TANGENTIAL BOUNDARY BEHAVIOR OF HARMONIC EXTENSIONS OF L^p POTENTIALS

Alexander Nagel, Walter Rudin
University of Wisconsin

Joel H. Shapiro
Michigan State University

INTRODUCTION

describe, mostly without proofs, some results on the boundary havior of harmonic functions in classes modeled on the space $\mathfrak{D}$ of nctions harmonic in the open unit disc with finite Dirichlet in- gral. Detailed proofs will appear elsewhere [6].

For our purposes, $\mathfrak{D}$ is best regarded as the space of Poisson tegrals of functions f square integrable on the unit circle T, th the additional restriction:

$$\sum_{-\infty}^{\infty} |n| \, |\hat{f}(n)|^2 < \infty, \tag{1}$$

ere $\hat{f}(n)$ is the nth Fourier coefficient of f. Such f are somewhat re regular than "typical" L^2 functions, but still not necessarily ntinuous, or even bounded. We study how the additional regularity f affects the boundary behavior of its harmonic extension u.

This type of problem was considered by Salem and Zygmund], who showed that the Fourier series of each $f \in L^2(T)$ satisfy- g (1) converges at each point of T, with the possible exception of

---

Research partially supported by the National Science undation.

t of logarithmic capacity zero. This implies that each $u \in \mathfrak{D}$ a radial limit at each point of T, with the possible exception uch a set. In fact, the same is true for nontangential limits.

Our contribution is to show that if larger classes of ptional sets—intermediate between "log-capacity zero" and sure zero"—are allowed, then the functions in $\mathfrak{D}$ will converge in regions that meet the unit circle *tangentially*. We describe precise relationship between the curvature of the approach on and the size of the exceptional sets for convergence within e regions. In particular, we show that each $u \in \mathfrak{D}$ has at st every $\zeta \in T$ a limit as $z \to \zeta$ through an approach region ng *exponential* contact with T at ζ.

To make these matters precise, consider the following ons in the open unit disc U. If $c > 0$ and $\gamma \geq 1$, let

$$\mathcal{A}_{\gamma,c}(\varphi) = \left\{ re^{i\theta} \in U : 1 - r > c\left|\sin\left(\frac{\theta - \varphi}{2}\right)\right|^{\gamma}\right\},$$

e if $\gamma > 0$, let

$$\mathcal{E}_{\gamma,c}(\varphi) = \left\{ re^{i\theta} \in U : 1 - r > \exp\left[-c\left|\sin\left(\frac{\theta - \varphi}{2}\right)\right|^{-\gamma}\right]\right\}.$$

$\mathcal{A}_{\gamma,c}(\varphi)$ has *order of contact* γ with T at $e^{i\varphi}$, while $\mathcal{E}_{\gamma,c}(\varphi)$ *exponential contact*. Note that the regions $\mathcal{A}_{1,c}(\varphi)$ are the l nontangential approach regions.

We say a function u defined in U has *$\mathcal{A}_\gamma$-limit L at $e^{i\varphi}$* if $\to L$ as $z \to e^{i\varphi}$ within $\mathcal{A}_{\gamma,c}(\varphi_0)$ for every $c > 0$. A similar nition applies to $\mathcal{E}_\gamma$-limits. We can state the classical results his language: if $f \in L^2(T)$ and u is the Poisson integral of f, u has an $\mathcal{A}_1$-limit at almost every point of T. If, in addition, tisfies (1), then u has an $\mathcal{A}_1$-limit at every point of T, with possible exception of a set of logarithmic capacity zero. Our lts for the space $\mathfrak{D}$ can be stated as follows.

eorem 1. Suppose $u \in \mathfrak{D}$. Then

(a) *U has an $\mathcal{E}_1$-limit at almost every point of T.*

(b) *If, in addition, $0 < \beta < 1/2$ and $\gamma = (1 - 2\beta)^{-1}$, then U has an $\mathfrak{A}_\gamma$-limit at every point of T, with the possible exception of a set of $C_{\beta,2}$-capacity zero.*

Here the capacity $C_{\beta,2}$ is the analogue for the unit circle the corresponding Bessel capacity on R^n (see [5], for example). coincides with the classical capacity $C_{1-2\beta}$ discussed by Kahane d Salem in [3; Chapter 3, p. 33]. In particular, $C_{1/2,2}$ is the garithmic capacity, so when $\beta = 1/2$, our result coincides with e previously mentioned one of Salem and Zygmund.

The correct setting for this work is a more general one, tivated by the fact that $\mathfrak{D} = P[K * L^2]$, where P is the Poisson tegral for the unit disc,

$$K(\theta) = \left|\sin \frac{\theta}{2}\right|^{-1/2} \sim \sum_{-\infty}^{\infty} (|n| + 1)^{-1/2} e^{in\theta},$$

d $*$ denotes the convolution on T. We prove a generalization of eorem 1 valid for the classes $P[K * L^p]$ where $1 \leq p < \infty$ and K is positive, integrable, even function on $[-\pi, \pi]$ that is decreasing $(0, \pi]$. As a by-product of our work, we answer a question of S. Shapiro and A. L. Shields concerning the zeros of holomorphic nctions in classes $P[K * L^2]$. These results are stated in detail the next section.

As the reader has probably guessed, our tangential convernce theorems follow from weak-type estimates on maximal functions sociated with our approach regions. For $p > 1$, we use Hansson's rong-type capacity inequality [2] to obtain strong-type maximal timates, which are in turn crucial to the proof of part (b) of eorem 1 and its generalizations. We state these results presely in Secs. 3 and 4. In Sec. 5 we show how the strong-type imal estimates enter into the proof of Theorem 1, part (b); and Sec. 6 we discuss Carleson measures for the classes $P[K * L^p]$.

NGENTIAL CONVERGENCE THEOREMS

prefer to work in the upper half-space R_+^{n+1} of $(n + 1)$-dimensional .idean space, instead of the unit disc; so R^n replaces the unit :le as the boundary. We write L^p for $L^p(R^n)$ and denote points $?_+^{n+1}$ by (x, y) with $x \in R^n$ and $y > 0$. P will denote the Poisson egral for R_+^{n+1}, with $\{P_y : y > 0\}$ the corresponding Poisson iels on R^n. $*$ will denote convolution on R^n.

ntial Spaces and Dirichlet-Type Spaces

ose K is a *kernel* on R^n; that is, K is positive, integrable, ally symmetric, and $K(x)$ decreases as $|x|$ increases. For $p < \infty$, let

$$L_K^p = \{K * F : F \in L^p\}$$

te the space of L^p potentials associated with K, and let

$$h_K^p = P[f] : f \in L_K^p$$

he corresponding space of harmonic extensions to R_+^{n+1}. Every h_K^p hus an analogue of the Dirichlet space $\mathfrak{D}$ of Sec. 1. In parti- r, it follows from Plancherel's theorem that h_K^2 consists of the ionic extensions of functions $f \in L^2$ satisfying the additional itions

$$\int_{R^n} |\hat{K}(\lambda)|^{-2} |\hat{f}(\lambda)|^2 d\lambda < \infty \qquad (2)$$

e ^ denotes the Fourier transform on L^2.

To avoid trivialities, we always assume that $K \notin L^q$, where $+ q^{-1} = 1$. Thus L_K^p, and therefore h_K^p, always contains unbounded tions.

proach Regions

r $(x, y) \in R_+^{n+1}$, let

$$K_y(x) = P[K](x, y) = P_y * K(x)$$

d set

$$r(y) = r_{K,p}(y) = \|K_y\|_q^{-p/n}, \quad y > 0.$$

r $x_0 \in R^n$ and $\beta > 0$, define

$$\Omega(x_0) = \Omega_{K,\beta}^p(x_0) = \{(x, y) \in R_+^{n+1} : |x - x_0| < \beta r(y)\}.$$

us $\Omega(x_0)$ is the region in R_+^{n+1} with spherical cross section of dius $\beta r(y)$ at height y above R^n. Since $K \notin L^q$, we know that $y) \to 0$ as $y \to 0^+$, so $\Omega(x_0)$ approaches R^n only at the point x_0 ts "vertex"). Moreover, standard estimates of Poisson integrals ow that $y^{-1}r(y) \to 0$ as $y \to 0^+$, so the boundary of $\Omega(x_0)$ actually proaches R^n *tangentially* at x_0.

We define Ω_K^p *limits* of functions defined on R_+^{n+1} exactly in the last section.

pacity

llowing Meyers [5], if $1 < p < \infty$, we define the (K, p)-capacity of subset E of R^n as follows. Let $T_{K,p}(E)$ denote those nonnegative $\in L^p$ for which $K * F \geq 1$ everywhere on E. Note that since K and F e both positive, the convolution $K * F$ make sense (possibly $+\infty$) every point of R^n.

The *capacity* of E is

$$C_{K,p}(E) = \inf\{\|F\|_p^p : F \in T_{K,p}(E)\},$$

ere $\|\cdot\|_p$ denotes the L^p norm.

It is easy to check that $C_{K,p}$ is subadditive and monotone creasing on the subsets of R^n, and that a subset E has capacity ro if and only if $K * F \equiv \infty$ on E for some nonnegative F in L^p , Sec. 2].

This last comment shows that the members of the potential ›e L_K^p are defined and finite at $C_{K,p}$-almost every point of R^n.

ı Results

›an now state our main results on tangential convergence. In › follows, $1 \le p < \infty$, $p^{-1} + q^{-1} = 1$, $K \notin L^q$, $f \in L_K^p$, and $P[f]$.

›orem 2. *There is a set E of Lebesgue measure zero such that u Ω_K^p limit $f(x)$ at x for every $x \in R^n \backslash E$.*

›orem 3. *If, in addition, $p > 1$ and $K = H * G$ where H and G are › kernels, then there is a set E with $C_{H,p}(E) = 0$ such that u Ω_G^p limit $f(x)$ at x for each $x \in R^n \backslash E$.*

Thus Theorem 3 shows precisely how the degree of tangential ›ergence influences the size of the exceptional set.

ıples: Bessel Potentials

most important class of kernels are the Bessel kernels $K = g_\alpha$ ›ned for $0 < \alpha \le n$ by

$$\hat{g}_\alpha(\lambda) = (1 + |\lambda|^2)^{-\alpha/2}, \quad \lambda \in R^n.$$

corresponding capacities $B_{\alpha,p}$ make sense for $\alpha p \le n$ and are ›ed *Bessel capacities*. Observe that $g_\alpha * g_\beta = g_{\alpha+\beta}$. Properties ›hese kernels are worked out in detail in (for example) Meyers Sec. 7, p. 279]. The analogous kernels for the unit circle are ones considered in Kahane and Salem [3, Chapter 3]:

$$\Phi_\alpha(\theta) = \left|\sin\frac{\theta}{2}\right|^{\alpha-1} \sim \sum(|n| + 1)^{-\alpha} e^{in\theta}$$

$0 < \alpha < 1$, and

$$\Phi_1(\theta) = -\log\left|\sin\frac{\theta}{2}\right| \sim \sum(|n| + 1)^{-1} e^{in\theta}.$$

$y \to 0^+$, we have the following asymptotic estimates on the radius $_p(y) = r_{\alpha,p}(y)$, defined in "Approach Regions" of Sec. 2:

$$r_{\alpha,p}(y) \sim \begin{cases} y^{1-(\alpha p/n)} & \text{if } \alpha p < n \\ (-\log y)^{-1/n(q-1)} & \text{if } \alpha p = n,\ p > 1 \\ (-\log y)^{-1/n} & \text{if } \alpha = n,\ p = 1 \end{cases} \tag{3}$$

re $p^{-1} + q^{-1} = 1$. By analogy with the work of Sec. 1, we define roach regions for $c > 0$: for $\gamma \geq 1$,

$$\mathfrak{a}_{\gamma,c} = \{(x, y) \in R_+^{n+1} : y > c|x - x_0|^\gamma\},$$

le if $\gamma > 0$, then

$$\mathcal{E}_{\gamma,c} = \{(x, y) \in R_+^{n+1} \quad y > \exp(-c|x - x_0|^{-\gamma})\}.$$

n estimate (3) shows that the regions $\Omega^p_{g_{\alpha,\beta}}$ essentially coincide h the classes of regions:

$$\mathfrak{a}_{\gamma,c} \text{ if } \alpha p < n, \text{ where } \gamma = n/(n - \alpha p),$$
$$\mathcal{E}_{\gamma,c} \text{ if } \alpha p = n \text{ and } p > 1, \text{ where } \gamma = n(q - 1),$$

$$\mathcal{E}_{n,c} \text{ if } \alpha = n \text{ and } p = 1.$$

s Theorems 2 and 3 have the following corollaries, which gener- ze Theorem 1. Here $f \in L^p_{g_\alpha}$ where $\alpha p \leq n$, so $u = P[f] \in h^p_{g_\alpha}$; and e." refers to Lebesgue measure on R^n.

ollary 1.

(a) *If* $\alpha p < n$ *and* $p > 1$, *then* u *has* $\mathfrak{a}_{n/(n-\alpha p)}$*-limit* $f(x)$ *at a.e.* x *in* R^n.

(b) *If* $\alpha p = n$ *and* $p > 1$, *then* u *has* $\mathcal{E}_{n(q-1)}$*-limit* $f(x)$ *at a.e.* x *in* R^n.

(c) *If* $\alpha = n$ *and* $p = 1$, *then* U *has* $\mathcal{E}_n$*-limit* $f(x)$ *at a.e.* x *in* R^n.

ollary 2. Suppose in addition to the hypotheses above, that $p < \infty$, $\alpha p \leq n$, *and* $\alpha = \tau + \kappa$, *where* τ *and* κ *are positive* nbers. *Then* u *has* $\mathfrak{A}_{n/(n-\tau p)}$*-limit* $f(x)$ *at each* $x \in R^n$ *with the* ssible exception of a set of $B_{\kappa,p}$ *capacity zero.*

nark. If we return to the unit circle and the case $p = 2$, then noted earlier, the capacity $B_{\alpha,2}$ corresponds to the classical pacity $C_{1-2\alpha}$ of Kahane and Salem [3, Chapter 3] for $0 < \alpha \leq 1/2$, th the understanding that C_0 is logarithmic capacity. Note that index in Kahane and Salem's capacity refers to the exponent sociated with the *kernel*, while the index α of the Bessel capacity fers to the exponent associated with the *Fourier transform* of the rnel.

plication to Zeros of h_K^p Functions

next result generalizes one proved by H. S. Shapiro and A. L. ields in the case $n = 1$, $p = 2$ for special kernels and holomorphic nctions [8, Theorem 3], and it answers a question posed by them , p. 224].

eorem 4. *Suppose* $1 \leq p < \infty$ *and suppose* $(y_j)_1^\infty$ *is a sequence of* *sitive numbers with* $\sum r_{K,p}^n(y_j) = \infty$. *Then there exists a sequence* $_j)_1^\infty$ *in* R^n *such that no nontrivial function in the class* h_K^p *nishes at each point* (x_j, y_j) *of* R_+^{n+1}.

oof. The hypothesis on (y_j) ensures that we can choose open lls B_j of radius $r_{K,p}(y_j)$ such that each point of R^n lies in finitely many B_j. Let x_j be the center of B_j and set $= (x_j, y_j) \in R_+^{n+1}$. If $x \in R^n$, then x belongs to some sequence , B_{j_2}, ... of balls; hence z_{j_1}, z_{j_2}, ... all belong to $\Omega_{K,1}^p(x)$. So $u \in h_K^p$ vanishes at each z_j, then it has Ω_K^p limit zero at each x which it has an Ω_K^p limit. By Theorem 2 this happens for almost ery x in R^n. Since u is the Poisson integral of its boundary nction, this implies $u \equiv 0$, which completes the proof.

u is a complex valued function defined on R_+^{n+1}, define for
$\leq p < \infty$ and $\beta > 0$:

$$\mathfrak{M}_{K,p,\beta}u(x_0) = \sup\{|u(x, y)| : (x, y) \in \Omega^p_{K,\beta}(x_0)\}.$$

$u = P[f]$ we will also write this as $\mathfrak{M}_{K,p,\beta}f(x_0)$. Our fundamental
timate on this tangential maximal function comes from a direct
mparison with the following L^p-variant of the Hardy-Littlewood
ximal function. For $f \in L^p$ and $x_0 \in R^n$, let

$$M_pF(x_0) = \sup_{r>0}\left\{\frac{1}{m(B_r(x_0))}\int_{B_r(x_0)}|F|^p\,dm\right\}^{1/p},$$

ere $B_r(x_0)$ is the ball of radius r in R^n, centered at x_0, and m is
besgue measure on R^n. Clearly M_p is subadditive, and by the
ual Hardy-Littlewood maximal theorem, it is of weak type (p, p).
have:

*ma. Suppose $1 \leq p < \infty$ and $\beta > 0$. Then there exists
$A(K, p, \beta) > 0$ such that if $f = K * F$ for $F \in L^p$, then*

$$\mathfrak{M}_{K,p,\beta}f(x_0) \leq AM_pF(x_0)$$

every $x_0 \in R^n$.

This lemma immediately gives the following weak-type esti-
e, which by standard arguments yields Theorem 2.

*orem 5. For each $1 \leq p < \infty$ and $\beta > 0$ there exists
$A(K, p, \beta) < \infty$ such that for every $f = K * F$ with $F \in L^p$:*

$$m\{x \in R^n : \mathfrak{M}_{K,p,\beta}f(x) > \lambda\} \leq \left(\frac{A\|F\|_p}{\lambda}\right)^p$$

every $\lambda > 0$.

Theorem 5 is "best possible" in the following sense.
$_{p,\beta}$ is not of weak type (r, r) for any $r < p$. Moreover, if Ω is

region in R_+^{n+1} that approaches R^n only at 0, and $\Omega(x_0) = \Omega + x_0$, n the Ω-maximal function

$$M_\Omega f(x_0) = \sup\{|u(x, y)| : (x, y) \in \Omega(x_0)\}$$

$= P[f]$) is of weak type (p, p) on L_K^p only if there is a "slab" R_+^{n+1}:

$$S = \{(x, y) \in R_+^{n+1} : y < y_0\},$$

a $\beta > 0$ such that

$$\Omega \cap S \subset \Omega_{K,\beta}^p(0) \cap S.$$

We remark that the lemma is a consequence of the following volution inequality, whose proof is fairly straightforward. If $p < \infty$, then there exists $A = A(n, p) < \infty$ such that if $F \in L^p$ *K is a nonnegative, radial decreasing function on* R^n, *then*

$$|K * F(x)| \le AM_pF(x_0)\left[|x - x_0|^{n/p} \|K\|_q + \|K\|_1\right]$$

all x *and* x_0 *in* R^n.

The lemma follows from this inequality upon replacing K by and taking the supremum of both sides for $|x - x_0| < r_{K,p}(y)$ and 0.

TRONG-TYPE ESTIMATES

$1 < p < \infty$ we have the following improvement of Theorem 5.

eorem 6. Suppose $1 < p < \infty$ *and* $\beta > 0$. *Then there exists* $= A(K, p, \beta) < \infty$ *such that if* $f = K * F$ *for* $F \in L^p$, *then* ${}_{p,\beta}f \in L^p$ *and* $\|\mathfrak{M}_{K,p,\beta}f\| \le A\|F\|_p$.

Since M_p is not of strong type (p, p), this result shows at $\mathfrak{M}_{K,p,\beta}$ is not "equivalent" to M_p if $1 < p < \infty$. As we will see Sec. 5, Theorem 6 is the main step in the proof of Theorem 3.

ward the Proof of Theorem 6

need the following geometry. For $x_0 \in R^n$, let

$$\Gamma(x_0) = \{(x, y) \in R_+^{n+1} : |x - x_0| < y\}$$

note the right circular cone in R_+^{n+1} with vertex x_0; and for u ntinuous on R_+^{n+1}, let

$$Nu(x_0) = \sup\{|u(x, y)| : (x, y) \in \Gamma(x_0)\}$$

the usual nontangential maximal function. For E an open subset R^n, let

$$S(E) = R_+^{n+1} \setminus \bigcup_{x \notin E} \Gamma(x),$$

e usual "Carleson set" over E, and for F an open subset of R_+^{n+1}, $\leq p < \infty$, and $\beta > 0$, let

$$J(F) = J_{K,\beta}^p(F) = \{x \in R^n : \Omega_{K,\beta}^p(x) \cap F \neq \emptyset\}.$$

$J(F)$ is a sort of "shadow" of F on R^n. Then we have the follow- g two results, the first of which is an interesting geometric wer bound for capacity. Assume $1 < p < \infty$ and $\beta > 0$.

oposition 1. There exists $A = A(K, p, \beta) < \infty$ such that for every en set $E \subset R^n$:

$$m\{J_{K,\beta}^p(S(E))\} \leq AC_{K,p}(E).$$

oposition 2. If u is continuous in R_+^{n+1} and $1 < p < \infty$, then for ery $\lambda > 0$:

$$m\{\mathfrak{M}_{K,p,\beta}u > \lambda\} \leq AC_{K,p}\{Nu > \lambda\}$$

ere $A = A(K, p, \beta) < \infty$.

Proposition 2 follows from Proposition 1 upon observing that $E = \{Nu > \lambda\}$, then $J_{K,\beta}^p(S(E))$ is an open set containing $_{K,p,\beta}u > \lambda\}$. Proposition 1 follows from the weak-type estimate of orem 5 and the fact that if $F \in T_{K,p}(E)$, then (since $K * F \geq 1$ E) $P[K * F] \geq \lambda_0$ on $S(E)$, where λ_0 depends only on the dimension nd not on E. We describe a similar argument in more detail in next section.

Finally, we require:

sson's Strong-Type Estimate [2, Theorem 2.4]. *If* $1 < p < \infty$, *n there exists* $A = A(K, p) < \infty$ *such that for each* $F \in L^p$:

$$\int_0^\infty C_{K,p}\{|K * F| > \lambda\} d(\lambda^p) \leq A\|F\|_p^p.$$

cial cases of this result have previously been obtained by Adams and Maz'ya [4]. Note that the estimate is a significant rovement over the trivial weak-type estimate

$$C_{K,p}\{|K * F| > \lambda\} \leq \left(\frac{\|F\|_p}{\lambda}\right)^p, \tag{4}$$

ch is just a statement of the fact that for every $F \in L^p$ and 0, the function $|F|/\lambda$ belongs to $T_{K,p}(\{|K * F| > \lambda\})$.

of of Theorem 6

pose $f = K * F$ for $F \in L^p$, and write $u = P[f]$, $v = P[F]$. Then it easy to see that $Nu \leq K * Nv$, so

$$\begin{aligned}
\|\mathfrak{M}_{K,p,\beta}u\|_p^p &= \int_0^\infty m\{\mathfrak{M}_{K,p,\beta}u > \lambda\} d(\lambda^p) \\
&\leq A\int_0^\infty C_{K,p}\{Nu > \lambda\} d(\lambda^p) \qquad \text{(Prop. 2)} \\
&\leq A\int_0^\infty C_{K,p}\{K * Nv > \lambda\} d(\lambda^p) \\
&\leq A\|Nv\|_p^p \qquad \text{(Hansson's estimate)} \\
&\leq A\|F\|_p^p,
\end{aligned}$$

re the last inequality follows from the Hardy-Littlewood non-gential maximal theorem.

:APACITY VS. CURVATURE: PROOF OF THEOREM 3

:all that Theorem 3 describes how the curvature of our approach ;ions influences the capacity of the exceptional sets for conver- ıce of h_K^p functions within these regions. Theorem 3 follows by ındard arguments from part (b) of the next result, which in turn .lows from part (a) and Hansson's estimate.

ɜorem 7. *Suppose* $1 < p < \infty$, $\beta > 0$, *and* $K = H * G$, *where* H *and* G *ɜ also kernels.*

(a) *Then there exists* $A = A(p, \beta, G, H) < \infty$ *such that*

$$C_{H,p}\{\mathfrak{M}_{G,p,\beta}u > \lambda\} \leq AC_{K,p}\{Nu > \lambda\}$$

· every u *continuous on* R_+^{n+1} *and every* $\lambda > 0$.

(b) *If, in addition,* $u = P[K * F]$ *where* $F \in L^p$, *then*

$$\int_0^\infty C_{H,p}\{\mathfrak{M}_{G,p,\beta}u > \lambda\}d(\lambda^p) \leq A\|F\|_p^p$$

· A *as in part* (a).

Note that part (a) above is a capacitary analogue of Propo- .ion 2 of the last section. It follows from a similar analogue Proposition 1, which we state and prove in detail.

ɔposition 3. Under the hypotheses of Theorem 7 we have for every ın subset E *of* R^n:

$$C_{H,p}\{J_{G,\beta}(S(E))\} \leq AC_{K,p}(E),$$

re $A = A(p, \beta, G, H) < \infty$.

of. Write $M_G = M_{G,\beta,p}$ and $J_G = J_{G,\beta}^p$. Suppose $F \in T_{K,p}(E)$ and te $u = P[K * F]$ and $v = P[G * F]$. Since $K * F \geq 1$ on E, we know t $u \geq \lambda_0$ on $S(E)$, where $\lambda_0 > 0$ is independent of E. Thus

$$J_G(S(E)) \subset \{\mathfrak{M}_G u > \lambda_0\}$$

1 so

$$\begin{aligned} C_{H,p}\{J_G(S(E))\} &\le C_{H,p}\{\mathfrak{M}_G u > \lambda_0\} \\ &= C_{H,p}\{\mathfrak{M}_G P[H * G * F] > \lambda_0\} \\ &\le C_{H,p}\{H * \mathfrak{M}_G P[G * F] > \lambda_0\} \\ &\le \left(\frac{\|\mathfrak{M}_G v\|_p}{\lambda_0}\right)^p \qquad (v = P[G * F]) \\ &\le \left(\frac{A\|F\|_p}{\lambda_0}\right)^p, \end{aligned}$$

:re the next-to-last inequality follows from the trivial weak-
)e estimate (4) of Sec. 4, and the last inequality follows from
: strong-type estimate Theorem 6. Take the infimum of the right
le of the above inequality as F ranges through $T_{K,p}(E)$. The proof
complete.

:ARLESON MEASURES

call a finite positive Borel measure μ on R_+^{n+1} a *Carleson measure* · h_K^p if

$$\int |u|^p \, d\mu < \infty$$

: every $u \in h_K^p$. Stengenga [9] has characterized the (K, p) :leson measures for $1 < p < \infty$ as follows: they are precisely those :or which there exists $A > 0$ such that

$$\mu(S(E)) \le AC_{K,p}(E) \tag{5}$$

: every open subset E of R^n. Stengenga also gives examples that ›w that if (5) holds merely for open balls E, then μ need not be :arleson measure.

Using Theorem 6 and standard arguments, we can obtain a more ›metric condition on μ that is *sufficient* for it to be a Carleson ısure for h_K^p. To state our result, we define for every open sub- : E of R^n a sort of "(K, p, β)-Carleson region" over E:

$$Q^p_{K,\beta}(E) = R^{n+1}_+ \setminus \bigcup_{x \notin E} \Omega^p_{K,\beta}(x).$$

orem 8. Suppose $1 < p < \infty$, $\beta > 0$; *and suppose there exists* ∞ *such that*

$$\mu\{Q^p_{K,\beta}(B)\} \le Am(B)$$

every open ball B *in* R^{n+1}_+. *Then* μ *is a Carleson measure for* h^p_K.

However, this condition is *not* necessary. Using an idea municated to us by Stengenga, we can show that for each kernel K $1 < p < \infty$, there exists a Carleson measure μ for h^p_K that does satisfy the hypothesis of the theorem.

ERENCES

Adams, D. R. On the existence of capacitary strong type estimates in R^n. *Ark. Mat.* 14 (1976):125-140.

Hansson, K. Imbedding theorems of Sobolev type in potential theory. *Math. Scand.* 45 (1979):77-102.

Kahane, J. P., and Salem, R. Ensembles parfaits et séries trigonométriques. *Actualités Sci. Ind.* No. 1301. Paris: Hermann, 1963.

Maz'ya, V. G. On capacitary estimetes of strong type for fractional norms. In Russian. *Zap. Sem. LOMI Leningrad* 70 (1977): 161-168.

Meyers, N. G. A theory of capacities for potentials of functions in Lebesgue classes. *Math. Scand.* 26 (1970):255-292.

Nagel, A.; Rudin, W.; and Shapiro, J. H. Tangential boundary behavior of functions in Dirichlet-type spaces. *Annals of Math.* 115 (1982).

Salem, R., and Zygmund, A. Capacity of sets and Fourier series. *Trans. Amer. Math. Soc.* 59 (1946):23-41.

Shapiro, H. S., and Shields, A. L. On the zeros of functions with finite Dirichlet integral, and some related function spaces. *Math. Zeit.* 80 (1962):217-229.

Stengenga, D. A. Multipliers of the Dirichlet space. *Ill. J. Math.* 24 (1980):113-139.