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# Quasi wavelets and quasi interpolating wavelets

G.W. Wei [1]

*Department of Computational Science, National University of Singapore, Singapore 119260, Singapore*

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**Abstract**

This Letter introduces two families of new wavelets, quasi wavelets and quasi interpolating wavelets. It is found that the mathematical regularization of Shannon's orthonormal interpolating wavelet leads to an interplay between a quasi wavelet and a quasi interpolating wavelet. The resulting quasi wavelet scaling function preserves the interpolation property but it does not satisfy the wavelet normalization requirement. Whereas the quasi interpolating wavelet scaling function satisfies the normalization requirement but it is no longer rigorously interpolative. Both quasi wavelets and quasi interpolating wavelets are functions of the Schwartz class. Their numerical performance is extremely similar to each other for data interpolation, differentiation and for solving partial differential equations. 

Wavelet analysis is a new mathematical branch which has found success in a variety of science and engineering disciplines, particularly in telecommunications and electronics engineering. Although wavelets themselves were recognized in one form or another by scientists in various disciplines before the 1980s, a geophysicist, Morlet [1], was the first to introduce the word ''wavelet'' and constructed it by translation and dilation of one fixed function in the early 1980s. Since then, many constructions of wavelets have been introduced in both mathematical analysis and signal processing literature. In fact, the fruitful interaction between these communities is largely responsible for the success of wavelets. A mathematical framework was developed by the ''French school'' [2–4]. In 1985, Meyer discovered a method to construct orthogonal wavelet bases for $L^2(R)$ with a certain order of regularity, which has significant advantage over the earlier Haar wavelets [5]. A major breakthrough was the construction by Daubechies [6] of orthogonal, compactly supported wavelets. In signal processing, pyramidal schemes and sub-band coding, or more precisely, quadrature mirror filters were developed by several engineerings in the 1970s to reduce quantization noise [7]. Mallat and Meyer made a decisive step in the theory of wavelets in 1987 when they proposed a fast algorithm for the computation of wavelet coefficients [8]. Their work unified the earlier signal processing techniques with wavelet theory in terms of a multiresolution analysis. Orthogonal wavelets were further generalized to interpolating wavelets [9], wavelet packets [10], and biorthogonal [11] or semiorthogonal wavelets [12]. (In the latter case wavelets on different scales are orthogonal, whereas wavelets on the same scales are not.) Interpolating wavelets are particularly efficient from a computational point of view [9]. Wavelet packets can be used to provide adapted wavelet transforms which can be optimized accord-

[1] E-mail: cscweigw@nus.edu.sg


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ing to specific problems. Biorthogonality allows the construction of symmetric wavelets, hence linear phase filters, in a flexible manner. Examples are the construction of compactly supported semiorthogonal wavelets and recursive filter banks. Sweldens [13] has provided a robust lifting scheme for a custom-design construction of biorthogonal wavelets.

The functions introduced by Morlet are a different class of wavelets; they are frames. Like bases, frames are also building blocks in Hilbert spaces, with which one can obtain representations for an $L^2$ function; but, unlike bases, frames are not necessarily linearly independent. Frames can also be generated from one single function, by translation, modulation, dilation, or combination thereof. They are intimately connected to windowed Fourier transform or short-time Fourier transform and sampling theory in reproducing kernel Hilbert spaces. The latter involves interpolating wavelets and plays an important role in wavelet theory and signal processing. Basically, a sampling scheme can be regarded as a (discrete) interpolating wavelet transform which is one of the most efficient approaches for representing a function in the Hilbert space. The sampling kernels, such as Shannon's sampling kernel [14] and generalized Lagrange sampling kernel, are interpolating wavelets. It is interesting to note that a family of sampling kernels actually provides a sequence of functions that converge to the Dirac delta distribution. Recently we have analyzed the physical applicability of various sampling kernels, which includes the examination of the generalized Lagrange sampling kernel [15] and wavelet analysis of Dirichlet's sampling kernel [16,17]. This analysis was later applied to Shannon's sampling kernel [15,16,18]. These kernels are poorly localized in the coordinate domain and are discontinuous in their Fourier representations. A regularization technique developed in an earlier work [19] was applied to efficiently suppress the computational bandwidth of the sampling scheme and to obtain regularized interpolating kernels. The purpose of this Letter is to introduce two new wavelets, namely, quasi wavelets and quasi interpolating wavelets, and to discuss their relations. The numerical performance of these new wavelets is examined by real physical applications.

The formal theory of orthogonal wavelets on $L^2(R)$ has been presented in many books [20–27]. An orthogonal wavelet system is usually generated from a single function, a mother wavelet $\psi$, by a standard operation of translation $[T_n\psi(x)=\psi(x-n)]$ and dilation $[D_m\psi(x)=2^{-m/2}\psi(2^{-m}x)]$

$$\psi_{mn}(x)=D_mT_n\psi(x)=2^{-m/2}\psi\left(\frac{x}{2^m}-n\right),\quad m,n\in Z, \tag{1}$$

where the symbol $Z$ denotes the set of all integers. It is convenient to introduce a scaling function (father wavelet), $\phi(x)$, with a normalization,

$$\hat{\phi}(0)=\int\phi(x)\,\mathrm{d}x=1. \tag{2}$$

Then the wavelet expansion of $L^2(R)$ can be phrased rigorously in terms of a multiresolution analysis, i.e. a nested sequence of subspaces $\{V_m\}$, $m\in Z$ such that

1. $\{T_n\phi(x)=\phi(x-n)\}$ is an orthogonal basis of $V_0$;
2. $\cdots\subset V_1\subset V_0\subset V_{-1}\subset\cdots\subset L^2(R)$;
3. $f(x)\in V_m\leftrightarrow f(2x)\in V_{m-1}$;
4. $\bigcap_m V_m=\{0\}$ and $\overline{\bigcup_m V_m}=L^2(R)$.

Since $\phi\in V_0\subset V_{-1}$, it can be expressed as superposition of $\{\phi_{-1,n}\}$, which span an orthogonal basis for $V_{-1}$

$$\phi(x)=\sum_n a_n\phi_{-1,n}, \tag{3}$$

where $\{a_n=\langle\phi,\phi_{-1,n}\rangle\}$ is a set of finite coefficients.

For an orthogonal system, the subspace $V_{-1}$ can be further split into its orthogonal projection in $V_0$ and a complement $W_0$

$$V_{-1}=V_0\oplus W_0, \tag{4}$$

where $W_0$ is a subspace spanned by orthogonal mother wavelets $\{\psi\}$. In general, $\psi_{mn}$, $n\in Z$ is an orthogonal basis of $W_{-m}$ and

$$\oplus_{m\in Z}W_m=L^2(R). \tag{5}$$

It follows that $\psi_{mn}$ $(m,n\in Z)$ is an orthogonal basis of $L^2(R)$. Similarly to Eq. (3), the mother wavelet can also be expressed as a superposition of $\{\phi_{-1,n}\}$

$$\psi(x)=\sum_n b_n\phi_{-1,n}, \tag{6}$$

where $b_n=(-1)^{1-n}a_{1-n}$ are expansion coefficients.

One of the most important wavelets is Shannon's wavelet and its scaling function (father wavelet) is given by

$$\phi(x) = \frac{\sin \pi x}{\pi x}. \tag{7}$$

This satisfies Eq. (2) and in general, its Fourier transform is given by the characteristic function $\hat{\phi}(\omega) = \chi_{[-1/2,1/2)}$. It is easy to see that

$$\sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega + n)|^2 = 1. \tag{8}$$

This equation implies that the sequence of function $\{\phi(x-n)\}_{n=-\infty}^{\infty}$ is orthonormal. The expansion of Eq. (3) is realized by a self-similar relation

$$\phi(x) = \sum_{n=-\infty}^{\infty} \frac{\sin(n\pi/2)}{n\pi/2} \phi(2x-n). \tag{9}$$

Shannon's scaling functions are intimately related to Dirichlet's continuous delta sequence kernels [28]

$$\lim_{\alpha \to \infty} \langle \frac{\sin \alpha x}{\pi x}, \eta(x) \rangle = \eta(0), \tag{10}$$

where $\eta$ is a test function. Moreover, Shannon's mother wavelet can be calculated from Eq. (6)

$$\psi(x) = \frac{\sin 2\pi x - \sin \pi x}{\pi x}, \tag{11}$$

which is an ideal band pass filter in its Fourier representation

$$\hat{\psi}(\omega) = \chi_{[-1,1]}(\omega) - \chi_{[-1/2,1/2]}(\omega). \tag{12}$$

Other multiresolution analysis properties can easily be verified.

The most important property of Shannon's scaling function and wavelet is its use as a reproducing kernel

$$K(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}. \tag{13}$$

Reproducing means

$$f(x) = \int_{-\infty}^{\infty} f(y) \frac{\sin \pi(x-y)}{\pi(x-y)} \mathrm{d}y, \quad \forall f \in B_\pi^2, \tag{14}$$

where $\forall f \in B_\pi^2$ indicates that, in its Fourier representation, the $L^2$ function $f$ vanishes outside the interval $[-\pi,\pi]$. Here $B_\pi^2$ is the Paley–Wiener reproducing kernel Hilbert space which is a subspace of the Hilbert space $L^2(R)$. It is noted that not every Hilbert space is a reproducing kernel Hilbert space. In particular, the space $L^2(R)$ is not a reproducing kernel Hilbert space. The Paley–Wiener reproducing kernel Hilbert space has a very useful *sampling basis*

$$S_n(x) = K(x,y_n) = \frac{\sin \pi(x-y_n)}{\pi(x-y_n)},$$

$$y_n = n, \quad \forall n \in Z. \tag{15}$$

It provides a *discrete representation* of every function in $B_\pi^2$

$$f(x) = \sum_{n \in Z} f(y_n) S_n(x), \quad \forall f \in B_\pi^2. \tag{16}$$

This is recognized as Shannon's sampling theorem [14]. $S_n$ is the scaling function of Shannon's wavelet and it satisfies the interpolation condition

$$S_n(x_m) = \delta_{n,m}. \tag{17}$$

Note that Shannon's wavelet, Eq. (11) is obviously interpolative on $Z$. Computationally, being interpolative is of particular importance for numerical accuracy and simplicity.

Both $S_n(x)$ and its associated wavelet play a crucial role in information theory and signal processing. However, their utility is limited by the fact that $\hat{\phi}(\omega)$ and $\hat{\psi}(\omega)$ are discontinuous functions. As a consequence, $\phi(x)$ and $\psi(x)$ are delocalized in coordinate space. Based on a regularization procedure introduced in Ref. [19], we discussed a Gaussian weighted Shannon's sampling kernel [16–18]

$$\Phi_\sigma(x) = \frac{\sin \pi x}{\pi x} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \sigma > 0. \tag{18}$$

On a grid, this is discretized as an interpolating kernel

$$\Phi_{\sigma,\Delta}(x-y_n) = \frac{\sin \frac{\pi}{\Delta}(x-y_n)}{\frac{\pi}{\Delta}(x-y_n)} \times \exp\left(-\frac{(x-y_n)^2}{2\sigma^2}\right). \tag{19}$$

It is noted that if $\Delta$ is chosen as the spatial mesh size, $\Phi_{\sigma,\Delta}(x-y_n)$ retains the interpolation property, Eq. (17), which is of particular merit for numerical calculations. In applications, the best results are usually obtained if the window size $\sigma$ varies as a function of the central frequency $\pi/\Delta$, such that $r=\sigma/\Delta$ is a parameter chosen in computations. An immediate benefit of the regularized Shannon's scaling function Eq. (18) is that its Fourier transform is no longer discontinuous. The numerical utility of this and a few other regularized interpolating expressions have been examined in some detail [16–18]. However, some important properties were overlooked.

The normalization of $\Phi_\sigma(x)$ is

$$\begin{aligned}\hat{\Phi}_{\sigma,\Delta}(0) &= \int \Phi_{\sigma,\Delta}(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\sin\frac{\pi}{\Delta}x}{\pi x}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x \\ &= \sqrt{2\pi}\,\frac{\sigma}{\Delta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(2k+1)}\left(\frac{\pi\sigma}{\sqrt{2}\,\Delta}\right)^{2k}. \end{aligned} \tag{20}$$

This can be rewritten as

$$\begin{aligned}\hat{\Phi}_{\sigma,\Delta}(0) &= \mathrm{erf}\left(\frac{\pi\sigma}{\sqrt{2}\,\Delta}\right) \\ &= 1-\sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma}\exp\left(-\frac{\sigma^2\pi^2}{2\Delta^2}\right) \\ &\quad\times\int_0^{\infty}\exp\left(-\frac{t^2}{2\sigma^2}-\frac{\pi t}{\Delta}\right)\mathrm{d}t \\ &= 1-\mathrm{erfc}\left(\frac{\pi\sigma}{\sqrt{2}\,\Delta}\right), \end{aligned} \tag{21}$$

where $\mathrm{erf}(z)=(2/\sqrt{\pi})\int_0^z \mathrm{e}^{-t^2}\mathrm{d}t$ is the error function and erfc is the complementary error function. Obviously, for a given grid spacing $\Delta$, $\mathrm{erfc}(\pi\sigma/\sqrt{2}\,\Delta)$ is positively defined for all $\sigma>0$, and $\hat{\Phi}_{\sigma,\Delta}(0)$ is always less than unity except at the limit of $\sigma\to\infty$. In view of Eq. (2), $\Phi_{\sigma,\Delta}(x)$ does *not* really satisfy the requirement of a wavelet scaling function. However, when we choose $r=\sigma/\Delta \gg \sqrt{2}/\pi$, which is the case in practical applications, the residue term, $\mathrm{erfc}(\pi\sigma/\sqrt{2}\,\Delta)$, approaches zero extremely fast. Consequently, $\hat{\Phi}_{\sigma,\Delta}(0)$ is extremely close to unity. Therefore, we shall call the regularized Shannon's scaling function $\Phi_{\sigma,\Delta}$ a *quasi-scaling function* and its corresponding wavelets generated by the difference of two quasi-scaling functions

$$\Psi_\sigma(x) = \frac{\sin 2\pi x}{\pi x}\exp\left(-\frac{2x^2}{\sigma^2}\right) - \frac{\sin \pi x}{\pi x}\exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{22}$$

are called *quasi wavelets*. In the Fourier presentation, $\hat{\Phi}_\sigma$ is given by the convolution of two quantities, the Gaussian and the characteristic function, $\chi_{[-1/2,1/2)}$. The latter is an ideal low pass filter. As a consequence, the quasi scaling function is a *smoothed low pass filter*. Correspondingly, the quasi wavelet $\hat{\Psi}_\sigma$ is a *smoothed high pass filter* with infinite order of regularity.

Obviously, the quasi scaling function, Eq. (18) can be rescaled to satisfy condition (2),

$$\bar{\Phi}_\sigma(x) = \hat{\Phi}_{\sigma,1}^{-1}(0)\,\frac{\sin\pi x}{\pi x}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \sigma>0, \tag{23}$$

with its discrete kernel form

$$\bar{\Phi}_{\sigma,\Delta}(x-y_n) = \hat{\Phi}_{\sigma,\Delta}^{-1}(0)\,\frac{\sin\frac{\pi}{\Delta}(x-y_n)}{\frac{\pi}{\Delta}(x-y_n)} \times\exp\left(-\frac{(x-y_n)^2}{2\sigma^2}\right). \tag{24}$$

However, this ''true'' scaling function is no longer interpolative

$$\bar{\Phi}_{\sigma,\Delta}(x_m-x_n) \neq \delta_{n,m}. \tag{25}$$

As a consequence, the corresponding wavelets

$$\bar{\Psi}_\sigma(x) = \hat{\Phi}_{\sigma,1}^{-1}(0)\left[\frac{\sin 2\pi x}{\pi x}\exp\left(-\frac{2x^2}{\sigma^2}\right) - \frac{\sin\pi x}{\pi x}\exp\left(-\frac{x^2}{2\sigma^2}\right)\right] \tag{26}$$

lose their interpolation property too. However, this can be made arbitrarily close to the true interpolation because the factor $\hat{\Phi}^{-1}_{\sigma,1}(0)$ is *almost* unity when $\sigma$ is sufficiently large. We shall call these $\overline{\Psi}_\sigma(x)$and $\overline{\Phi}_\sigma(x)$ *quasi interpolating wavelets* and *quasi interpolating scaling functions*, respectively.

Conceptually, both quasi wavelets and quasi interpolating wavelets are two different families of wavelets with much different mathematical properties. It follows that after the regularization, there is an interplay between the normalization and the interpolation. One can either have a family of quasi wavelets or a family of quasi interpolating wavelets. Hence, the regularity of the original Shannon's orthonormal interpolating wavelets and their scaling functions is improved at the cost of their normalization or interpolation. It is noted that these quasi interpolating wavelets can be made as close to true interpolating as one wishes by an appropriate choice of ratio $r=\sigma/\Delta$. A major advantage of our new wavelets is their infinite regularity. Numerically, they are extremely powerful for a wide range of applications.

The normalization $\hat{\Phi}^{-1}_{\sigma,\Delta}(0)$ as a function of the difference ratio $r$ is listed in Table 1. These are computed with 5000 terms according to the expression

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\,e^{-z^2}\sum_{k=0}\frac{2^k}{(2k+1)!!}z^{2k+1}$$

and the results are well converged in the sense of double precision. When $r$ is very small, $\hat{\Phi}^{-1}_{\sigma,\Delta}(0)$ is also small so that quasi wavelets and quasi interpolating wavelets are very different. This difference becomes smaller when $r$ becomes larger. When $r=1.8$, the difference between quasi wavelets and quasi interpolating wavelets diminishes to the size of computer round off error for a single precision computation. Similarly, when $r=2.6$, the difference between a quasi wavelet and a quasi interpolating wavelets has reduced to the size of computer round off error for a double precision computation.

Table 1
The normalization factor $\hat{\Phi}^{-1}_{\sigma,\Delta}(0)$ for different ratios $r=\sigma/\Delta$

| Ratio | $\hat{\Phi}^{-1}_{\sigma,\Delta}(0)$ |
|---|---|
| 0.2 | 0.470204705280702 |
| 0.4 | 0.791114911884962 |
| 0.6 | 0.940564166413682 |
| 0.8 | 0.988038365036296 |
| 1.0 | 0.998319683663473 |
| 1.2 | 0.999836694348003 |
| 1.4 | 0.999989086256835 |
| 1.6 | 0.999999500613110 |
| 1.8 | 0.999999984403280 |
| 2.0 | 0.999999999668295 |
| 2.2 | 0.999999999995205 |
| 2.4 | 0.999999999999953 |
| 2.6 | 0.999999999999999 |

It is noted that although the normalization factor $\hat{\Phi}^{-1}_{\sigma,\Delta}$ is extremely close to unity when $r$ is close to 2.6, conceptually, quasi wavelets and quasi interpolating wavelet remains very different. Moreover, they differ dramatically from Shannon's wavelets. The latter can be used only for the $L^2$ class or better behaved functions. Whereas the new wavelets discussed in this work, i.e. quasi wavelets and quasi interpolating wavelets, are Schwartz class functions [29] (Schwartz class functions are infinitely differentiable and rapidly decay asymptotically), and they can be used as integral kernels for all tempered distributions [29]. The examples of tempered distributions are all polynomials (which behave badly asymptotically) and all $L^1$, $L^2$ functions (which may have discontinuities) and even the periodic delta distribution $\sum_{k=-\infty}^{\infty}\delta(x-2\pi k)$ (which is singular on the real line). However, neither $\sum_{k=-\infty}^{\infty}\delta^{(k)}(x-2\pi k)$ nor $e^{|x|}$ are tempered. Actually, our wavelets are applicable even to exponentially growing functions such as $e^{|x|}$.

The rest of this Letter is devoted to numerical examinations of these two families of wavelets. Numerically, both quasi wavelets (quasi scaling functions) and quasi interpolating wavelets (quasi interpolating scaling functions) are extremely useful for various physical applications. We first examine their performance for numerical representations of a function off a grid and its first and second order derivatives on a grid. The off-grid values of a function are of particular important for numerical interpolation and data fitting. Accuracy of on-grid differentiation is the key to solving a differential equation. These are done in a manner similar to Shannon's sampling, Eq. (16), using quasi wavelet scaling functions (QWSF), Eq. (19), and quasi interpolating wavelet scaling functions (QIWSF), Eq. (24), in the place of Shannon's sampling kernel. The infinite summation in Eq. (16) is truncated to a subset of grid points of

Table 2
$L_\infty$ errors for sampling and derivative predictions

| Ratio | QWSF | | | QIWSF | | |
|---|---|---|---|---|---|---|
| | $f$ | $f^{(1)}$ | $f^{(2)}$ | $f$ | $f^{(1)}$ | $f^{(2)}$ |
| 1.0 | 3.88(−03) | 1.71(−01) | 1.34(+01) | 2.48(−03) | 1.65(−01) | 1.35(+01) |
| 1.5 | 2.03(−05) | 1.03(−03) | 6.61(−02) | 1.83(−05) | 1.02(−03) | 6.62(−02) |
| 2.0 | 2.02(−08) | 9.59(−07) | 6.12(−05) | 2.00(−08) | 9.57(−07) | 6.12(−05) |
| 2.5 | 3.42(−12) | 1.51(−10) | 9.99(−09) | 3.41(−12) | 1.51(−10) | 9.99(−09) |
| 3.0 | 7.77(−16) | 5.33(−15) | 2.33(−13) | 1.55(−15) | 4.22(−15) | 2.06(−13) |
| 3.5 | 7.77(−16) | 3.77(−15) | 1.94(−13) | 1.22(−15) | 6.22(−15) | 2.11(−13) |
| 4.0 | 7.77(−16) | 3.55(−15) | 2.31(−13) | 8.88(−16) | 4.00(−15) | 2.34(−13) |
| 4.5 | 4.83(−15) | 8.38(−13) | 3.43(−11) | 4.52(−15) | 8.30(−13) | 3.44(−11) |
| 5.0 | 9.60(−13) | 1.30(−10) | 4.33(−09) | 9.60(−13) | 1.30(−10) | 4.33(−09) |
| 5.5 | 4.69(−11) | 5.37(−09) | 1.48(−07) | 4.69(−11) | 5.37(−09) | 1.48(−07) |
| 6.0 | 8.95(−10) | 9.03(−08) | 2.10(−06) | 8.95(−10) | 9.03(−08) | 2.10(−06) |

$\{-W,-W+1,\dots,W\}$, centered around the point of interest, $x$. We choose $W=32$ in these calculations. A badly behaved function, $f(x)=e^{-x/6}[\cos x \sin 4x]$, which grows exponentially for $x<0$, is selected for our test. Forty equally spaced grid points in the interval $[0,\pi]$ are used for this calculation and the $L_\infty$ errors for off-grid values are calculated based on 55 grid points in the interval for a variety of ratios, see Table 2. The $L_\infty$ errors of predicting first order derivative, $f^{(1)}$, and second order derivative, $f^{(2)}$, based on the same 40 grid points are also presented in Table 2. It might appear that the quasi wavelet is better because it is interpolative. Our calculation shows that the two systems have extremely similar behavior over a wide range of $\sigma/\Delta$ ratios.

We next examine the behaviors of our new wavelets for numerical solutions of partial differential equations. A benchmark problem for testing numerical algorithms is the Ornstein–Uhlenbeck process which is a stationary Markov process and its corresponding Fokker–Planck equation describes a drift-diffusion system

$$\frac{\partial f(x,t)}{\partial t}=\gamma\frac{\partial[xf(x,t)]}{\partial x}+D\frac{\partial^2 f(x,t)}{\partial x^2}, \qquad (27)$$

where $\gamma$ and $D$ are positive constants. The Ornstein–Uhlenbeck process has been used to simulate a laser field far below (or above) its threshold [30], to describe an overdamped oscillator in the presence of colored Gaussian noise [31], and to model the velocity relaxation of a Rayleigh gas [32]. Mathematically, Eq. (27) is an initial value problem and can be solved analytically

$$f(x,t)=\left[\frac{\gamma}{2D\pi\sqrt{(1-e^{-2\gamma(t-t_0)})}}\right]\times\exp\left[-\frac{\gamma\left(x-x_0\,e^{-2\gamma(t-t_0)}\right)^2}{2D(1-e^{-2\gamma(t-t_0)})}\right]. \qquad (28)$$

Hence, the Ornstein–Uhlenbeck Fokker–Planck

Table 3
$L_\infty$ errors for solving the Ornstein–Uhlenbeck Fokker–Planck Equation

| | Ratio | Time | | |
|---|---|---|---|---|
| | | 0.2 | 2.0 | 10.0 |
| QWSF | 1.0 | 4.38(−01) | 5.50(−01) | 5.64(−01) |
| | 1.4 | 2.51(−02) | 9.36(−03) | 2.63(−02) |
| | 1.8 | 1.60(−03) | 2.82(−05) | 4.60(−05) |
| | 2.2 | 1.50(−04) | 3.72(−08) | 2.29(−08) |
| | 2.6 | 1.79(−05) | 7.72(−11) | 5.11(−12) |
| | 3.0 | 4.23(−06) | 8.71(−11) | 1.31(−13) |
| | 3.4 | 3.02(−06) | 8.71(−11) | 1.31(−13) |
| | 3.8 | 3.09(−06) | 8.71(−11) | 1.31(−13) |
| QIWSF | 1.0 | 4.36(−01) | 5.49(−01) | 5.64(−01) |
| | 1.4 | 2.51(−02) | 9.36(−03) | 2.63(−02) |
| | 1.8 | 1.60(−03) | 2.82(−05) | 4.60(−05) |
| | 2.2 | 1.50(−04) | 3.72(−08) | 2.29(−08) |
| | 2.6 | 1.79(−05) | 7.72(−11) | 5.11(−12) |
| | 3.0 | 4.23(−06) | 8.71(−11) | 1.31(−13) |
| | 3.4 | 3.02(−06) | 8.71(−11) | 1.31(−13) |
| | 3.8 | 3.09(−06) | 8.71(−11) | 1.31(−13) |

equation is also computationally important and has been used for testing new numerical schemes [33]. In the present computations, $\gamma$ and $D$ are chosen as 0.25 and 0.125 respectively. A total of 100 grid points in the interval $[-5.5,5.5]$ are used in the computation and the initial function is a unit impulse function located at $-0.55$. The fourth order Runge–Kutta is used for the time integration with a time increment of 0.01. The computational bandwidth $W$ is fixed at 40 in these calculations. The performance of the quasi wavelet scaling function and the quasi interpolating wavelet scaling function is compared for a wide range of ratios ($r=\sigma/\Delta$). The $L_\infty$ errors for a range of propagation times and $r$ values are listed in Table 3. It is evident that by using a relatively small number of grid points and a reasonable time increment, both the quasi wavelets and quasi interpolating wavelets are able to provide an accuracy close to the computer round off limit for solving a dynamical equation. As listed in Table 3, two wavelet families have almost identical behavior for a wide range of $\sigma/\Delta$ ratios.

In conclusion, we introduce quasi wavelets and quasi interpolating wavelets for numerical applications. It is found that the mathematical regularization of Shannon's orthonormal interpolating wavelet leads to an interplay between a quasi wavelet and a quasi interpolating wavelet. The resulting quasi wavelet scaling function preserves the interpolating property but it does not satisfy the wavelet normalization requirement. The quasi interpolating wavelet scaling function, however, satisfies the normalization requirement but it is no longer rigorously interpolative. Although these two new families of wavelets are distinctive conceptually, their numerical performance is extremely similar for datum interpolation, differentiation and for solving partial differential equations. Various parameter regions are discussed in which two wavelet families are *numerically* identical to each other. They provide almost machine accuracy for interpolating an exponentially growing function, which cannot be done by Shannon's wavelet system. Their accuracy for numerically solving the Ornstein–Uhlenbeck Fokker–Planck equation reaches 13 significant figures. Although the two families of wavelets introduced in this Letter are functions of the Schwartz class with an infinite order of regularity, the present procedure is very general and can be used to generate other quasi wavelets (or quasi interpolating wavelets) from various existing well-known wavelet systems with a controlled order of regularity. This will be discussed in our future work.

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