigma}\sqrt{2\pi}\,\sigma(M+\tfrac{1}{2})\}^{-1}\phi(x) = \left[\frac{\sin(M+\frac{1}{2})\frac{2\pi}{L}x}{(2M+1)\sin\frac{\pi}{L}x}\right]\delta(x) = \delta(x). \qquad (11)$$

We note that neither of these expression rigorously implies that $\phi(x)$ itself approaches a delta function in the same limits; however, the DAF parameters can be chosen in such a way so that this is an excellent approximation. This behavior is characteristic of DAFs. Further discussion of the point here would divert us from our main purpose, but we will consider this matter in more detail in a subsequent publication. In general, when the scaling function or father wavelet is also a member of a generalized delta sequence, we shall call it a "wavelet-DAF". Therefore, expression (5) is called Dirichlet–Gabor wavelet-DAF (DGWD). Accordingly, the corresponding mother wavelet, such as the expression in Eq. (8), is called a "DAF-wavelet".

Somewhat related discussions have been given previously. For example, Walter [24] has investigated wavelet resolutions of the Dirac delta function. However, his discussion was restricted to orthonormal wavelet systems, such as the Haar, Franklin and Shannon wavelets [1,2,24], which do not have optimal time-frequency localization nor both the smoothness and decaying properties.

In the remainder of this work we shall focus the discussion on using the DGWDs for numerical solutions of partial differential equations using a spatial discretization of the wavelet-DAFs and a fourth order Runge–Kutta scheme for the time propagations.

To proceed, we discretize the wavelet-DAF approximation of the function

$$f(x) \approx \int_{-\infty}^{\infty} f(y)\phi(x-y)\,dy, \qquad (12)$$

by choosing a grid with equal spacing $\Delta = L/(2M+1)$. In principle, the integral requires an infinite discrete grid, but in practice, the Gaussian weight allows one to restrict the sum to a finite bandwidth about the point x,

$$f(x) \approx \sum_{k=-W}^{W} f(x_k)e^{-(x-x_k)^2/2\sigma^2} \times\left[\frac{\sin\left(\frac{\pi}{\Delta}(x-x_k)\right)}{(2M+1)\sin\left(\frac{\pi}{\Delta}\frac{x-x_k}{2M+1}\right)}\right], \qquad (13)$$

where W is the computational bandwidth. The derivative operators on a grid are approximated, in DAF-fashion, as

$$\frac{\partial^q}{\partial x^q} f(x) \approx \sum_k f(x_k) \left\{ e^{-(x-x_k)^2/2\sigma^2} \times \left[\frac{\sin\left(\frac{\pi}{\Delta}(x-x_k)\right)}{(2M+1)\sin\left(\frac{\pi}{\Delta}\frac{x-x_k}{2M+1}\right)} \right] \right\}^{(q)}, \qquad (14)$$

where the superscript (q) attached to the DGWD represents the qth derivative with respect to x. Eqs. (13) and (14) serve as the basis for solving various partial differential equations. In the present computations, we take $M = 35$, $W = 62$ and $\sigma/\Delta = 8.8$ for all cases.

To demonstrate the robust nature and test the accuracy of the DGWDs for solving partial differential equations, we shall consider the numerical solution of the Korteweg–de Vries (KdV) equation [25] and the generalized KdV equation. The reason for choosing these equations is twofold: first, the KdV equation is extremely important in theoretical physics and applied mathematics because it admits solitary wave solutions. Such solutions are ubiquitous in nature and describe a wide range of physical, chemical and biological phenomena. Second, despite the fact that an enormous amount of effort has been expended in the last few decades to understand these nonlinear systems, there is still not a general, systematic computational approach which is applicable to all KdV equations. Spurious soliton phase shifts have been found to be induced by various numerical methods [26]. In particular, the most sophisticated adaptive wavelet method [13] has encountered difficulties in obtaining the basic solutions.

The generalized KdV equation is

$$\frac{\partial u(x,t)}{\partial t} + \alpha u^n(x,t)\frac{\partial u(x,t)}{\partial x} + \beta \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \qquad (15)$$

where α and β are constants. The $n = 1$ case is the well-known KdV equation which admits the soliton solution

$$u(x,t) = 3\eta\,\mathrm{sech}^2(Ax - Bt + D), \qquad (16)$$

where $\eta > 0$ is the soliton amplitude, $A = \frac{1}{2}(\alpha\eta/\beta)^{1/2}$, $B = \eta\beta A$ and D is the initial phase. Another important case of Eq. (15) is $n = 2$, different forms of which are best understood by making use of the Miura transformation [27]. It has a number applications, such as in nonlinear optics [28]. The analytical solution for this case (for $\alpha = 6$ and $\beta = 1$) is $u(x,t) = 2\eta\,\mathrm{sech}[2\eta(x - 4\eta^2 t)]$, with the initial soliton wave $u(x,0) = 2\eta\,\mathrm{sech}(2\eta x)$.

Table 1
Errors of the numerical solutions for the KdV equation ($n = 1$, $\tau = 0.01$)

	SC		DGWD	
t	L_2	L_∞	L_2	L_∞
	$\Delta = 0.0333$		$\Delta = 0.0342$	
0.25	5.94(−03)	2.80(−03)	4.81(−04)	5.58(−04)
0.50	7.56(−03)	4.53(−03)	5.15(−04)	8.59(−04)
0.75	8.70(−03)	4.85(−03)	4.42(−04)	8.32(−04)
1.00	9.49(−03)	5.85(−03)	4.37(−04)	6.31(−04)

We first consider the numerical solution of Eq. (15) with an initial solitary wave packet $u(x,0) = 3\eta\,\mathrm{sech}^2(Ax + D)$. The values of the parameters were chosen as $\eta = 0.3, D = -6, \alpha = 1$ and $\beta = 4.84 \times 10^{-4}$. Our results are compared with those of Sanz-Serna and Christie (SC) [26] obtained using their modified Petrov–Galerkin method. As shown in Table 1, for the case $\tau = 0.01$, the DGWD results are about 5 to 10 times more accurate than those of SC, while using a much larger coordinate grid spacing Δ. The accuracy and reliability of our DGWD method are also confirmed by numerically solving the generalized KdV equation (15) with $n = 2$. In the present calculation, we choose $\eta = 0.4, \Delta = 0.203$ and $\tau = 0.001$. The results for the generalized KdV equation are listed in Table 2. Our DGWD approach achieves even higher accuracy here than in the previous case.

In various test calculations, we have found that the DGWD method provides extremely accurate results for a number of other important nonlinear partial differential equations. Included are the nonlinear Schrödinger equation, the Sine–Gordon equation, Burgers' equation and the Navier–Stokes equations. These results will be presented elsewhere [21]. Unlike for other wavelet methods, the present DGWD approach is *extremely* simple and robust: one simply uses the DGWD, Eqs. (13) and (14) for representing spatial derivatives and the Runge–Kutta scheme for the time propagation. As expected for wavelet methods,

Table 2
Errors of the numerical solutions for the generalized KdV equation ($n = 2$, $\tau = 0.001$, $\Delta = 0.203$)

t	L_2	L_∞
0.25	7.44(−12)	3.39(−12)
0.5	1.36(−11)	5.84(−12)
1.0	2.55(−11)	9.99(−12)
2.0	4.87(−11)	1.98(−11)
4.0	9.07(−11)	3.10(−11)
8.0	1.62(−10)	4.95(−11)
10.0	1.93(−10)	5.68(−11)

the DGWD matrix representations of both a function and its derivatives are banded and the corresponding bandwidths can be easily optimized for a given application. This endows the DGWDs with great efficiency for large-scale computations, of the sort frequently encountered in molecular dynamical, biological and engineering problems

In conclusion, we have constructed the DGWDs by appropriately combining Dirichlet kernels with the Gabor functions. The resulting DGWDs have several important properties. First, they can be used as DAF-L^2 kernels, as demonstrated by solving the KdV equation, for which the previous adaptive wavelet method [13] encountered difficulties. This makes use of the fact that they are well-behaved generalized delta sequences, which follows the behavior that when σ approaches zero or M approaches infinity in certain manner, they behave like Dirac delta functions [22,23]. Second, the DGWDs are scaling functions or father wavelets. A family of time-frequency optimized Dirichlet Gabor mother wavelets, or DAF-wavelets, has been generated from the DGWDs. These DAF-wavelets have great potential for applications in signal and image processing and many other engineering fields. These topics are under investigation.

It is clear that ours is a general procedure for generating various wavelets. Basically, all that is needed is to combine a group of Gabor functions in such a manner that they belong to a class of generalized delta sequences, which are DAFs. The generalized delta sequences are normalized to unity in $\mathbb{R}$ (or other desired domains), and thus serve as scaling functions (father wavelets) or wavelet-DAFs. These scaling functions (father wavelets) can be used both for generating mother wavelets or DAF-wavelets and as wavelet-DAFs for solving partial differential equations on a grid. The resulting DAF-wavelets can be used for wavelet transforms (including a multiresolution analysis). Using this procedure, we find that wavelet-DAFs generated as a windowed de la Vallée Poussin Kernel, $2/3\exp(-x^2/2\sigma^2)[\cos(ax) - \cos(2ax)]/(ax)^2$, and windowed modified Dirichlet kernel,

$$\begin{aligned}\phi(x) &= \frac{1}{M}e^{-x^2/2\sigma^2}\left[\sum_{k=0}^{M}\cos(kax)\right.\\&\qquad\left.-\frac{1+\cos(Max)}{2}\right]\\&= \frac{1}{M}e^{-x^2/2\sigma^2}\left[\frac{\sin(Max)}{2\tan\frac{ax}{2}}\right],\end{aligned} \tag{17}$$

also work very well for solving various partial differential equations [21]. We point out that it is straightforward to construct Hermite DAF-wavelets and interpolating Lagrange DAF-wavelets [21] by using our Hermite-DAF [22] and Lagrange DAF [23], respectively.

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