ZEROS OF FUNCTIONS IN WEIGHTED BERGMAN SPACES

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1. INTRODUCTION

Throughout this paper, Δ denotes the open disc in the complex plane, and μ is a finite, positive Borel measure on the half-open interval $[0, 1)$, giving nonzero mass to each interval $[r, 1)$ for $0 \leq r < 1$. In other words, μ places no mass at the point 1, but has 1 in its closed support. We denote the rotation-invariant extension of μ to Δ by μ_Δ: this is the unique finite (positive) Borel measure on Δ such that

$$\int f d\mu_\Delta = \int \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it})dt \right\} d\mu(r)$$

for each f continuous on the closed unit disc.

For $0 < p < \infty$, the *Bergman space* A^p_μ is the collection of functions f holomorphic in Δ such that $\|f\|_p^p = \int |f|^p d\mu_\Delta < \infty$. This paper deals with the zero sets of functions in A^p_μ. For f holomorphic in Ω and not $\equiv 0$, the *zero set of* f *in* Ω, denoted by Z(f) (or by $Z_\Omega(f)$ when we wish to emphasize the domain of f) is the sequence of zeros of f in Ω, ordered by increasing moduli, and repeated according to multiplicity. If $f \in A^p_\mu$ then we call Z(f) an A^p_μ-*zero set.*

We are going to show that *for* $0 < p < \infty$ *there is an* A^p_μ *zero set not contained in any* A^q_μ *zero set for* $q > p$; and that A^p_μ *has a disjoint pair of zero sets whose union is not a zero set.* Both these results have recently been obtained by Charles Horowitz [7] for the case $d\mu(r) = (1 - r)^\alpha dr$ $(\alpha > -1)$, and the first one has been proved for the Hardy spaces of balls and polydiscs in $\mathbb{C}^n$ by J. Miles [10] and W. Rudin [11]. While Horowitz and Miles employ infinite products to produce the required zero sets explicitly, Rudin constructs gap series with prescribed growth properties and leaves the zeros to fend for themselves. Rudin's idea plays the fundamental role in this paper: we will see that it provides a simple and flexible means for "constructing" the desired zero sets in Bergman spaces.

In particular, our results have relevance to operator theory. It is known that on Hilbert space every injective subnormal unilateral weighted shift not similar to the standard unweighted one is unitarily equivalent to the operator of multiplication by z on some A^2_μ (the standard unweighted shift is unitarily equivalent to multiplication by z on the Hardy space H^2) [2; Theorem 8], [13; Proposition 25, page 84]. It follows quickly from this and the existence of two disjoint A^2_μ zero sets whose union lies in no zero set that every such weighted shift has a pair of nontrivial invariant subspaces with trivial intersection.

It is well known that all these results fail in the Hardy spaces H^p, which can be regarded as the limiting case where μ has unit mass at $\{1\}$. Since a sequence

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(z_n) is an H^p zero set if and only if $\sum (1 - |z_n|) < \infty$ [1; Theorem 2.3, page 18], it follows that all H^p spaces have the same zero sets, and the union of two H^p zero sets is again one. In addition, it follows from Beurling's theorem [5; Chapter 7] and the factorization theory for H^p spaces [1; Section 2.4] and [5; Chapter 5] that any two nontrivial invariant subspaces of the standard unweighted unilateral shift have nontrivial intersection.

The arguments used here work equally well in annuli, where they yield information about invariant subspaces of *bilateral* weighted shifts, and even give results about Bergman spaces of rather general plane domains.

The paper is organized as follows. In the next section we state a general result (Theorem 2.1) from which all else follows. We show how this result yields the desired A^p_μ zero sets, prove a version in several variables, and discuss unilateral weighted shifts. The result itself is proved in Section 3. The fourth section contains an extension of the main theorem to annuli, applications to bilateral weighted shifts, and results on zero sets of Bergman spaces in more general domains. Some further remarks and open problems are recorded in the final section.

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2. MAIN RESULT AND FIRST APPLICATIONS

Recall that μ is a finite positive Borel measure on $[0, 1)$ with 1 in its closed support, and μ_Δ is the rotation-invariant extension of μ to the open unit disc Δ. We are actually going to deal with spaces more general than A^p_μ.

Call a function ϕ on $(-\infty, \infty)$ *admissible* if it is strictly positive, convex, increasing, unbounded, and

$$(2.1)\qquad \begin{array}{lll} \text{(a)} & \sup \phi(t+1)/\phi(t) < \infty & (-\infty < t < \infty); \\ \text{(b)} & \lim \phi(t) = 0 & (t \to -\infty). \end{array}$$

For ϕ admissible, let A^ϕ_μ denote the collection of functions f holomorphic in Δ such that $\int \phi(\log|f|)\, d\mu_\Delta < \infty$. In particular, $A^p_\mu = A^\phi_\mu$ for $\phi(t) = e^{pt}$ $(0 < p < \infty)$.

Suppose ϕ is admissible, and let $\Phi(t) = \phi(\log|t|)$. Then Φ is continuous on the real line, increases to ∞ on the positive axis, and vanishes at the origin (by 2.1b). The monotonicity of ϕ and (2.1a) quickly yield constants $p > 0$ and $M \geq 1$ such that $\phi(t) \leq Me^{pt}$ for $t \geq 0$, and $\Phi(2t) \leq M\Phi(t)$ for all t. The first inequality shows that A^ϕ_μ is contained in some A^p_μ, while the second yields a sort of subadditivity for Φ:

$$(2.2)\qquad \Phi(s+t) \leq M[\Phi(s) + \Phi(t)] \qquad (\text{all } s, t).$$

In particular, A^ϕ_μ is a vector space under pointwise operations. We can now state the main result of the paper.

THEOREM 2.1. *Suppose ϕ and ψ are admissible functions with*

$$\lim \psi(t)/\phi(t) = \infty \qquad (t \to +\infty).$$

Then there is a function $f \in A_\mu^\phi$ *such that for any positive integer* n, *any bounded holomorphic function* b *on* Δ, *and any holomorphic function* h *on* Δ,

$$(f^n + b)h \notin A_\mu^{\psi_n},$$

where $\psi_n(t) = \psi(t/n)$.

This result with ϕ and ψ slightly more general was proved by Rudin [10] for Hardy spaces of balls and polydiscs in $\mathbb{C}^k$ $(k > 1)$. Our proof is an adaptation of his, and will be given in the next section, after the promised applications to A_μ^p zero sets.

COROLLARY 2.2. *For* ϕ *and* ψ *as in Theorem* 2.1, *there is a function* f *in* A_μ^ϕ *such that* $Z(f - c) \not\subseteq Z(g)$ *for any complex number* c *and any* $g \in A_\mu^\psi$.

Proof. The function f in the conclusion of Theorem 2.1 does the job. If not, then for some complex number c and some $g \in A_\mu^\psi$, the quotient $h = g/(f - c)$ is holomorphic in Δ. Hence $(f - c)h \in A_\mu^\psi$, contradicting the conclusion of Theorem 1.1 with $n = 1$ and $b \equiv c$.

Thus no level set of f lies in any A_μ^ψ zero set. In particular, Z(f) is an A_μ^ϕ zero set that is not contained in any A_μ^ψ zero set. Specializing to the case $\phi(t) = e^{pt}$, and $\psi(t) = (2 + p^2 t^2)e^{pt}$, we have ϕ and ψ obeying the hypotheses of the corollary, and $A_\mu^\psi \subseteq A_\mu^q$ for every $q > p$. Thus *for* $0 < p < \infty$, *there is an* A_μ^p *zero set not contained in any* A_μ^q *zero set for* $q > p$. (*Cf.* [11], where all of these comments were recorded for Hardy spaces on the unit ball of $\mathbb{C}^k$).

COROLLARY 2.3. *Suppose* ϕ *is admissible and* $\phi(2t)/\phi(t) \to \infty$ *as* $t \to +\infty$. *Then there exist two disjoint* A_μ^ϕ *zero sets whose union is contained in no* A_μ^ϕ *zero set.*

Proof. Let $\psi(t) = \phi(2t)$. Then ϕ and ψ satisfy the hypotheses of Theorem 2.1, so there exists $f \in A_\mu^\phi$ such that $(f^2 - 1)h \notin A_\mu^{\psi_2} = A_\mu^\phi$ for any h holomorphic on Δ. Clearly, $Z_1 = Z(f + 1)$ and $Z_2 = Z(f - 1)$ are disjoint A_μ^ϕ zero sets; we claim that their union Z is contained in no A_μ^ϕ zero set. Suppose otherwise; that is, $Z \subseteq Z(g)$ for some $g \in A_\mu^\phi$. Then since $Z = Z(f^2 - 1)$, we have $h = g/(f^2 - 1)$ holomorphic in Δ. Hence $(f^2 - 1)h \in A_\mu^\phi$, contrary to the choice of f.

Note that if $\phi(2t)/\phi(t)$ is bounded then A_μ^ϕ is an algebra, hence the union of each pair of A_μ^ϕ zero sets is again a zero set. The function $\phi(t) = \log(e^t + 1)$ furnishes an example where this happens; here the algebra A_μ^ϕ can be regarded as an "area analogue" of the Nevanlinna class.

Corollary 2.3 has an application to operator theory. A bounded (linear) operator T on complex separable Hilbert space H is called a *weighted shift* if there are an orthonormal basis (e_n) and a bounded scalar sequence (w_n) such that

$$Te_n = w_n e_{n+1} \quad \text{for all } n.$$

If the index n runs over the nonnegative integers, T is called *unilateral*, while if n runs over all integers, T is *bilateral*. Clearly T is injective (one-to-one) if and only if no w_n vanishes.

Weighted shifts are closely connected with analytic function theory (for an excellent survey of this, see [13]); in particular, the Bergman spaces A^2_μ are related to *subnormal* weighted shifts. Recall that a bounded operator on Hilbert space H is subnormal if it is the restriction to H of a normal operator N on a larger Hilbert space which contains H as a closed N-invariant subspace. In our situation, the operator M_z of multiplication by the independent variable z, $(M_z f)(z) = zf(z)$, is a normal operator on the Hilbert space $L^2(\mu_\Delta)$, and A^2_μ is a closed subspace [2; Theorem 1, page 473] invariant under M_z. Thus M_z is a subnormal operator on A^2_μ, and an easy calculation [13; page 84] shows that it is unitarily equivalent to the unilateral weighted shift with weight sequence

$$(2.3) \qquad w_n = \beta(n+1)/\beta(n), \quad \beta(n)^2 = \int r^{2n}\,d\mu(r), \qquad n = 0, 1, 2, \cdots.$$

The "unweighted" shift $w_n = 1$ $(n \geq 0)$ has a similar description: it is unitarily equivalent to M_z on the Hardy space H^2.

It turns out that these examples exhaust all possibilities: every injective, subnormal unilateral weighted shift not similar to the unweighted one is unitarily equivalent to M_z on some A^2_μ [2; page 481, Theorem 8], [4; pp. 895-896].

COROLLARY 2.4. *If an injective subnormal unilateral weighted shift is not similar to the unweighted one, then it has a pair of nontrivial closed invariant subspaces with trivial intersection.*

Proof. According to the previous remarks, we need only prove the result for M_z on A^2_μ. By Corollary 2.2 there exist disjoint A^2_μ zero sets Z_1 and Z_2 whose intersection Z contains no A^2_μ zero set. Then $S_j = \{f \in A^2_\mu : Z(f) \supseteq Z_j\}$ is a nontrivial, closed, M_z-invariant subspace (j = 1, 2), and

$$S_1 \cap S_2 = \{f \in A^2_\mu : Z(f) \supset Z\} = \{0\},$$

which completes the proof.

We close this section with an application to Bergman spaces in several complex variables, using the following notation to aid the discussion. For two points $z = (z_1, \cdots, z_k)$ and $w = (w_1, \cdots, w_k)$ in $\mathbb{C}^k$, let zw be the coordinatewise product $zw = (z_1 w_1, z_2 w_2, \cdots, z_k w_k)$. Denote the open unit polydisc (Cartesian product of k open unit discs) in $\mathbb{C}^k$ by Δ^k, the closed unit polydisc (product of k closed unit discs) by $\overline{\Delta}^k$, and the k-torus (product of k unit circles) by T^k. A domain $\Omega \subset \mathbb{C}^k$ is called a *complete Reinhardt domain* if $wz \in \Omega$ whenever $z \in \Omega$ and $w \in \overline{\Delta}^k$. These are the domains naturally associated with convergence of power series in k complex variables [6; Section 2.4, page 34].

For Ω a domain in $\mathbb{C}^k$ and ϕ an admissible function, let $A^\phi(\Omega)$ denote the collection of functions f holomorphic on Ω such that $\int \phi(\log|f|)\,dV < \infty$, where V is 2k-dimensional Lebesgue measure on Ω.

COROLLARY 2.5. *Suppose Ω is a bounded, complete Reinhardt domain in $\mathbb{C}^k$. Then Theorem* 2.1, *Corollary* 2.2, *and Corollary* 2.3 *all remain true when Δ is replaced by Ω and A^ϕ_μ, $A^{\psi_n}_\mu$, and A^ψ_μ are replaced by $A^\phi(\Omega)$, $A^{\psi_n}(\Omega)$, and $A^\psi(\Omega)$, respectively.*

Proof. Define $P: \mathbb{C}^k \to \mathbb{C}$ by $P(z) = z_1$, and $Q: \mathbb{C} \to \mathbb{C}^k$ by

$$Q(z_1) = (z_1, 0, 0, \cdots, 0).$$

Then $P(\Omega)$ is a bounded open disc in the plane centered at the origin; without loss of generality, we may assume it is the open unit disc Δ. Then VP^{-1} is a finite, positive, rotation-invariant Borel measure on Δ which gives positive mass to every nonvoid open subset. Thus $VP^{-1} = \mu_\Delta$, the rotation-invariant extension of a finite, positive Borel measure μ on $[0, 1)$ which has 1 in its closed support.

Thus Theorem 2.1 produces an $f \in A_\mu^\phi$ such that $(f^n + b)h \notin A_\mu^{\psi_n}$ for any integer $n > 0$, and for any b and h holomorphic on Δ, with b bounded. We claim that $F = f \circ P$ plays the same role on Ω. The standard change-of-measure formula [3; page 163, Theorem C] shows that $\int \phi(\log|F|)\,dV = \int \phi(\log|f|)\,d\mu_\Delta < \infty$, so $F \in A^\phi(\Omega)$.

Suppose that for some $n > 0$, and for B and H holomorphic in Ω, with B bounded, we have $G = (F^n + B)H \in A^{\psi_n}(\Omega)$. Then letting g, b, and h be $G \circ Q$, $B \circ Q$, and $H \circ Q$, respectively, we have $g = (f^n + b)h$, where b and h are holomorphic on Δ, and b is bounded. But the next lemma is going to show that the function $g \circ P(z) = G(z_1, 0, 0, \cdots, 0)$ is in $A^{\psi_n}(\Omega)$, so $g \in A_\mu^{\psi_n}$ by the change-of-measure formula, contrary to the choice of b. This proves the analogue of Theorem 2.1 for Ω, once we have stated and proved the missing lemma. The analogues of Corollary 2.2 and Corollary 2.3 follow as before.

LEMMA. *Suppose Ω is a bounded, complete Reinhardt domain, ϕ is an admissible function,* $G \in A^\phi(\Omega)$, *and* $g(z) = G(z_1, 0, 0, \cdots, 0)$ $(z \in \Omega)$. *Then* $g \in A^\phi(\Omega)$.

Proof. For each $\xi \in \Omega$ the function $w(z, \xi) = \phi(\log|F(z\xi)|)$ is an n-subharmonic function of z (*i.e.*, subharmonic in each variable separately) in a neighborhood of $\bar{\Delta}^n$, since $F(z\xi)$ is analytic in z and ϕ is convex. Thus

$$W(z) = \int_\Omega w(z, \xi)\,dV(\xi)$$

is n-subharmonic on Δ^n. Now the rotation-invariance of Ω shows that

$$W(z_1, \cdots, z_n) = W(|z_1|, \cdots, |z_n|) \qquad (z \in \Delta^n),$$

so the maximum principle applied in each variable separately shows that W increases as the modulus of each variable increases. In particular,

$$W(r, 0, 0, \cdots, 0) \le W(r, r, \cdots, r) \qquad (0 \le r < 1);$$

that is,

$$\int_\Omega \phi(\log|g(r\xi)|)\,dV(\xi) \le \int_\Omega \phi(\log|G(r\xi)|)\,dV(\xi) \qquad \text{for } 0 \le r < 1.$$

The change of variable $z = r\xi$ and the monotone convergence theorem show that this last inequality persists even when $r = 1$, which is the desired result.

Finally, we remark that when $\partial\Omega$ is smooth enough we can replace the measure V by the surface area measure σ, and give the above arguments for suitably defined

Hardy spaces. In this case the induced measure σP^{-1} lives on the *closed* unit disc, so it is the rotation-invariant extension of a measure μ on the closed unit interval. The proof we have just given still works, provided $\mu\{1\} = 0$. For example, this will be the case when Ω is the unit ball in $\mathbb{C}^k$ (the situation considered in [11]), but *not* when $\Omega = \Delta^k$. We leave the details to the reader.

3. PROOF OF THEOREM 2.1

Recall that ϕ and ψ are admissible functions for which

$$\lim_{t \to +\infty} \psi(t)/\phi(t) = \infty . \tag{3.1}$$

Let $\Phi(t) = \phi(\log|t|)$, and $\Psi_n(t) = \psi_n(\log|t|) = \psi(n^{-1}\log|t|)$. Observe that the "sub-additive" property (2.2) of Φ yields

$$\Phi(t_1 + t_2 + \cdots + t_n) \leq \sum_{k=1}^{n} M^k \Phi(t_k), \tag{3.2}$$

where $M \geq 1$ depends only on ϕ, and not on the numbers $t_1, t_2, \cdots, t_n$ $(n = 1, 2, \cdots)$.

LEMMA. *There exist sequences* (t_k) *and* (a_k) *of positive numbers increasing to* ∞, *and* (n_k) *of positive integers increasing to* ∞, *and* (r_k) *and* (ρ_k) *with*

$$0 < r_1 < \rho_1 < r_2 < \rho_2 < \cdots \to 1$$

such that if $u_k(z) = a_k z^{n_k}$ *and* $R_k = \{r_k \leq |z| \leq \rho_k\}$, *then for* $k \geq 2$ *the following conditions hold:*

(3.3)

(a) $\quad t_k \geq 4 \sum_{j=1}^{k-1} a_j$ *and* $t > t_k \Rightarrow \Psi(t)/\Phi(t) > kM^k$;

(b) $\quad \int \Phi(u_k)\, d\mu_\Delta = 1/k^2 M^k$;

(c) $\quad \int_{R_k} \Phi(u_k)\, d\mu_\Delta \geq 1/2k^2 M^k$;

(d) $\quad |u_k| \geq t_k$ *on* R_k;

(e) $\quad |u_k| \leq |u_{k-1}|/5$ *on* $\bigcup_{j=1}^{k-1} R_j$.

Deduction of Theorem 2.1. Assuming the lemma, we are going to show that the series

$$f(z) = \sum_{1}^{\infty} u_k(z) \tag{3.4}$$

defines a holomorphic function with the desired properties.

To verify its convergence, fix $k \geq 1$ and observe that on R_k we have from (3.3e) that $|u_{k+j}| \leq |u_k|/5^j$ for $j = 1, 2, \cdots$; hence

$$\sum_{j=1}^{\infty} |u_{k+j}| \leq |u_k|/4 \text{ on } R_k, \qquad k = 1, 2, \cdots. \tag{3.5}$$

Thus the series (3.4) converges uniformly on compact subsets of Δ to a holomorphic function f.

To see that $f \in A^{\phi}_{\mu}$, use (3.2) and (3.3c) to obtain:

$$\int \Phi\left(\sum_2^m u_k\right) d\mu_\Delta \leq \int \sum_2^m M^k \Phi(u_k)\, d\mu_\Delta \leq \sum_2^m k^{-2}.$$

Thus by Fatou's lemma and the pointwise convergence of $\sum u_k$ on Δ;

$$\int \Phi(f)\, d\mu_\Delta \leq \int \Phi(u_1)\, d\mu_\Delta + \sum_2^{\infty} k^{-2} < \infty.$$

Hence $f \in A^{\phi}_{\mu}$.

The rest of the proof hinges on the fact that on R_k the function f is large, and essentially the same as its k^{th} term u_k. To make this precise, we write $f = S_k + u_k + T_k$, where $S_k = \sum_1^{k-1} u_j$ and $T_k = \sum_{k+1}^{\infty} u_j$, $k = 2, 3, \cdots$. Then on R_k we see from (3.5) that $|T_k| \leq |u_k|/4$, while from (3.3a,d),

$$|S_k| \leq \sum_1^{k-1} |a_j| \leq t_k/4 \leq |u_k|/4.$$

Thus

$$|u_k|/2 \leq |f| \leq 3|u_k|/2 \text{ on } R_k \qquad (k = 2, 3, \cdots). \tag{3.6}$$

To complete the proof, we introduce the notations

$$L(g;\, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|g(re^{i\theta})|\, d\theta$$

and

$$M_\eta(g;\, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \eta(\log|g(re^{i\theta})|)\, d\theta$$

for g holomorphic in Δ, $0 \leq r < 1$, and η admissible.

Fix b and h holomorphic in Δ with b bounded, and let $g = (f^n + b)h$. We want to show that $g \notin A^{\psi_n}_{\mu}$; that is, $\int M_{\psi_n}(g;\, r)\, d\mu(r) = \int \Psi_n(g)\, d\mu_\Delta = \infty$. Let

$B = \sup_{|z|<1} |b(t)|$, and $A = \inf_{r>r_1} L(h; r)$. Then $A > -\infty$, since $L(h; r)$ is a convex function of $\log r$. Since $t_k \uparrow \infty$, we can choose K so large that $(t_K/4)^n > B$. Fix $k > K$ and $r_k \leq r < \rho_k$. Then letting C denote a positive constant, independent of r but possibly varying from line to line, we apply Jensen's convexity theorem to obtain

$$M_{\psi_n}(g; r) \geq \psi_n\{L(g; r)\} = \psi_n\{L(f+b; r) + L(h; r)\}$$

$$\geq \psi_n\{L(f+b; r) + A\} \geq C\psi_n\{L(f+b; r)\},$$

where the last inequality follows from the boundedness of $\psi(t+1)/\psi(t)$. Now (3.6) and (3.3d) yield the following estimates, valid on the annulus R_k:

$$|f^n + b| \geq |f|^n - B \geq (|u_k|/2)^n - (t_k/4)^n \geq (|u_k|/2)^n - (|u_k|/4)^n \geq (|u_k|/4)^n.$$

For $r_k \leq r \leq \rho_k$ this inequality yields

$$\psi_n\{L(f^n + b; r)\} \geq \psi_n\{L((u_k/4)^n; r)\} \geq C\psi_n\{nL(u_k; r)\}$$

$$= C\psi\{L(u_k; r)\} = CM_\psi(u_k; r),$$

where the last line follows because $|u_k(re^{i\theta})|$ does not depend on θ. Putting all of this together, we get $M_{\psi_n}(g; r) \geq CM_\psi(u_k; r)$ for $r_k \leq r \leq \rho_k$; hence

$$\int_{R_k} \Psi_n(g)\, d\mu_\triangle \geq C \int_{R_k} \Psi(u_k)\, d\mu_\triangle.$$

But (3.3a,d) implies that $\Psi(u_k) \geq kM^k\Phi(u_k)$ on R_k. Together with the previous inequality, this yields $\int_{R_k} \Psi_n(g)\, d\mu_\triangle \geq CkM^k \int_{R_k} \Phi(u_k)\, d\mu_\triangle \geq C/2k$ for all $k \geq K$.

Thus $\int_\triangle \Psi_n(g)\, d\mu_\triangle \geq \sum_{k=K}^\infty \int_{R_k} \Psi_n(g)\, d\mu_\triangle \geq \sum_{k=K}^\infty C/2k = \infty$, which completes the deduction of the theorem from the lemma.

Proof of lemma. The lemma follows from a simple measure-theoretic result proved in [11; pp. 59-60]: *given* $\alpha > 0$, *there exists a sequence* (c_n) *of positive numbers such that*

(3.7)

(a) $\int \Phi(c_n z^n)\, d\mu_\triangle = \alpha$;

(b) $c_n \uparrow \infty$;

(c) $\lim_n c_n z^n = 0$ *uniformly on compact subsets of* $\triangle$;

(d) $\lim_{n\to\infty} \int_{\{|c_n z^n| > t\}} \Phi(c_n z^n)\, d\mu_\triangle = \alpha$ *for each* $t > 0$.

We prove the lemma by induction. Choose any positive numbers t_1, a_1, n_1; and any $0 < r_1 < \rho_1 < 1$. Suppose $k \geq 2$, and suppose the five sequences have been successfully chosen for all indices less than or equal to $k - 1$. There is no difficulty in choosing t_k to satisfy (3.3a). Let $\alpha = 1/k^2 M^k$ in (3.7), and choose $c_n \uparrow \infty$ accordingly. By (3.7c,d) we can choose n_k so large that, letting $a_k = c_{n_k}$, we have

$$(3.8) \qquad |a_k z^{n_k}| \leq |a_{k-1} r_1^{n_{k-1}}|/5, \qquad r_1 \leq |z| \leq \rho_{k-1},$$

and $\int_{\{|a_k z^{n_k}|>t_k\}} \Phi(a_k z^{n_k})\, d\mu_\Delta > 1/2k^2 M^k$. Define r_k by the equation $\{|a_k z^{n_k}| > t_k\} = \{|z| > r_k\}$, and observe that $r_k > \rho_{k-1}$ by (3.3a) and (3.8). Finally, choose $r_k < \rho_k < 1$ so that $\int_{\{r_k \leq |z| \leq \rho_k\}} \Phi(a_k z^{n_k})\, d\mu_\Delta(z) > 1/2k^2 M^k$.
The proof of the lemma, and hence of the theorem, is now complete.

4. OTHER DOMAINS, BILATERAL SHIFTS

Let μ be a finite positive Borel measure on the interval $(a, 1)$ with $0 \leq a < 1$. We assume that μ has the endpoints a and 1 in its closed support. If $a = 0$ we further assume that μ has moments of all negative orders:

$$(4.1) \qquad \int r^n d\mu(r) < \infty, \qquad n < 0.$$

Let $R = R_a$ be the annulus $\{a < |z| < 1\}$, let μ_R be the rotation-invariant extension of μ to R, and for each admissible function ϕ let $A_\mu^\phi(R)$ be the collection of functions f holomorphic in R such that $\int \phi(\log|f|)\, d\mu_R < \infty$. Note that (4.1) and the growth condition $\phi(t) = O(e^{pt})$ insure that the functions z^n belong to $A_\mu^\phi(R_a)$ for all integers n (positive and negative), even when $a = 0$.

In this section we are going to extend Corollaries 2.2 and 2.3 to rather general plane domains, and prove an analogue of Corollary 2.4 for bilateral weighted shifts. Everything depends on the following observation: *the proof of Theorem* 2.1 *goes through unchanged when the unit disc* Δ *is replaced by the annulus* R_a. In fact the only property of Δ that we used, other than its circular symmetry about the origin, was the convexity of $L(g; r)$ for g holomorphic in Δ. But this holds as well for g holomorphic in R_a and $a < r < 1$ [1; page 10, "Remark".], so we have:

THEOREM 4.1. *For* $0 \leq a < 1$, *Theorem* 2.1 *as well as Corollaries* 2.2 *and* 2.3 *remain true if* A_μ^ϕ, A_μ^ψ *and* $A_\mu^{\psi_n}$ *are replaced by their counterparts on* R_a. *The function* f *can still be taken to be holomorphic on the open unit disc.*

The analogue of Corollary 2.4 is a result about *bilateral* weighted shifts. It is not too difficult to see that the operator M_z of multiplication by z on $A_\mu^2(R_a)$ is unitarily equivalent to the bilateral weighted shift with weight sequence given by (2.3), where now n runs through *all* integers. Clearly, M_z is subnormal on $A_\mu^2(R_a)$.

Two other subnormal bilateral shifts come to mind. The first is M_z on $L^2(T)$, where T is the unit circle with normalized Lebesgue measure. In this case M_z is the unweighted shift: $w_n = 1$ for all n. The second is M_z on $H^2(R_a)$, the Hilbert space of functions f holomorphic on R_a for which

$$\|f\|_2^2 = \frac{1}{2\pi}\left\{\sup_{a<r_1<r_2<1} \int_{-\pi}^{\pi} |f(r_1 e^{i\theta})|^2 d\theta + \int_{-\pi}^{\pi} |f(r_2 e^{i\theta})|^2 d\theta\right\} < \infty .$$

It turns out that every injective bilateral subnormal weighted shift is unitarily equivalent to one of the above operators M_z [13; Proposition 27, pp. 87-88], and is invertible if and only if it is not equivalent to M_z on $A^2_\mu(R_0)$.

Now a subspace that is invariant under both an invertible operator and its inverse is called *doubly invariant.* A version of Beurling's theorem for H^p spaces of annuli (see [12; Section 11 and 12] or [14]) shows that each pair of nontrivial doubly invariant subspaces of M_z on $H^2(R_a)$ $(a > 0)$ has nontrivial intersection. For M_z on $L^2(T)$ this is no longer the case, and the analogue of Corollary 2.3 for $A^2_\mu(R_a)$ shows that it also fails on that space. In summary:

COROLLARY 4.2. *If an invertible, injective, subnormal bilateral weighted shift is not similar to* M_z *on* $H^2(R_a)$ $(a > 0)$, *then it has a pair of nontrivial, closed, doubly invariant subspaces with trivial intersection.*

Theorem 4.1 also gives information about zero sets in Bergman spaces of plane domains other than discs and annuli. If Ω is a plane domain and $0 < p < \infty$, then the Bergman space $A^p(\Omega)$ is the space of functions f holomorphic on Ω such that $\int_\Omega |f|^p\, dA < \infty$, where dA is planar Lebesque measure. Call a component Γ_0 of the boundary of Ω *free* if it lies in an open subset of the plane which intersects no other boundary component, and *nontrivial* if it is not a single point.

COROLLARY 4.3. *Suppose* Ω *is a plane domain with a nontrivial free boundary component. Then for* $0 < p < \infty$,

(a) *there is an* $A^p(\Omega)$ *zero set not contained in any* $A^q(\Omega)$ *zero set for* $q > p$, *and*

(b) *there are two disjoint* $A^p(\Omega)$ *zero sets whose union is contained in no* $A^p(\Omega)$ *zero set.*

Proof. Let S denote the Riemann sphere, let Γ be the nontrivial free boundary component of Ω, and let $\Omega_0 = S\setminus\Gamma$. Then Ω_0 is a simply connected domain conformally equivalent to the unit disc Δ; and under any such equivalence Γ corresponds to the unit circle T in the sense that whenever a sequence in Ω_0 eventually leaves every compact subset, its image in Δ does the same.

Let U be a one-to-one conformal map taking Ω_0 onto Δ, and let V be its inverse. Then $U(\Omega)$ is a subdomain of Δ which, because Γ is free, contains an annulus $R = R_a$ for some $0 < a < 1$. Let $d\mu(r) = r\,dr$ on the interval $a < r < 1$, so $A^p_\mu(R) = A^p(R)$.

Proof of (a). Theorem 4.1 provides an $f \in A^p(R)$ *holomorphic on* Δ, with $fh \notin A^q(R)$ for any $q > p$ and h holomorphic in R. We claim that $Z = Z_\Omega(f \circ U)$ is the desired zero set.

Since the derivative U' is holomorphic in the simply connected domain Ω_0 and never vanishes there, it has a holomorphic $1/p^{\text{th}}$ root, which we denote by $(U')^{1/p}$. Let $F(z) = f(U(z))U'(z)^{2/p}$, $z \in \Omega_0$ (recall that f is holomorphic on Δ). Then the change of variable $w = U(z)$ yields

$$\int_\Omega |F|^p\,dA \leq \int_{\Omega_0} |F|^p\,dA = \int_\Delta |f|^p\,dA \leq \int_R |f|^p\,dA + \pi a^2 \max_{|w|\leq a} |f(w)| < \infty.$$

so $F \in A^p(\Omega)$ and $Z(F) = Z$.

Now suppose Z lies in an $A^q(\Omega)$ zero set for some $q > p$. Then there exists H holomorphic on Ω such that $HF = G \in A^q(\Omega)$. Let $g(w) = G(V(w))V'(w)^{2/q}$ for $w \in U(\Omega)$. Then the change of variable $z = V(w)$ yields

$$\int_R |g|^q\,dA \leq \int_{U(\Omega)} |g|^q\,dA = \int_\Omega |G|^q\,dA < \infty,$$

so $g \in A^q(R)$. But the equation $HF = G$ yields

$$(H \circ V)\,f(V')^{-2/p} = (H \circ V)(F \circ V) = G \circ V = g(V')^{-2/q}$$

on $U(\Omega)$, hence on R. In particular, if $h = (H \circ V)(V')^{-2/p+2/q}$, then $g = fh$ on R, with h holomorphic on R and $g \in A^q(R)$. But this contradicts the choice of f; hence Z cannot be contained in any $A^q(\Omega)$ zero set and the proof of (a) is complete.

Proof of (b). The idea is the same as before. Choose $f \in A^p(R)$ by Theorem 4.1 so that $(f^2 - 1)h \notin A^p(R)$ for any h holomorphic on R, and so that f is holomorphic on Δ. Let $f_1 = f + 1$, $f_2 = f - 1$, and $Z_j = Z_\Omega(f_j \circ U)$ for $j = 1, 2$. We claim that Z_1 and Z_2 are disjoint $A^p(\Omega)$ zero sets whose union Z contains no $A^p(\Omega)$ zero set. Clearly, Z_1 and Z_2 are disjoint, and if we let $F_j = (f_j \circ U)(U')^{2/p}$ on Ω_0 $(j = 1, 2)$, then as before F_1 and $F_2 \in A^p(\Omega)$, and $Z_j = Z(F_j)$ $(j = 1, 2)$. Thus Z_1 and Z_2 are $A^p(\Omega)$ zero sets.

Suppose $Z = Z_1 \cup Z_2 = Z_\Omega(F_1 F_2)$ is contained in an $A^p(\Omega)$ zero set. Then there exists H holomorphic on Ω such that $F_1 F_2 H = G \in A^p(\Omega)$. Letting $g = (G \circ V)(V')^{2/p}$, we have $g \in A^p(R)$, and on $U(\Omega)$ we have $g = (f^2 - 1)h$, where $h = (V')^{-2/p}(H \circ V)$ is holomorphic on $U(\Omega)$, hence on R. This contradicts the choice of f, shows that Z lies in no $A^p(\Omega)$ zero set, and completes the proof of the corollary.

5. SOME REMARKS AND PROBLEMS

In this section we use the following notations. For g holomorphic in Δ, let $M_p^p(g; r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\,d\theta$ $(0 < p < \infty)$, and $M_\infty(g; r) = \max_{|z|\leq r} |g(z)|$. The letter C is used as in Section 3: it denotes a positive constant which may increase with successive appearances in a proof, but never depends on r. Finally, if α and β are complex-valued functions on some set, then $\alpha \sim \beta$ means that both α/β and β/α are bounded on that set.

(i). *The function* b *need not be bounded.* The proof of Theorem 2.1 goes through virtually without change if we merely require of b that $M_\infty(b; r) = o(M_\infty(f; r))$, $r \to 1^-$. It would be of interest to have better results in this direction. For example, does the conclusion of Theorem 2.1 still hold if we only require $b \in A_\mu^\psi$?

(ii). *The space* A_α^p. This is the special case $d\mu(r) = (1 - r)^\alpha\,dr$ $(\alpha > -1)$ considered by Horowitz in [7]. For these spaces we can write down an explicit function f that works in Theorem 2.1 (with $\phi(t) = e^{pt}$, $\psi(t) = e^{pt}$, $q > p$), and illustrates the point made in (i) above.

PROPOSITION 5.1. *For* $\alpha > -1$ *and* $0 < p < q < \infty$, *choose* $\lambda > 0$ *so that* $p < (\alpha + 1)/\lambda < q$. *Then the function* $f(z) = \sum_{k=0}^{\infty} 2^{\lambda k} z^{2^k}$ *is in* A_α^p, *but*

$$(f^n + b)\,h \notin A_\alpha^{q/n}$$

for any integer $n > 0$, *any* h *holomorphic in* Δ, *and any* b *holomorphic in* Δ *with* $M_\infty(b;\, r) = o(1 - r)^{-\lambda})$, $r \to 1-$.

Proof. Let $r_k = e^{-\lambda/2^k}$, so as $k \to \infty$ we have $r_k \to 1-$, and

$$(1 - r_k) \sim -\log r_k = \lambda/2^k. \tag{5.1}$$

Let $u_k(z) = 2^{\lambda k} z^{2^k}$. Then fairly straightforward estimates [6; Sec. 8.5, pp. 96-97] show that

$$|f(z)| \sim |u_k(z)| \sim (1 - |z|)^{-\lambda} \tag{5.2}$$

for $|z| = r_k$ $(k = 1, 2, \cdots)$. Since $M_p(f;\, r)$ increases with r [1; Theorem 1.5, page 9] and $|u_k|$ is rotation-invariant, we have from (5.1) and (5.2) that for $r_{k-1} \le r \le r_k$,

$$M_p^p(f;\, r) \le M_p^p(f;\, r_k) \le CM_p^p(u_k;\, r_k) = CM_\infty(u_k;\, r_k)^p \le C(1 - r_k)^{-\lambda p} \le C2^{\lambda p k}.$$

Moreover, by (5.1), $\int_{r_{k-1}}^{r_k} (1 - r)^\alpha\, dr \sim (1 - r_{k-1})^{\alpha+1} \sim 2^{-(\alpha+1)k}$, so letting $A = M_\infty(f;\, r_0)$, we have

$$\int_0^1 M_p^p(f;\, r)(1 - r)^\alpha\, dr \le A + \sum_{k=1}^{\infty} M_p^p(f;\, r_k) \int_{r_{k-1}}^{r_k} (1 - r)^\alpha\, dr$$

$$\le A + C \sum_{k=1}^{\infty} 2^{[\lambda p - (\alpha+1)]k} < \infty,$$

the last sum being finite because $\alpha + 1 > \lambda p$. Thus $f \in A_\alpha^p$.

To show that $(f^n + b)\,h \notin A_\alpha^{g/n}$, we proceed as in the proof of Theorem 2.1 to show that if $g = (f^n + b)\,h$, then for $r_{k-1} \le r \le r_k$,

$$M_{q/n}^{q/n}(g;\, r) \ge CM_q^q(f;\, r_{k-1}) \ge C2^{\lambda q k},$$

where the last inequality follows from (5.1) and (5.2). Then a calculation like the one we just performed shows that

$$\int M_q^q(g;\, r)(1-r)^\alpha\, dr \geq \sum_{k=1}^{\infty} M_q^q(g;\, r_{k-1}) \int_{r_{k-1}}^{r_k} (1-r)^\alpha\, dr$$

$$\geq C \sum_{k=1}^{\infty} 2^{[\lambda q-(\alpha+1)]k} = \infty,$$

where now the sum diverges because $\alpha + 1 < \lambda q$. Thus $g \notin A_\alpha^q$, and the proof is complete.

(iii). *More general domains.* It would be of interest to see whether Corollaries 4.2 and 4.3 hold in any plane domain (or for that matter any domain of holomorphy in $\mathbb{C}^k$) for which the corresponding Bergman spaces are nontrivial.

(iv). *Subsets of zero sets.* In addition to proving Corollaries 2.2 and 2.3 for the spaces A_α^p, Horowitz showed in [7] that *any subset of an* A_α^p *zero set is again a zero set.* We do not know whether this result holds for all A_μ^p, and the methods of this paper give no information about the problem.

Horowitz [8] has also used his methods to show that every A_α^p function is a product of two A_α^{2p} functions. Is this true for A_μ^p as well?

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