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Tangential boundary behavior of function in Dirichlet-type spaces

By ALEXANDER NAGEL, WALTER RUDIN, and JOEL H. SHAPIRO

I. Introduction

1.1. This paper deals with the tangential boundary behavior of harmonic functions that belong to analogues of the classical Dirichlet space D. This is the space of all holomorphic functions defined in the unit disc U for which

$$\|h\|_D^2 = |h(0)|^2 + \frac{1}{\pi}\iint_U |h'(z)|^2\, dx\, dy$$

is finite. For $h \in D$, we are interested in the existence of limits, and in estimates of the supremum, within regions $\Omega \subset U$ that allow tangential approach to the boundary T of U.

For example, let

$$\mathcal{E}_{\gamma,c}(\theta) = \left\{re^{i\varphi}: 1 - r > \exp\left\{-c\left|\sin\frac{\varphi-\theta}{2}\right|^{-\gamma}\right\}\right\}.$$

This region makes *exponential contact* with the unit circle T at $e^{i\theta}$. For any f defined in U, set

$$\left(\mathfrak{M}_{\mathcal{E},\gamma,c}f\right)(\theta) = \sup\left\{|f(z)| : z \in \mathcal{E}_{\gamma,c}(\theta)\right\}.$$

We show that *there are constants* $A_c < \infty$ *so that, for every* $h \in D$,

$$\|\mathfrak{M}_{\mathcal{E},1,c}\|_2 \le A_c \|h\|_D$$

and lim $h(z)$ *exists as* $z \to e^{i\theta}$ *within* $\mathcal{E}_{1,c}(\theta)$, *for almost all* $\theta \in [0, 2\pi]$.

This contrasts of course strongly with the behavior of bounded holomorphic functions: if L is any curve in U that approaches the point 1 tangentially, then there is a bounded holomorphic function in U whose limit along $e^{i\theta}L$ exists for *no* $\theta \in [0, 2\pi]$. (See [19], [20; p. 280], [5; p. 43].)

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Such results about tangential boundary behavior extend to a large class of function spaces. In Sections II and III we study spaces of Poisson integrals of potentials on $\mathbf{R}^n$. However, our results apply also to various spaces of holomorphic functions in U, and we state some of them below as Theorem A.

In addition to the exponential contact regions $\mathcal{E}_{\gamma, c}$ we define regions

$$\mathcal{Q}_{\gamma, c}(\theta) = \left\{re^{i\varphi} : 1 - r > c\left|\sin\frac{\varphi - \theta}{2}\right|^{\gamma}\right\}$$

which have *order of contact* γ. We say that a function h, defined in U, *has* $\mathcal{Q}_\gamma$*-limit* L *at* $e^{i\theta}$ if $h(z) \to L$ as $z \to e^{i\theta}$ within $\mathcal{Q}_{\gamma, c}(\theta)$ for every c.

$\mathcal{E}_\gamma$-limits are defined analogously.

Finally, in addition to the maximal operator $\mathfrak{M}_{\mathcal{E}, \gamma, c}$ we define

$$\left(\mathfrak{M}_{\mathcal{Q}, \gamma, c}h\right)(\theta) = \sup\left\{|h(z)| : z \in \mathcal{Q}_{\gamma, c}(\theta)\right\}.$$

THEOREM A. *Suppose* $1 \le p < \infty$, $pq = p + q$, $F \in L^p(T)$, $0 < \alpha < 1$, *and*

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F(t)\,dt}{(1 - e^{-it}z)^{1-\alpha}} \qquad (z \in U).$$

(a) *If* $\alpha p < 1$ *and* $\gamma = 1/(1 - \alpha p)$ *then the* $\mathcal{Q}_\gamma$*-limit of* h *exists almost everywhere on* T.

(b) *If* $\alpha p = 1$ *then the* $\mathcal{E}_{q-1}$*-limit of* h *exists almost everywhere on* T.

(c) *If* $\mathfrak{M}h = \mathfrak{M}_{\mathcal{Q}, \gamma, c}h$ *or* $\mathfrak{M}_{\mathcal{E}, \gamma, c}h$ *is the corresponding maximal function, then*

$$\|\mathfrak{M}h\|_p \le A\|F\|_p$$

when $1 < p < \infty$, *and*

$$m\{\mathfrak{M}h > \lambda\} \le A\lambda^{-1}\|F\|_1$$

when $p = 1$, $0 < \lambda < \infty$.

We remark that Theorem A deals with the Dirichlet space D when $p = 2$ and $\alpha = 1/2$, since $h \in D$ if and only if there is an $f \in L^2(T)$ such that

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F(t)\,dt}{(1 - e^{-it}z)^{1/2}}.$$

We also note that the function h in Theorem A is continuous on $\bar{U}$ when $\alpha p > 1$. In Section 5.7 we explain how Theorem A follows from the results of Sections II and III.

The existence of tangential limits of holomorphic functions of certain types was studied by Kinney [12]. His results imply that every $h \in D$ has $\mathcal{Q}_\gamma$-limits almost everywhere on T, for every finite γ, and also contain information about the capacities of the exceptional sets.

Section IV contains applications of our main results to Carleson-type measures for the potential spaces under consideration.

Section V contains applications to Bessel potentials. It also contains generalizations of the Fejér-Riesz inequalities (Theorem 5.5 (c)), some information about zero sets (Theorem 5.10), and a proof that the Cauchy integral of every function of bounded variation on T has $\mathcal{E}_1$-limits almost everywhere on T (Theorem 5.8).

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1.2. *Notations and terminology.* For the most part, we shall not deal with the unit disc, but with half-spaces of arbitrary dimension. We shall use the following notations.

(i) $\mathbf{R}^n$ is euclidean n-space, with norm $|x| = (\Sigma x_i^2)^{1/2}$, and

$$\mathbf{R}_+^{n+1} = \{(x, y): x \in \mathbf{R}^n, y > 0\}.$$

The dimension n will usually not be explicitly mentioned in any other notation.

$B = \{x: |x| < 1\}$ is the open unit ball of $\mathbf{R}^n$.

(ii) For $E \subset \mathbf{R}^n$, $m(E)$ is the Lebesgue measure of E. Instead of dm, we often write $dx, dt, \ldots.$

$L^p = L^p(m)$, with norm $\|F\|_p = (\int |F|^p \, dm)^{1/p}$ if $1 \le p < \infty$. The index conjugate to p will always be q: $pq = p + q$. Convolutions are defined on $\mathbf{R}^n$ by

$$(F * G)(x) = \int_{\mathbf{R}^n} F(x-t)G(t)\, dt,$$

and by

$$(F * \mu)(x) = \int_{\mathbf{R}^n} F(x-t)\, d\mu(t)$$

if μ is a measure.

(iii) The letter K and the word *kernel* denote a nonnegative L^1-function which is *radial* and *decreasing*; i.e., $K(x) = K(t)$ if $|x| = |t|$ and $K(x) \le K(t)$ if $|x| \ge |t|$.

Also, $K(0) = \infty$ (we are not interested in bounded K), and we usually normalize so that $\|K\|_1 = 1$.

(iv) L_K^p is the space of all *K-potentials* associated to the exponent p. This means that L_K^p consists of all convolutions

$$f = K * F$$

with $F \in L^p$. The norm

$$\|f\|_{K,p} = \inf\{\|F\|_p : f = K * F\}$$

makes L^p_K into a Banach space that is isometrically isomorphic to L^p/N, where N consists of all F for which $K * F = 0$. (It should be noted that $N = \{0\}$ for many kernels K; in that case, the above inf can be ignored.)

Clearly, $L^p_K \subset L^p$. On the other hand, every $f \in L^p$ lies in some L^p_K (see Section 3.14).

(v) The Poisson kernel for $\mathbf{R}^{n+1}_+$ is

$$P_y(x) = \frac{c_n y}{\left(|x|^2 + y^2\right)^{(n+1)/2}} \qquad (x \in \mathbf{R}^n,\ y > 0)$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-(n+1)/2}$ is so chosen that $\|P_y\|_1 = 1$ for $0 < y < \infty$.

$K_y(x) = (P_y * K)(x)$ is the harmonic extension of K to R^{n+1}_+.

Note that h^p_K is the space of all Poisson integrals $P[f]$ of functions $f \in L^p_K$. These are the *Dirichlet-type spaces* of our title. Thus, saying that $u \in h^p_K$ means that

$$u(x, y) = P[f](x, y) = \left(P_y * f\right)(x) = \left(K_y * F\right)(x)$$

for some $f \in L^p_K$, $F \in L^p$, and all $(x, y) \in \mathbf{R}^{n+1}_+$.

We norm h^p_K by setting

$$\|u\|_{K,p} = \|f\|_{K,p}$$

if $u = P[f]$, $f \in L^p_K$.

(vi) If K is a kernel, $1 \le p < \infty$, $pq = p + q$, $0 < \beta < \infty$, and $x_0 \in \mathbf{R}^n$, we define the *approach region*

$$\Omega^p_{K,\beta}(x_0) = \left\{(x, y) \in \mathbf{R}^{n+1}_+ : |x - x_0|^{n/p}\|K_y\|_q < \beta\right\}$$

and the associated maximal function

$$\left(\mathfrak{M}_{K,p,\beta}f\right)(x_0) = \sup\{|u(x, y)| : (x, y) \in \Omega^p_{K,\beta}\}$$

where $u = P[f]$.

These approach regions are suggested by Theorem 2.5. Some of their geometric properties are described in Proposition 2.11. When K is a Bessel kernel, they are closely related to the regions $\mathcal{A}$ and $\mathcal{E}$ defined above; see Theorem 5.5.

1.3. *The Main Result*. This is Theorem 3.8. It asserts, for $1 < p < \infty$, that *the maximal function operators* $\mathfrak{M}_{K,p,\beta}$ *are bounded from* L^p_K *to* L^p: *For every kernel* K,

$$\|\mathfrak{M}_{K,p,\beta}f\|_p \le A\|f\|_{K,p}$$

where $A = A(K, p, \beta) < \infty$.

The proof uses the corresponding weak-type inequality (which is much easier, holds also when $p = 1$, and suffices to give pointwise convergence results; see Theorems 2.7, 2.9) combined with a strong capacitary estimate due to Hansson (Theorem 3.7) and with the Hardy-Littlewood maximal theorem.

The L^p-boundedness of $\mathfrak{M}_{K,p,\beta}$ is needed to obtain

(a) capacitary results about factored kernels $K = H * G$ (Theorems 3.12, 3.13) and

(b) a simple geometric condition which implies that a measure is a Carleson measure for h_K^p (Theorem 4.5).

If we combine Theorem 3.8 with Proposition 2.15, we obtain examples of maximal function operators that are bounded on a *closed* interval of p's, and that are of strong type whenever they are of weak type. Theorem 3.8 can thus not be proved by dominating $\mathfrak{M}_{K,p,\beta}$ by any of the standard maximal functions which are only of weak type at the endpoint of their range of boundedness.

II. Inequalities of weak type and tangential convergence

A simple convolution inequality (Theorem 2.3) leads in a natural way to the discovery of the approach regions $\Omega_{K,\beta}^p$ that are associated to kernels K and exponents p. The corresponding maximal function operators $\mathfrak{M}_{K,p,\beta}$ turn out to be of weak type (p, p) (Theorem 2.7). Consequently, every $u \in h_K^p$ has Ω_K^p-limits at almost all points of $\mathbf{R}^n$ (Theorem 2.9). These results are shown to be best possible in the following sense: for fixed p, no larger regions work (Proposition 2.13) and, at least for a certain class of kernels K, $\mathfrak{M}_{K,p,\beta}$ is not of weak type (p_1, p_1) for any $p_1 < p$ (Proposition 2.15).

2.1 *Definition.* If $1 \le p < \infty$, $F \in L^p$, and $x \in \mathbf{R}^n$, define

$$(M_pF)(x) = \sup_{0<r<\infty} \left\{ \frac{1}{m(rB)} \int_{B(x,r)} |F|^p \, dm \right\}^{1/p} \tag{1}$$

where $B(x, r)$ is the open ball with center x and radius r, and $rB = B(0, r)$. Note that

$$M_pF = (M_1 |F|^p)^{1/p}. \tag{2}$$

It is well known that M_1 is of weak type $(1, 1)$ [18, p. 5]. Therefore M_p is of weak type (p, p): There is a constant $A_p < \infty$ such that

$$m\{M_pF > \lambda\} \le A_p \lambda^{-p} \|F\|_p^p \tag{3}$$

for every $F \in L^p$.

2.2 Lemma. *If $F \in L^1$, and $g \geq 0$ is radial and decreasing, then*

$$\int_{\mathbf{R}^n} |F| g \, dm \leq (M_1 F)(0) \int_{\mathbf{R}^n} g \, dm.$$

Proof. This is obvious if $g = g_r$, the characteristic function of rB, hence for $g = \Sigma c_r g_r$, where the c_r are positive constants. The general case follows from the monotope convergence theorem. (See [18; p. 63].)

2.3. Theorem. *Suppose $1 \leq p < \infty$. There is a constant $A = A(n, p) < \infty$ such that*

$$|(K * F)(x)| \leq A\{(M_p F)(x_0) |x - x_0|^{n/p} \|K\|_q + (M_1 F)(x_0) \|K\|_1\}$$

whenever $F \in L^p$, K is a nonnegative radial decreasing function on $\mathbf{R}^n$, $x_0 \in \mathbf{R}^n$, and $x \in \mathbf{R}^n$

Proof. Take $x_0 = 0$, without loss of generality. Fix x. Then

$$|(K * F)(x)| \leq \int_{\mathbf{R}^n} K(x - t) |F(t)| \, dt = I + II,$$

where I and II are integrals over $\{|t| \leq 2|x|\}$ and $\{|t| > 2|x|\}$, respectively. Since

$$\int_{rB} |F|^p \, dm \leq r^n m(B) [(M_p F)(0)]^p$$

holds for all $r > 0$, Hölder's inequality shows that

$$I \leq [2^n |x|^n m(B)]^{1/p} (M_p F)(0) \|K\|_q.$$

In II, $|x - t| \geq \frac{1}{2}|t|$, hence $K(x - t) \leq K(\frac{1}{2}t)$, so that

$$II \leq \int_{\mathbf{R}^n} K(\tfrac{1}{2}t) |F(t)| \, dt \leq 2^n \|K\|_1 (M_1 F)(0),$$

by Lemma 2.2.

These two estimates prove the theorem.

2.4. *Remarks.* (a) Observe that the convolution of two nonnegative radial decreasing functions is (obviously) radial, and that it is also decreasing. This is clear for the convolution of the characteristic functions of two balls centered at 0, hence for linear combinations with positive coefficients of such characteristic functions, hence in general, by the monotone convergence theorem.

(b) If K is a kernel, and $K_y = K * P_y$, as in Section 1.2 (v), Remark (a) shows that Theorem 2.3 holds with K_y in place of K.

By Hölder's inequality, $M_1 F \leq M_p F$.

By Fubini's theorem, $\|K_y\|_1 = \|K\|_1 \|P_y\|_1 = \|K\|_1$.
Hence Theorem 2.3 implies the following:

2.5. THEOREM. *If* $1 \le p < \infty$, $F \in L^p$, *and* u *is defined in* $\mathbf{R}^{n+1}_+$ *by*

$$u(x, y) = (P_y * K * F)(x) = (K_y * F)(x),$$

then

$$|u(x, y)| \le A(M_p F)(x_0)\left[\|K\|_1 + |x - x_0|^{n/p} \|K_y\|_q\right]$$

for all $x_0 \in \mathbf{R}^n$, $x \in \mathbf{R}^n$.

2.6. *Remark*. For fixed $x_0 \in \mathbf{R}^n$, Theorem 2.5 shows that any bound on

$$(1) \qquad |x - x_0|^{n/p} \|K_y\|_q$$

gives a bound on $u(x, y)$. Letting $\Omega^p_{K,\beta}(x_0)$ consist of all $(x, y) \in \mathbf{R}^{n+1}_+$ for which (1) is less than β, we obtain the approach regions defined in Section 1.2 (vi).

Since M_p is of weak type (p, p), Theorem 2.5 shows that the same is true of the maximal function operators $\mathfrak{M}_{K,p,\beta}$ associated to $\Omega^p_{K,\beta}$ as in Section 1.2 (vi):

2.7. THEOREM. *There is a constant* $A = A(n, K, p, \beta) < \infty$ *such that*

$$(1) \qquad m\{\mathfrak{M}_{K,p,\beta} f > \lambda\} \le A\lambda^{-p} \|f\|^p_{K,p}$$

for all $f \in L^p_K$ *and all* $\lambda \in (0, \infty)$.

Recall that $\|f\|_{K,p} = \inf\{\|F\|_p : f = K * F\}$.

Proof. If $u = P[f]$ and $f = K * F$, Theorem 2.5 shows that

$$(2) \qquad |u(x, y)| \le A(M_p F)(x_0)(1 + \beta)$$

in $\Omega^p_{K,\beta}(x_0)$. Thus

$$(3) \qquad \mathfrak{M}_{K,p,\beta} f \le A(1 + \beta) M_p F.$$

Since M_p is of weak type (p, p), (3) implies (1).
As usual, this weak-type estimate leads to a convergence theorem:

2.8. *Definition*. A function u with domain $\mathbf{R}^{n+1}_+$ is said to have Ω^p_K-limit L at a point $x_0 \in \mathbf{R}^n$ if it is true for every $\beta < \infty$ that $u(x, y) \to L$ as $(x, y) \to (x_0, 0)$ within $\Omega^p_{K,\beta}$.

2.9. THEOREM. *If* $1 \le p < \infty$, $f \in L^p_K$, *and* $u = P[f]$, *then, for almost all* $x_0 \in \mathbf{R}^n$, *the* Ω^p_K*-limit of* u *exists at* x_0 *and equals* $f(x_0)$.

Since the derivation of 2.9 from 2.7 is perfectly standard, we omit the details.

2.10. The content of Theorems 2.7 and 2.9 will become clearer as soon as we describe the shape of the regions $\Omega^p_{K,\beta}(x_0)$. For simplicity, we take $x_0 = 0$, and just write $\Omega^p_{K,\beta}$. For fixed $y > 0$, Definition 1.2 (vi) shows that the set of all $x \in \mathbf{R}^n$ for which $(x, y) \in \Omega^p_{K,\beta}$ is an open ball, centered at 0, whose radius is $\beta^{p/n} r(y)$, where

$$r(y) = \|K_y\|_q^{-p/n}.$$

Thus $\Omega^p_{K,\beta}$ is determined by $r(y)$.

2.11. PROPOSITION. *If $1 \le p < \infty$ and $K \notin L^q$, then*
(a) *$r(y) \to 0$ as $y \to 0$,*
(b) *$r(y_1) \le r(y_2)$ when $y_1 \le y_2$,*
(c) *$r(y)/y \to \infty$ as $y \to 0$,*
(d) *there are constants γ_1, γ_2 such that*

$$\frac{r(y)}{y} > \gamma_1 > 0 \qquad (0 < y < \infty)$$

and

$$\frac{r(y)}{y} < \gamma_2 < \infty \qquad (1 \le y < \infty).$$

The assumption $K \notin L^q$ is a natural one. For if $K \in L^q$ then every $f \in L^p_K$ is continuous, so that $P[f]$ is continuous on the closure of $\mathbf{R}^{n+1}_+$.

The geometric meaning of (c) is that $\Omega^p_{K,1}$ is *tangential*; (d) says that $\Omega^p_{K,1}$ contains the cone $\{|x| < \gamma_1 y\}$ and that the part of $\Omega^p_{K,1}$ where $y > 1$ lies in the cone $\{|x| < \gamma_2 y\}$.

Proof. Since $K \notin L^q$ and since $K_y(x)$ converges to $K(x)$ almost everywhere as $y \to 0$, Fatou's lemma shows that $\|K_y\|_q \to \infty$ as $y \to 0$. This gives (a).

If $y_2 = y_1 + y$, $y > 0$, $y_1 > 0$, then $P_{y_2} = P_{y_1} * P_y$, so that

$$\|K_{y_2}\|_q = \|P_y * K_{y_1}\|_q \le \|P_y\|_1 \|K_{y_1}\|_q = \|K_{y_1}\|_q. \tag{1}$$

This proves (b).

The proofs of (c) and (d) use the relation

$$[y/r(y)]^{n/p} = y^{n/p}\|K_y\|_q = y^{n/p}\|K * P_y\|_q. \tag{2}$$

Choose $\varepsilon > 0$. Split K into $K = H + G$, where $\|H\|_1 < \varepsilon$, $G \in L^\infty$, $\|G\|_1 < 1$. Then $\|G * P_y\|_\infty \le \|G\|_\infty$, $\|G * P_y\|_1 < 1$, *so that* $\|G * P_y\|_q$ is bounded. On the other hand,

$$\|H * P_y\|_q < \varepsilon \|P_y\|_q = \varepsilon y^{-n/p}\|P_1\|_q. \tag{3}$$

Thus $y/r(y) \to 0$ as $y \to 0$. This proves (c).

Similarly, $[y/r(y)]^{n/p} \leq \|P_1\|_q$ for all y. This gives the first half of (d). For the second half, note that $P_{2y} = P_y * P_y$ gives $K_{2y} = K_y * P_y$. Since $y^n P_{2y}(x)$ is an increasing function of y, for each x, it follows, when $y \geq 1$, that

(4) $$K_2(0) \leq y^n K_{2y}(0) \leq y^n \|K_y\|_q \|P_y\|_p = y^{n/p} \|K_y\|_q \|P_1\|_p.$$

Thus $[y/r(y)]^{n/p} \geq K_2(0)/\|P_1\|_p$. This proves (d).

2.12. Our next proposition will show that Theorem 2.7 is optimal with regard to the size of the approach regions. To formulate this precisely, recall that $\Omega^p_{K,\beta}(x_0)$ consists of all (x, y) such that

(1) $$|x - x_0| < \beta^{p/n} r(y)$$

and compare this with another region

(2) $$\Omega(x_0) = \{(x, y): |x - x_0| < \rho(y)\},$$

where ρ is some positive continuous function. Let

(3) $$(\mathfrak{N} f)(x_0) = \sup\{|u(x, y)| : (x, y) \in \Omega(x_0)\}$$

where, as usual, $u = P[f]$.

2.13. Proposition. *If* $1 \leq p < \infty$ *and* $\mathfrak{N}$ *is of weak type* (p, p), *then* $\rho(y)/r(y)$ *is bounded on* $0 < y < \infty$.

Proof. The hypothesis about $\mathfrak{N}$ is, explicitly, that there is a constant $A_0 < \infty$ such that

(1) $$m\{\mathfrak{N} f \geq \lambda\} \leq A_0 \lambda^{-p} \|f\|^p_{K,p}$$

for all $f \in L^p_K$ and all $\lambda \in (0, \infty)$.

Pick $F \in L^p$, $F \geq 0$, $\|F\|_p = 1$, let $f = K * F$, $u = P[f]$. If $|x| < \rho(y)$ then $(0, y) \in \Omega(x)$, so that $u(0, y) \leq (\mathfrak{N} f)(x)$. The set $\{\mathfrak{N} f \geq u(0, y)\}$ contains therefore a ball of radius $\rho(y)$. Hence

(2) $$\rho(y)^n m(B) \leq m\{\mathfrak{N} f \geq u(0, y)\} \leq A_0 u(0, y)^{-p}$$

by (1), since $\|f\|_{K,p} \leq 1$. Thus

(3) $$A\rho(y)^{-n/p} \geq u(0, y) = (K_y * F)(0).$$

Taking the supremum over all admissible F leads from (3) to

(4) $$A\rho(y)^{-n/p} \geq \|K_y\|_q = r(y)^{-n/p};$$

hence $\rho(y) \leq A^{p/n} r(y)$.

2.14. Let us now see how $r(y)$ is affected by changes in p; for emphasis, write

$$r_p(y) = \|K_y\|_q^{-p/n} \tag{1}$$

in place of $r(y)$.

If $1 < q < \infty$, then

$$\|K_y\|_q^p = \left\{\int_{R^n} K_y^{q-1} K_y \, dm\right\}^{1/(q-1)}. \tag{2}$$

Since $K_y \, dm$ is a probability measure, the right side of (2) is an increasing function of q.

It follows that $r_p(y)$ is an increasing function of p.

Hypothesis (1) of the following proposition is stronger. We do not know whether it holds for every K. It does hold for the Bessel kernels; see Proposition 5.3.

2.15. PROPOSITION. *Assume that $p_1 < p$ and that*

$$\limsup_{y\to 0} \frac{r_p(y)}{r_{p_1}(y)} = \infty. \tag{1}$$

Then $\mathfrak{M}_{K,p,\beta}$ is not of weak type (p_1, p_1).

This shows another direction in which Theorem 2.7 is optimal.

Proof. Apply Proposition 2.13, with r_{p_1}, r_p, $\mathfrak{M}_{K,p,\beta}$ in place of r, ρ, $\mathfrak{M}$.

2.16. *Remark.* If $\tilde{K}$ is any nonnegative measurable function that is majorized by some radial decreasing $K \in L^1$, then Theorems 2.7 and 2.9 hold for the potentials $f = \tilde{K} * F$ in place of $K * F$, and for the approach regions $\Omega_{K,\beta}^p$, simply because $\tilde{K} * |F| \le K * |F|$.

2.17. *Potentials of measures.* In all of the preceding work, L^1 could have been replaced without any difficulty by the larger space of all complex Borel measures μ on $\mathbf{R}^n$, since, as is well known, the maximal function

$$(M\mu)(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{m(B(x,r))} \tag{1}$$

satisfies the weak-type $(1, 1)$ inequality

$$m\{M\mu > \lambda\} \le \frac{A}{\lambda}\|\mu\| \qquad (0 < \lambda < \infty). \tag{2}$$

Here $\|\mu\| = |\mu|(\mathbf{R}^n)$, where $|\mu|$ is the total variation measure associated to μ.

Theorems 2.5 and 2.7 extend therefore to measures: If

(3) $$f(x) = (K * \mu)(x) = \int_{R^n} K(x - t)\, d\mu(t)$$

and $u = P[f]$, *then*

(4) $$|u(x, y)| \leq A(M\mu)(x_0)\left[1 + |x - x_0|\, \|K_y\|_\infty\right]$$

and the Ω_K^1-limit of u *exists at almost every* $x_0 \in \mathbf{R}^n$, *where*

(5) $$\Omega_{K,\beta}^1(x_0) = \left\{|x - x_0|^{1/n} \|K_y\|_\infty < \beta\right\}.$$

2.18 *Remark*. Suppose that Ω is as in Section 2.12, that $\mathcal{N}$ is the corresponding maximal function, and that $\rho(y)/r(y)$ is unbounded as $y \to 0$. Proposition 2.13 shows then that $\mathcal{N}$ is not of weak type (p, p). A version of Stein's theorem, due to Sawyer [7; p. 13], implies therefore that there exists $f \in L_K^p$ such that $\mathcal{N}f = \infty$ on a set of positive measure. In particular, $P[f]$ does not have Ω-limits almost everywhere.

III. Capacities, and inequalities of strong type

To every potential space L_K^p corresponds a capacity $C_{K,p}$. These capacities are crucially used in our proof of the main result of this section (Theorem 3.8), although the statement of that theorem does not mention them. The theorem states that the maximal function operators $\mathcal{M}_{K,p,\beta}$ are of strong type (p, p) when $p > 1$, not merely of weak type, as asserted by Theorem 2.7. Hansson's recent strong type capacitary inequality (Theorem 3.7) is what enables us to pass from 2.7 to 3.8.

We also obtain a modified version of Theorem 2.9, for "factored" kernels $K = H * G$, where both H and G are kernels (Theorem 3.13). In that case, the approach regions Ω_G are narrower than the Ω_K's, simply because $\|K\|_q \leq \|G\|_q$. To compensate for this, the set on which a function $u \in h_K^p$ fails to have Ω_G^p-limits turns out to be not just of measure zero (as in Theorem 2.9) but to have (H, p)-capacity zero.

We now define these capacities, for $1 < p < \infty$.

3.1. *Definition*. Suppose K is a kernel, as in Section 1.2, $1 < p < \infty$, and $E \subset \mathbf{R}^n$. Define $T(K, p, E)$ as the set of all $F \in L^p$ such that $F \geq 0$ on $\mathbf{R}^n$ and

(1) $$(K * F)(x) \geq 1 \quad \text{for every } x \in E.$$

(Note that $(K * F)(x)$ exists, in $[0, \infty]$, for *every* $x \in \mathbf{R}^n$, since $K \geq 0$ and $F \geq 0$.) The letter T stands for "test function".

Following Meyers [14], the (K, p)-*capacity of* E is defined to be

(2) $$C_{K,p}(E) = \inf\{\|F\|_p^p : F \in T(K, p, E)\},$$

with the understanding that the inf of the empty set is $+\infty$.

3.2. Here are some simple properties of these capacities:

(a) $C_{K,p}$ is a translation-invariant, monotone, countably subadditive set function.

(b) Let χ_E denote the characteristic function of a set $E \subset \mathbf{R}^n$. Then $\chi_E \leq K * F$ for every $F \in T(K, p, E)$. Since $\|K * F\|_p \leq \|F\|_p$, we conclude:

If $C_{K,p}(E) = 0$ *then* $m(E) = 0$.

(c) Pick $F_0 \in L^p$, $F_0 \geq 0$, let $E = \{K * F_0 = \infty\}$. Then $\delta F_0 \in T(K, p, E)$ for every $\delta > 0$. Letting $\delta \to 0$, we see that $C_{K,p}(E) = 0$.

Consequently, if $F \in L^p$, then $(K * F)(x)$ exists, as a Lebesgue integral, except possibly on a set E with $C_{K,p}(E) = 0$.

Briefly: *The functions* $f = K * F \in L_K^p$ *are defined* $C_{K,p}$-almost everywhere.

(d) If $K \notin L^q$ it follows from (c) that every countable set has (K, p)-capacity 0.

(e) Let $K = H * G$, both H and G being kernels. If $F \in T(K, p, E)$ then obviously $G * F \in T(H, p, E)$. Since $\|G * F\|_p < \|F\|_p$, we conclude that

$$C_{H,p}(E) \leq C_{H*G,p}(E) \quad \text{for every } E \subset \mathbf{R}^n.$$

Some further notations are now needed:

3.3. *Definitions.* (a) If $x_0 \in \mathbf{R}^n$, then

$$\Gamma(x_0) = \{(x, y) \in \mathbf{R}_+^{n+1} : |x - x_0| < y\}.$$

(b) If $E \subset R^n$, then

$$S(E) = \mathbf{R}_+^{n+1} \setminus \bigcup_{x \notin E} \Gamma(x).$$

(c) If $u = P[f]$ and $x_0 \in \mathbf{R}^n$, then

$$(Nf)(x_0) = \sup\{|u(x, y)| : (x, y) \in \Gamma(x_0)\}.$$

(d) If $W \subset \mathbf{R}_+^{n+1}$ then $J_{K,\beta}^p(W)$ is the set of all $x \in \mathbf{R}^n$ for which W intersects $\Omega_{K,\beta}^p(x)$.

The sets $\Gamma(x_0)$ are cones. When $n = 1$, $S(E)$ is a union of triangles, one over each maximal segment contained in E. These triangles play the same role as the "Carleson squares" on which the original definition of Carleson measures was based.

When there is no need to specify p, K, β, we shall sometimes write J and Ω in place of $J_{K,\beta}^p$ and $\Omega_{K,\beta}^p$.

Nf is the nontangential maximal function to which the Hardy-Littlewood maximal theorem [18; p. 197] refers.

3.4 Lemma. *There is a constant $b > 0$, depending only on n, with the following property:*

If $f \geq 1$ on E, $f \geq 0$ on $\mathbf{R}^n$, $u = P[f]$, and $(x, y) \in S(E)$, then

$$u(x, y) > b.$$

Proof. If $\xi \in R^n$, $|\xi - x| < y$, then $(x, y) \in \Gamma(\xi)$; hence $\xi \in E$, by the definition of $S(E)$. Thus E contains the open ball with center x and radius y. If we set $x - \xi = yt$, it follows that

$$u(x, y) \geq \int_E P_y(x - \xi)\, d\xi \geq \int_B P_y(yt) y^n\, dt = \int_B P_1\, dm$$

where B is the unit ball of $\mathbf{R}^n$.

3.5. Lemma. *For $f \in L^1 + L^\infty$, and $0 < \lambda < \infty$,*

$$\{\mathfrak{M} f > \lambda\} \subset J(S(\{Nf > \lambda\})),$$

where $\mathfrak{M} = \mathfrak{M}_{K, p, \beta}$, $J = J^p_{K, \beta}$.

Proof. Assume x_0 satisfies $(\mathfrak{M} f)(x_0) > \lambda$. Then there is a point $(x, y) \in \Omega(x_0)$ at which $|u(x, y)| > \lambda$.

Next, let $E = \{Nf > \lambda\}$. If $\xi \in \mathbf{R}^n \setminus E$, then $(Nf)(\xi) \leq \lambda$, hence $|u| \leq \lambda$ in $\Gamma(\xi)$, so that $(x, y) \notin \Gamma(\xi)$. Consequently, $(x, y) \in S(E)$.

It follows that $\Omega(x_0)$ intersects $S(E)$. This says that $x_0 \in J(S(E))$.

These two lemmas, combined with Theorem 2.7, lead to some lower estimates for capacities.

3.6. Proposition. *There is a constant $A = A(K, p, \beta) < \infty$ such that*

$$m\left(J^p_{K,\beta}(S(E))\right) \leq A C_{K,p}(E) \tag{1}$$

for every $E \subset \mathbf{R}^n$, and

$$m\left\{\mathfrak{M}_{K,p,\beta} f > \lambda\right\} \leq A C_{K,p}\{Nf > \lambda\} \tag{2}$$

for every $f \in L^1 + L^\infty$.

Proof. Let $u = P[K * F]$, where $F \in T(K, p, E)$. By Lemma 3.4, $u > b$ on $S(E)$. If $x \in J(S(E))$ then $\Omega(x)$ intersects $S(E)$; hence $\mathfrak{M}(K * F)(x) > b$. In other words,

$$J(S(E)) \subset \{\mathfrak{M}(K * F) > b\}. \tag{3}$$

Our earlier weak-type estimate (Theorem 2.7) shows therefore that

$$m(J(S(E))) \le Ab^{-p} \| F \|_p^p. \tag{4}$$

If we replace the right side of (4) by its infimum over all $F \in T(K, p, E)$, we obtain (1).

Lemma 3.5, with $\{Nf > \lambda\} = E$, shows that (2) follows from (1).

3.7. HANSSON'S THEOREM. *For* $1 < p < \infty$, *there is a constant* $A = A(K, p) < \infty$ *such that*

$$\int_0^\infty C_{K,p}\{K * F > \lambda\}\, d(\lambda^p) \le A \| F \|_p^p$$

for every $F \in L^p$, $F \ge 0$.

We refer to [8; pp. 93–95] for the proof; see also [1], [13].

Now we can prove that the maximal function operators $\mathfrak{M}_{K,p,\beta}$ are of strong type (p, p) if $p > 1$.

3.8. THEOREM. *If* $p > 1$, *there is a constant* $A = A(K, p, \beta) < \infty$ *such that*

$$\| \mathfrak{M}_{K,p,\beta} f \|_p \le A \| f \|_{K,p} \tag{1}$$

for every $f \in L^p_K$. *Equivalently,*

$$\int_{\mathbf{R}^n} | \mathfrak{M}_{K,p,\beta}(K * F) |^p\, dm \le A^p \int_{\mathbf{R}^n} | F |^p\, dm \tag{2}$$

for every $F \in L^p$.

Proof. The well-known identity

$$\int_{\mathbf{R}^n} \psi^p\, dm = \int_0^\infty m\{\psi > \lambda\}\, d(\lambda^p)$$

which is valid for all measurable $\psi \ge 0$, when applied to $\psi = \mathfrak{M}_{K,p,\beta}(K * F)$, shows that the left side of (2) equals

$$\begin{aligned}\int_0^\infty m\{\mathfrak{M}(K * F) > \lambda\}\, d(\lambda^p) &\le A \int_0^\infty C_{K,p}\{N(K * F) > \lambda\}\, d(\lambda^p) \\ &\le A \int_0^\infty C_{K,p}\{K * NF > \lambda\}\, d(\lambda^p) \\ &\le A \| NF \|_p^p \le A \| F \|_p^p,\end{aligned}$$

which proves (2). In this chain of inequalities, we first used 3.6 (2), then the obvious fact that $N(K * f) \le K * NF$, then Hansson's theorem (with NF in place of F), and finally the Hardy-Littlewood maximal theorem.

3.9. *Remark.* Theorem 3.8 does not hold for $p = 1$. To avoid duplication, we postpone the proof of this to Theorem 4.8.

We now turn to the study of factored kernels $K = H * G$, where both H and G are kernels as defined in Section 1.2 (iii). Our objective is Theorem 3.13, which complements Theorem 2.9.

3.10. LEMMA. *If* $K = H * G$ *and* $1 < p < \infty$, *then*

$$C_{H,p}\{J^p_{G,\beta}(S(E))\} \leq AC_{K,p}(E), \tag{1}$$

where $A = A(H, G, p, \beta) < \infty$.

Proof. Put $V = J^p_{G,\beta}(S(E))$. We have to prove that

$$C_{H,p}(V) \leq A\|F\|^p_p \tag{2}$$

for every test function $F \in T(K, p, E)$. If $f = K * F$, Lemma 3.4 shows that $P[f] > b$ on $S(E)$. If we write $\mathfrak{M}$ for $\mathfrak{M}_{G,p,\beta}$ it follows, for every $x \in V$, that

$$b < (\mathfrak{M}f)(x) = \mathfrak{M}(H * G * F)(x) \leq H * \mathfrak{M}(G * F)(x).$$

Thus $b^{-1}\mathfrak{M}(G * F) \in T(H, p, V)$, so that

$$C_{H,p}(V) \leq b^{-p}\|\mathfrak{M}(G * F)\|^p_p. \tag{3}$$

Theorem 3.8, applied with G in place of K, shows that (3) implies (2).

Note that this proof used the L^p-boundedness of the operator $\mathfrak{M}$. The weak-type result 2.7 would not have sufficed.

3.11. PROPOSITION. *If* $K = H * G$, $1 < p < \infty$, *and* $f \in L^p_K$, *then*

$$C_{H,p}\{\mathfrak{M}_{G,p,\beta}f > \lambda\} \leq AC_{K,p}\{Nf > \lambda\}.$$

Proof. Combine Lemmas 3.5 and 3.10.

We now come to a modified version of Theorem 3.8.

3.12. THEOREM. *If* $K = H * G$, $1 < p < \infty$, *and* $f \in L^p_K$, *then*

$$\int_0^\infty C_{H,p}\{\mathfrak{M}_{G,p,\beta}f > \lambda\}\, d(\lambda^p) \leq A\|f\|^p_{K,p} \tag{1}$$

where $A = A(H, G, p, \beta) < \infty$. *Consequently*,

$$C_{H,p}\{\mathfrak{M}_{G,p,\beta}f > \lambda\} \leq A\lambda^{-p}\|f\|^p_{K,p}. \tag{2}$$

Proof. Since $f = K * F$ for some $F \in L^p$, we have $Nf \leq K * NF$. Proposition 3.11 implies therefore that the left side of (1) is at most

$$A\int_0^\infty C_{K,p}\{K * NF > \lambda\}\, d(\lambda^p) \leq A\|NF\|^p_p \leq A\|F\|^p_p$$

by the theorems of Hansson and of Hardy-Littlewood, as in the proof of Theorem 3.8. This proves (1).

Denote the left side of (2) by $g(\lambda)$. Then g is a decreasing function, so that

$$\int_0^\infty g(t)\,d(t^p) \geq \int_0^\lambda g(t)\,d(t^p) \geq g(\lambda)\lambda^p$$

Hence (1) implies (2).

Part (ii) of the following theorem is our principal reason for looking at the factored kernels $K = H * G$. Part (i) is classical (see [11], for example, for the case of the unit circle) and is included for comparison.

3.13. THEOREM. *Suppose* $K = H * G$, $1 < p < \infty$, $f \in L_K^p$, *and* $u = P[f]$.

(i) *There is a set* $E' \subset \mathbf{R}^n$, *with* $C_{K,p}(E') = 0$, *such that the nontangential limit of* u *exists and equals* $f(x)$ *at every* $x \in \mathbf{R}^n \setminus E'$.

(ii) *There is a set* $E'' \subset \mathbf{R}^n$, *with* $C_{H,p}(E'') = 0$, *such that the* Ω_G^p*-limit of* u *exists and equals* $f(x)$ *at every* $x \in \mathbf{R}^n \setminus E''$. ("$\Omega_G^p$*-limit*" *is defined in Section* 2.8.)

Proof. There exists $F \in L^p$ so that $f = K * F$. As explained in Section 3.2 (c), $f(x)$ is then unambiguously defined by this convolution outside some set Y with $C_{K,p}(Y) = 0$. Hence (see Section 3.2 (e)) $C_{H,p}(Y) = 0$ as well.

The pointwise convergence $C_{H,p}$-almost everywhere follows from the corresponding weak-type result 3.12 (2) by standard arguments.

3.14. *Remark*. The obvious inclusion $L_K^p \subset L^p$ has a partial converse which shows that every $f \in L^p$ lies in some potential space L_K^p:

($*$) *To every* $f \in L^p$ *corresponds a kernel* K *and an* $F \in L^p$ *such that* $f = K * F$.

This follows from Hewitt's proof [10] (where $f = K * F$ is obtained with $K \geq 0$, $K \in L^1$) of his generalization of Cohen's factorization theorem [4], if we take care to make K radial and decreasing. Since the proof works for every approximate identity one can, for instance, find a K (depending on f, of course) of the form

$$K(x) = \sum_{j=1}^{\infty} \varepsilon_j \exp\left(-t_j |x|^2\right)$$

where $t_j \to \infty$, $\varepsilon_j > 0$.

Every $f \in L^p$ *has thus tangential* Ω_K^p*-limits almost everywhere, for a suitable* K, *and the corresponding maximal functions* $\mathfrak{M}_{K,p,\beta} f$ *are in* L^p.

IV. Carleson measures

The following definition is an obvious analogue of the familiar one in which H^p-spaces occur in place of h_K^p. We recall that h_K^p consists of the Poisson integrals

u of potentials $K * F$, and that h_K^p is normed by

$$\|u\|_{K,p} = \inf\{\|F\|_p : u = P[K * F]\}.$$

4.1. *Definition.* A positive Borel measure μ on $\mathbf{R}_+^{n+1}$ is said to be a *Carleson measure for* h_K^p if there is a constant $A < \infty$ such that

$$\int_{\mathbf{R}_+^{n+1}} |u|^p \, d\mu \leq A \|u\|_{K,p}^p$$

for every $u \in h_K^p$.

Stegenga [17] obtained the following *capacitary* characterization of these Carleson measures.

4.2. THEOREM. *Assume* $1 < p < \infty$. *A positive Borel measure* μ *on* $\mathbf{R}_+^{n+1}$ *is a Carleson measure for* h_K^p *if and only if there is a constant* $A < \infty$ *such that*

$$\mu(S(E)) \leq A C_{K,p}(E)$$

for every open set $E \subset \mathbf{R}^n$.

Actually, Stegenga proved this only when $n = 1$, for certain kernels K. But since Hansson's theorem (3.7) is now available, the proof given on pp. 123–4 of [17] yields the theorem as stated here.

Theorem 4.5 below exhibits a *geometric* condition which implies that μ is a Carleson measure for h_K^p. This sufficient condition is, however, not necessary. Theorem 4.10 shows this.

Theorem 4.5 uses the following sets.

4.3. *Definition.* For $E \subset \mathbf{R}^n$, and K, p, β as before,

$$Q_{K,\beta}^p(E) = \mathbf{R}_+^{n+1} \setminus \bigcup_{x \notin E} \Omega_{K,\beta}^p(x).$$

Note that $Q(E)$ is related to Ω in the same way in which $S(E)$ is related to Γ in Definition 3.3.

Here are some properties of these sets; to simplify notation, we omit K, p, β, and take $\beta = 1$.

(a) *If* $E_1 \subset E_2$ *then* $Q(E_1) \subset Q(E_2)$. This is obvious.

(b) *If* $r(y)$ *is the radius defined in Section* 2.10, *then* $(x, y) \in Q(E)$ *if and only if* $|x - x'| \geq r(y)$ *for every* $x' \notin E$. This is also clear from the definitions.

(c) *If* E *is open,* $(x, y) \in Q(E)$, *and* $B(x)$ *is the largest open ball with center* x *that lies in* E, *then* $(x, y) \in Q(B(x))$.

To see this, let ρ be the radius of $B(x)$, assume $\rho < \infty$, without loss of generality. Then there is an $x' \notin E$ with $|x' - x| = \rho$. Since $(x, y) \in Q(E)$ and

$x' \notin E$, (b) gives

$$r(y) \leq |x' - x| = \rho.$$

For every $x'' \notin B(x)$, $|x'' - x| \geq \rho \geq r(y)$. Hence, using (b) again, $(x, y) \in Q(B(x))$.

4.4. PROPOSITION. *If $f \in L^p_K$, $u = P[f]$, then*

$$\{|u| > \lambda\} \subset Q\{\mathfrak{M}f > \lambda\}$$

where $\mathfrak{M} = \mathfrak{M}_{K,p,\beta}$, $Q = Q^p_{K,\beta}$, $0 < \lambda < \infty$.

Proof. Pick (x, y) so that $|u(x, y)| > \lambda$. Let $E = \{\mathfrak{M}f > \lambda\}$. If $x_1 \notin E$, then $(\mathfrak{M}f)(x_1) \leq \lambda$; hence $(x, y) \notin \Omega(x_1)$. This says that $(x, y) \in Q(E)$.

4.5. THEOREM. *Let μ be a positive Borel measure on $\mathbf{R}^{n+1}_+$. If $p > 1$, and if for some $\beta > 0$ there is a constant $A < \infty$ such that*

(1) $$\mu\big(Q^p_{K,\beta}(B)\big) \leq Am(B)$$

for every open ball $B \subset \mathbf{R}^n$, then μ is a Carleson measure for h^p_K.

Proof. Let $E \subset \mathbf{R}^n$ be open, $E \neq \mathbf{R}^n$. For each $x \in E$, let $B(x)$ be as in Section 4.3 (c). A standard covering lemma shows that there is an at most countable set $\{x_i\}$ in E so that the balls $B(x_i)$ are pairwise disjoint, and so that every $B(x)$ lies in some B_i, where B_i is the open ball with center x_i whose radius in 10 times the radius of $B(x_i)$. Then 4.3 (c) gives the first inclusion in

(2) $$Q(E) \subset \bigcup_{x \in E} Q(B(x)) \subset \bigcup_i Q(B_i).$$

The second follows from 4.3 (a) and the covering lemma. By (2) and (1),

(3) $$\begin{aligned} \mu(Q(E)) &\leq \sum m\big(Q(B_i)\big) \leq A \sum m(B_i) \\ &\leq A \cdot 10^n \sum_i m\big(B(x_i)\big) \leq A'm(E) \end{aligned}$$

for every open $E \subset \mathbf{R}^n$.

The rest of the proof is as in [18; p. 236]. Let $f \in L^p_K$, put $u = P[f]$, let $\mathfrak{M} = \mathfrak{M}_{K,p,\beta}$. The lower semicontinuity of $\mathfrak{M}f$ shows that $\{\mathfrak{M}f > \lambda\}$ is open. Hence (3) and Proposition 4.4 imply that

(4) $$\mu\{|u| > \lambda\} \leq A'm\{\mathfrak{M}f > \lambda\},$$

and therefore

$$\int_{\mathbf{R}_+^{n+1}} |u|^p \, d\mu = \int_0^\infty \mu\{|u| > \lambda\} \cdot d(\lambda^p)$$

$$\leq A' \int_0^\infty m\{\mathfrak{M}f > \lambda\} \, d(\lambda^p)$$

$$= A' \int_{\mathbf{R}^n} |\mathfrak{M}f|^p \, dm \leq A'' \|f\|_{K,p}^p.$$

Theorem 3.8 was used in the last inequality. Since $\|f\|_{K,p} = \|u\|_{K,p}$, the proof is complete.

4.6. *Example.* Fix $p \geq 1$, assume $K \notin L^q$, put

(1) $$V(y) = r^n(y) = \|K_y\|_q^{-p}$$

and let μ be the measure on $\mathbf{R}_+^{n+1}$ defined by

(2) $$\int_{\mathbf{R}^{n+1}} \psi \, d\mu = \int_0^1 \psi(0, y) \, dV(y)$$

for every continuous ψ with compact support.

The radius $r(y)$ is as in Section 2.10, so that $V(y)$ is the volume of the cross-section of $\Omega_{K,1}^p$ at level y, except for some multiplicative constant. By Proposition 2.11, V is an increasing function, and $V(y) \to 0$ as $y \to 0$.

We claim that μ satisfies the hypothesis of Theorem 4.5, i.e., that

(3) $$\mu(Q(B)) \leq Am(B)$$

for all balls $B \subset \mathbf{R}^n$, where $Q = Q_{K,1}^p$.

Let r be the radius of B, and choose δ so that $r = r(\delta)$. The left side of (3) is obviously largest when the center of B is at the origin. In that case (2) implies that

(4) $$\mu(Q(B)) = \int_0^\delta dV = V(\delta) = Am(B).$$

Thus (3) is proved.

We draw two conclusions: first, when $p > 1$ our μ is a Carleson measure for h_K^p, a fact which can be just as easily deduced from Theorem 4.2 and which generalizes the classical inequality of Fejér and Riesz [6; p. 46]; see Section 4.7. Secondly, we shall see in Section 4.8 that μ is not a Carleson measure for h_K^1. This proves that Theorem 4.5 fails for $p = 1$.

4.7. *The Fejér-Riesz inequality. If V is associated to K as in Example* 4.6, *and if* $1 < p < \infty$, *then*

$$\int_0^1 |u(0,y)|^p \, dV(y) \le A \|u\|_{K,p}^p \tag{1}$$

for every $u \in h_K^p$.

The classical case deals with H^p-functions in the unit disc, and asserts that

$$\int_0^1 |f(r)|^p \, dr \le A \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \, d\theta.$$

Note that (1) says something about the rate of growth of functions in h_K^p. Since $u = K_y * F$ for some $F \in L^p$, Hölder's inequality shows that $|u| \le \|K_y\|_q \|F\|_p$; hence

$$|u(0,y)|^p \le V(y)^{-1} \|u\|_{K,p}^p. \tag{2}$$

But this easy estimate (2) does not imply (1), nor does its strengthened form

$$\lim_{y \to 0} \|K_y\|_q^{-1} |u(0,y)| = 0, \tag{3}$$

which follows from (2) since the continuous functions are dense in L^p.

4.8. THEOREM. *If $K \notin L^\infty$, and $\beta > 0$, then there is an $f \in L_K^1$, $f > 0$, such that*

$$\int_{\mathbf{R}^n} (\mathfrak{M}_{K,1,\beta} f) \, dm = \infty \tag{1}$$

and

$$\int_0^1 u(0,y) \, dV(y) = \infty, \tag{2}$$

where $u = P[f]$ and $V(y) = \|K_y\|_\infty^{-1}$.

This shows that Theorems 3.8, 4.5, and 4.7 fail when $p = 1$.

Proof. Since K is unbounded, one sees that

$$\|K_y\|_\infty = K_y(0) = \int_{\mathbf{R}^n} P_y K \, dm$$

assumes all positive values as y ranges over $(0, \infty)$. Hence there corresponds to every $x \in \mathbf{R}^n \setminus \{0\}$ a number $y(x) > 0$ such that

$$V(y(x))^{-1} = K_{y(x)}(0) = \beta |x|^{-n}. \tag{3}$$

Now pick a positive continuous $F \in L^1$, with $\|F\|_1 = 1$, put $F_j(x) = j^n F(jx)$ for $j = 1, 2, 3, \ldots$, put $f_j = K * F_j$, $u_j = P[f_j]$. Since $K * F_j \to K$ as $j \to \infty$, $u_j \to$

$P[K]$; hence (3) shows that

$$\lim_{j\to\infty} u_j(0, y(x)) = \beta |x|^{-n} \qquad (x \neq 0). \tag{4}$$

Equivalently,

$$\lim_{j\to\infty} u_j(0, y) = V(y)^{-1} \qquad (0 < y < \infty). \tag{5}$$

By Definition 2.6, $(0, y) \in \Omega^1_{K,\beta}(x)$ if and only if $|x|^n \| K_y \|_\infty < \beta$. Thus $(0, y(x))$ is a boundary point of $\Omega(x)$. Hence (dropping subscripts and superscripts on $\mathfrak{M}$), we have

$$(\mathfrak{M} f_j)(x) \geq u_j(0, y(x)) \qquad (x \neq 0). \tag{6}$$

By (6), (4), and Fatou's lemma,

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} (\mathfrak{M} f_j)\, dm \geq \beta \int_{\mathbf{R}^n} |x|^{-n}\, dx = \infty. \tag{7}$$

Similarly, (5) leads to

$$\lim_{j\to\infty} \int_0^1 u_j(0, y)\, dV(y) \geq \int_0^1 \frac{dV(y)}{V(y)} = \infty \tag{8}$$

since $V(y) \to 0$ as $y \to 0$.

Passing to a subsequence, we may assume that the integrals on the left of (7) and (8) exceed j^3. Then $f = \Sigma j^{-2} f_j$ satisfies (1) and (2). This completes the proof.

As was mentioned earlier, the converse of Theorem 4.5 is false. We thank David Stegenga for sending us an example which showed this, and which led us to Theorem 4.10.

4.9. Lemma. (a) *If $1 < p < \infty$ and $u = P[f] \in h^p_K$, then there exists $F \in L^p$ such that $\| F \|_p = \| u \|_{K,p}$ and $u(x, y) = (K_y * F)(x)$.*

(b) *If $\tilde{F}$ is the radial decreasing rearrangement of $|F|$, where F is as in* (a), *and if $u_0(x, y) = (K_y * \tilde{F})(x)$, then*

$$\| u_0 \|_{K,p} \leq \| u \|_{K,p} \tag{1}$$

although

$$\int_{rB} |u(x, y)|^p\, dx \leq \int_{rB} u_0(x, y)^p\, dx \tag{2}$$

for all $y > 0$, $r > 0$,

Recall that $rB = \{x \in \mathbf{R}^n \colon |x| < r\}$.

Proof. The set of all $F \in L^p$ such that $f = K * F$ is convex and closed, hence weakly closed (because $1 < p < \infty$), hence contains an element of minimal norm.

This proves (a), and gives (1) since

$$\|u_0\|_{K,p} \le \|\tilde{F}\|_p = \|F\|_p = \|u\|_{K,p}. \tag{3}$$

To prove (2), choose $v \in L^q$ so that $\|v\|_q = 1$ and $v = 0$ outside rB. Let $\tilde{v}$ be the radial decreasing rearrangement of v. An inequality due to Sobolev [16; pp. 39–42] (see also [9; pp. 279–284]) shows that

$$|(K_y * F * v)(0)| \le (K_y * \tilde{F} * \tilde{v})(0). \tag{4}$$

In other words,

$$\left|\int_{\mathbf{R}^n} u(x, y)v(-x)\,dx\right| \le \int_{\mathbf{R}^n} u_0(x, y)\tilde{v}(-x)\,dx. \tag{5}$$

Since $\|\tilde{v}\|_q = \|v\|_q = 1$, (2) follows from (5).

4.10. Theorem. *If $1 < p < \infty$ and K is any kernel, then there exist Carleson measures for h_K^p that violate the sufficient condition of Theorem* 4.5.

Proof. For $0 < y < \infty$, define

$$\psi(y) = \inf\left\{\frac{C_{K,p}(tB)}{m(tB)} : 0 < t \le y\right\}. \tag{1}$$

Since $C_{K,p}(tB) \ge AV(t)$ and $V(t)/t^n \to \infty$ as $t \to 0$ (see 3.6 (1), with $E = tB$, Section 4.6, and 2.11 (c)), it follows that

$$\psi(y) \nearrow \infty \text{ as } y \searrow 0. \tag{2}$$

Choose β_i so that $\beta_i \nearrow \infty$.

Choose $y_i > 0$, so small that, setting

$$r_i = \beta_i^{p/n} r(y_i) \tag{3}$$

(see Section 2.10), we have

$$\psi(r_i) > 2^{i+n} \quad \text{and} \quad C_{K,p}(r_i B) < 2^{-i} \tag{4}$$

for $i = 1, 2, 3, \ldots$.

Define μ_i to be $\psi(r_i)$ times n-dimensional Lebesgue measure, restricted to the ball

$$\{(x, y_i) : |x| < r_i\}, \tag{5}$$

which is the intersection of Ω_{K,β_i}^p with the hyperplane $y = y_i$ and which therefore lies in $Q^p_{K,\beta_i}(2r_iB)$. Hence

$$\mu_i\big(Q^p_{K,\beta_i}(2r_iB)\big) > 2^i m(2r_iB) \tag{6}$$

and

$$\|\mu_i\| = \psi(r_i)m(r_iB) \le C_{K,p}(r_iB) < 2^{-i}. \tag{7}$$

We claim next that

$$\int_{\mathbf{R}^{n+1}_+} |u|^p \, d\mu_i \le A\|u\|^p_{K,p} \qquad (u \in h^p_K) \tag{8}$$

where A depends only on K and p.

By Lemma 4.9 it suffices to prove (8) under the additional assumption that $u = P[f]$ for some radial decreasing $f \in L^p_K$.

Put $E_\lambda = \{Nf > \lambda\}$. (See Definition 3.3.) The intersection of $S(E_\lambda)$ with the support of μ_i is then a ball of radius $\rho \le r_i$, E_λ itself is a ball of radius $> \rho$, and hence

$$\begin{aligned}\mu_i(S(E_\lambda)) &= \psi(r_i)m(\rho B) \le \psi(\rho)m(\rho B)\\ &\le C_{K,p}(\rho B) \le C_{K,p}(E_\lambda).\end{aligned} \tag{9}$$

Therefore, noting that $\{u > \lambda\} \subset S(E_\lambda)$, as in [17] and in the proof of Lemma 3.5, we see that

$$\int u^p \, d\mu_i = \int_0^\infty \mu_i\{u > \lambda\}' d(\lambda^p) \le \int_0^\infty C_{K,p}(E_\lambda)\, d(\lambda^p). \tag{10}$$

Now (8) follows from (10), exactly as in the proof of Theorem 3.8, by the theorems of Hansson and of Hardy-Littlewood.

By (7) and (8), the measure $\mu = \Sigma(2/3)^i\mu_i$ is a Carleson measure for h^p_K.

Finally, fix β. Then $\beta < \beta_i$ for all sufficiently large i, so that

$$Q^p_{K,\beta}(2r_iB) \supset Q^p_{K,\beta_i}(2r_iB), \tag{11}$$

and (6) shows therefore that

$$\begin{aligned}\mu\big(Q^p_{K,\beta}(2r_iB)\big) &\ge \left(\tfrac{2}{3}\right)^i \mu_i\big(Q^p_{K,\beta_i}(2r_iB)\big)\\ &> \left(\tfrac{4}{3}\right)^i m(2r_iB).\end{aligned}$$

Hence 4.5 (1) fails for this μ.

V. Examples and applications

In this section we investigate a special class of kernels, the so-called Bessel kernels, for which the regions $\Omega^p_{K,\beta}$ can be determined quite precisely, and we present several function-theoretic applications.

Our work will be simplified by the fact that the essential features of $\Omega^p_{K,\beta}$ depend only on the behavior of K near the origin. The following proposition makes this more explicit.

5.1. PROPOSITION. *Assume that* $1 \le p < \infty$, *that* K *and* H *are kernels, not in* L^q, *and that there are constants* $a, b, \varepsilon > 0$ *such that*

$$(1) \qquad 0 < a \le \frac{K(x)}{H(x)} \le b < \infty \quad \text{if } 0 < |x| < \varepsilon.$$

Then there are constants a', b' *such that*

$$(2) \qquad 0 < a' \le \frac{\|K_y\|_q}{\|H_y\|_q} \le b' < \infty$$

for all $y \in (0, \infty)$.

Consequently, every $\Omega^p_{K,\beta}$ *contains an* $\Omega^p_{H,\lambda}$ *and vice versa.*

Proof. The ratio in (2) stays bounded and stays away from 0 as $y \to \infty$, by Proposition 2.11 (d). Hence we only need to examine its behavior as $y \to 0$.

Write $K = K' + K''$, $H = H' + H''$, where K' and H' are the restrictions of K and H to $\{|x| < \varepsilon\}$. Then $K'' \in L^\infty \cap L^1$; hence $K'' \in L^q$, and therefore $\|K'' * P_y\|_q$ is a bounded function of y on $(0, \infty)$. The same is true of $\|H'' * P_y\|_q$. Since $\|K_y\|_q$ and $\|H_y\|_q$ tend to ∞ when $y \to 0$, it follows that the upper and lower limits of the ratio in (2) are unchanged if K and H are replaced by K' and H'. Since $aH' \subseteq K' \subseteq bH'$, (2) holds.

The assertion concerning the regions Ω_K and Ω_H follows from (2) and Definition 2.6.

COROLLARY. *Assume that* K *and* H *satisfy* (1). *A function* u *in* $\mathbf{R}^{n+1}_+$ *has then an* Ω^p_K*-limit at* $x_0 \in \mathbf{R}^n$ *if and only if* u *has the same* Ω^p_H*-limit at* x_0. *To each* β *corresponds a* γ *such that*

$$\mathfrak{M}_{K,p,\beta} f \le \mathfrak{M}_{H,p,\gamma} f$$

and vice versa.

We therefore lose nothing of importance if, in discussing h^p_K, we replace Ω_K by Ω_H.

5.2. *The Bessel Kernels* G_α. For $0 < \alpha \le n$, G_α is the function on $\mathbf{R}^n$ whose Fourier transform is

$$(1) \qquad \hat{G}_\alpha(\xi) = \left(1 + |\xi|^2\right)^{-\alpha/2} \qquad (\xi \in \mathbf{R}^n).$$

Each G_α is a positive radial decreasing L^1-function. Explicit formulas for $G_\alpha(x)$

may be found in [3; p. 416] and [18; p. 132]. These show that $G_\alpha(x)$ decays exponentially as $|x| \to \infty$, that

(2) $$G_\alpha(x) \approx |x|^{\alpha-n} \qquad \text{as } x \to 0$$

if $0 < \alpha < n$, and that

(3) $$G_n(x) \approx \log\frac{1}{|x|} \qquad \text{as } x \to 0.$$

Here and later, the notation

(4) $$u \approx v$$

means that u/v has a positive finite limit, and

(5) $$u \sim v$$

will mean that u/v and v/u are bounded.

In order to apply our general theorems about h_K^p, we have to estimate the norms $\|G_\alpha * P_y\|_q$ as $y \to 0$, for $1 < q \le \infty$.

Note that $G_\alpha \in L^q$ when $\alpha p > n$.

5.3. Proposition. *As* $y \to 0$,

(1) $$\|G_\alpha * P_y\|_q \sim y^{\alpha - n/p} \qquad \text{if } \alpha p < n,$$

(2) $$\|G_\alpha * P_y\|_q \sim \left(\log\frac{1}{y}\right)^{1/q} \qquad \text{if } \alpha p = n,\ p > 1,$$

and

(3) $$\|G_n * P_y\|_\infty \sim \log\frac{1}{y}.$$

Proof. To facilitate the computations, we use Proposition 5.1 and replace $G_\alpha(x)$ by the kernel

(4) $$H_\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-s}s^{\alpha-1}P_s(x)\,ds.$$

If s is replaced in (4) by $|x|s$, comparison of the resulting integral with 5.2 (2) and 5.2 (3) shows that $H_\alpha(x) \approx G_\alpha(x)$ as $x \to 0$. Therefore it suffices to estimate $\|H_\alpha * P_y\|_q$ as $y \to 0$.

Since $P_s * P_y = P_{s+y}$, (4) implies that

(5) $$(H_\alpha * P_y)(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-s}s^{\alpha-1}P_{s+y}(x)\,ds.$$

If we define

$$I_{t,\alpha}(y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-s} s^{\alpha-1} (s+y)^{-t}\, ds, \tag{6}$$

the explicit formula for the Poisson kernel shows that there exist $\gamma' > 0$, $\gamma'' < \infty$ (depending only on n) such that

$$\gamma' I_{\alpha+1,n+1}(|x|+y) \le (H_\alpha * P_y)(x) \le \gamma'' I_{\alpha,n}(|x|+y). \tag{7}$$

Since $I_{t,\alpha}(y) \approx y^{\alpha-t}$ if $0 < \alpha < t$, it follows that

$$(H_\alpha * P_y)(x) \sim (|x|+y)^{\alpha-n} \qquad (0 < \alpha < n). \tag{8}$$

Since $|x|^n P_y(x) \le c_n$ for all x, y, we also have

$$(H_\alpha * P_y)(x) \le c_n |x|^{-n}. \tag{9}$$

If $1 < p < \infty$, use (8) when $|x| < 1$, use (9) when $|x| \ge 1$, raise to the q-th power, and integrate. The result is (1) and (2), with H in place of G. Finally,

$$\| H_\alpha * P_y \|_\infty = (H_\alpha * P_y)(0) = c_n I_{n,\alpha}(y) \tag{10}$$

yields (3) and the case $p = 1$ of (1).

5.4. *Bessel potentials.* The potential spaces generated by the Bessel kernels G_α are usually denoted by $\mathfrak{L}_\alpha^p$, rather than by $L_{G_\alpha}^p$. Thus, when $1 < p < \infty$, $\mathfrak{L}_\alpha^p$ consists of all $f = G_\alpha * F$ where $F \in L^p$, and $\|f\|_{\alpha,p} = \|F\|_p$. (Note that the map $F \to G_\alpha * F$ is $1-1$.) However, when $p = 1$, we shall include potentials of measures. Thus $\mathfrak{L}_\alpha^1$ consists of all $f = G_\alpha * \mu$, where μ is a complex Borel measure on $\mathbf{R}^n$, and we define $\|f\|_{\alpha,1} = \|\mu\|$ (see Section 2.17).

Here is a synopsis of Theorems 2.7, 2.9, 3.8, and 4.7, for the Poisson integrals of Bessel potentials. The proof uses Proposition 5.3.

5.5. THEOREM. *Suppose* $1 \le p < \infty$, $0 < \alpha \le n$, $f \in \mathfrak{L}_\alpha^p$, *and* $u = P[f]$.

(a) *Except possibly on a set of* x_0*'s of measure* 0, $u(x, y)$ *converges to* $f(x_0)$ *when* $(x, y) \to (x_0, 0)$ *within the regions defined by*

$$y > c|x - x_0|^{n/(n-\alpha p)} \qquad \text{if } \alpha p < n, \tag{1}$$

$$y > \exp\{-c|x - x_0|^{-n(q-1)}\} \qquad \text{if } \alpha p = n,\ p > 1, \tag{2}$$

$$y > \exp\{-c|x - x_0|^{-n}\} \qquad \text{if } \alpha = n,\ p = 1, \tag{3}$$

where c is any positive number.

(b) *The corresponding maximal functions* $\mathfrak{M}f$ *are in* L^p *if* $1 < p < \infty$, *and are in weak* L^1 *if* $p = 1$.

(c) *The following inequalities hold when* $p > 1$:

(4) $$\int_0^1 |u(0, y)|^p y^{n-\alpha p-1} dy \le A \|f\|_{\alpha, p}^p \qquad \text{if } \alpha p < n,$$

(5) $$\int_0^{1/2} |u(0, y)|^p \frac{dy}{y |\log y|^p} \le A \|f\|_{\alpha, p}^p \qquad \text{if } \alpha p = n.$$

With regard to part (a), we observe that the exponential contact regions that correspond to the potential spaces $\mathfrak{L}_\alpha^p$ when $\alpha p = n$ are all distinct, with the sole exception of $\mathfrak{L}_{n/2}^2$ and $\mathfrak{L}_n^1$. On the other hand, when $\alpha p < n$, then the approach regions for $\mathfrak{L}_\alpha^p$ depend only on the product αp (called "weight" in [2]).

This contrasts with the behavior of the Bessel capacities $B_{\alpha, p}$: according to [2; p. 874], no two of these capacities have the same null sets, in the range $1 < p < \infty$, $0 < \alpha p \le n$.

5.6. *Kernels on the unit circle.* If $\mathbf{R}^n$ is replaced by the unit circle T, and $0 < \alpha \le 1$, the kernel G_α is replaced by

(1) $$g_\alpha(\theta) = 1 + \frac{1}{2} \sum_{n \ne 0} |n|^{-\alpha} e^{in\theta}.$$

When $0 < \alpha < 1$, the asymptotic relation

(2) $$g_\alpha(\theta) \approx \left|\sin\frac{\theta}{2}\right|^{\alpha-1}$$

holds as $\theta \to 0$ (see [20; p. 186]); also

(3) $$g_1(\theta) = 1 + \log\frac{1}{\left|2\sin\frac{\theta}{2}\right|}.$$

Thus g_α has essentially the same singularity as G_α when $n = 1$, so that the results of Theorem 5.5 (with $n = 1$) obviously hold in this context. To list them again would be repetitious. Instead, we shall now apply our general theorems to holomorphic functions in the unit disc.

5.7 *Cauchy integrals of potentials.* Let us define

(1) $$\tilde{g}_1(\theta) = \log\frac{1}{1 - e^{i\theta}}, \qquad \tilde{g}_\alpha(\theta) = (1 - e^{i\theta})^{\alpha-1}$$

if $0 < \alpha < 1$. These are not kernels in the sense in which we have used the term, but they are dominated by g_α and g_1 (see Section 5.6): there are constants A_α, for $0 < \alpha \le 1$, such that $|\tilde{g}_\alpha| \le A_\alpha g_\alpha$.

If $P_r(\theta)$ is the Poisson kernel for U, and $z = re^{i\theta}$, then

$$(P_r * \tilde{g}_1)(\theta) = \log \frac{1}{1-z}, \tag{2}$$

$$(P_r * \tilde{g}_\alpha)(\theta) = (1-z)^{\alpha-1} \qquad (0 < \alpha < 1) \tag{3}$$

and the Cauchy integral of $\tilde{g}_\alpha * F$ is the same as its Poisson integral, namely

$$(P_r * \tilde{g}_\alpha * F)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F(t)\,dt}{(1-e^{-it}z)^{1-\alpha}}. \tag{4}$$

Since $|\tilde{g}_\alpha| \le A_\alpha g_\alpha$, the proof of Theorem A (stated in the introduction) is now simply a matter of translating Theorems 2.7, 2.9, 3.8 into our current terminology; the approach regions are determined as in Theorem 5.5, with θ and $1 - r$ in place of x and y.

5.8 THEOREM. *If f is a function of bounded variation on T, then the $\mathcal{E}_1$-limit of its Cauchy integral*

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)\,dt}{1-e^{-it}z}$$

exists at almost every point of T.

The term $\mathcal{E}_1$-limit refers to the exponential contact regions that occur in Theorem A of the introduction.

Proof. Assume $\hat{f}(0) = 0$, without loss of generality. The derivative of f is a measure λ, with $\hat{\lambda}(0) = 0$. Setting $\mu = -i\lambda$, we have $\hat{\mu}(n) = n\hat{f}(n)$ for all integers n. On the other hand, if g_1 is as in Section 5.6, then

$$(g_1 * \mu)^\wedge(n) = \tfrac{1}{2}\hat{\mu}(n)/|n| \qquad (n \neq 0).$$

Thus

$$\tfrac{1}{2}\hat{f}(n) + (g_1 * \mu)^\wedge(n) = \begin{cases} \hat{\mu}(n)/n = \hat{f}(n) & \text{if } n > 0 \\ 0 & \text{if } n \le 0. \end{cases}$$

Consequently,

$$h(z) = \sum_1^\infty \hat{f}(n) z^n = \tfrac{1}{2}P[f](z) + P[g_1 * \mu](z).$$

Since $g_1 * \mu \in \mathfrak{L}_1^1$, Theorem 5.5 (3), adapted to U, shows that $P[g_1 * \mu]$ has an $\mathcal{E}_1$-limit almost everywhere on T. Since f is continuous except possibly at a countable subset of T, and since $P[f]$ extends continuously to any point of continuity of f, the proof is complete.

Our final application deals with spaces of holomorphic functions in the upper half-plane and with their zero sets. Since we shall work with general kernels K, we prefer the half-plane to the unit disc.

We define H_K^p to be the space of all *holomorphic* functions in $\mathbf{R}_+^2$ that belong to h_K^p, i.e., that are Poisson integrals of potentials in L_K^p.

5.9. Lemma. *If $u \in H_K^p$, $u \not\equiv 0$, $\beta > 0$, and E is the set of all $x \in \mathbf{R}$ such that $\Omega_{K,\beta}^p(x)$ contains infinitely many zeros of u, then $m(E) = 0$.*

Proof. The Ω_K^p-limit u^* of u exists almost everywhere (Theorem 2.9) and is therefore 0 at almost all points of E. Hence $m(E) = 0$, since u^* cannot be 0 on a set of positive measure.

5.10. Theorem. *Suppose K is a kernel on $\mathbf{R}$, $1 \le p < \infty$, and $\{y_i\}$ is a sequence of positive numbers such that*

(1)
$$\sum_i \|K_{y_i}\|_q^{-p} = \infty.$$

Then there exist real numbers x_i such that 0 is the only function in H_K^p that vanishes at every $z_i = x_i + iy_i$.

Proof. Put $r(y) = \|K_y\|_q^{-p}$, as in Section 2.10. The width of $\Omega_{K,1}^p$ at height y is then $2r(y)$. Since $\Sigma r(y_i) = \infty$, there exists $\{x_i\}$ so that every real x belongs to infinitely many segments I_i, where

(2)
$$I_i = \{x \colon |x - x_i| < r(y_i)\}.$$

But (2) says precisely that $z_i \in \Omega_{K,1}^p(x)$. Every $\Omega_{K,1}^p(x)$ contains thus infinitely many z_i, so that the theorem follows from Lemma 5.9.

5.11. *Remark.* When transferred to the unit disc, Theorem 5.10 gives an affirmative answer to a question raised by Shapiro and Shields [15; p. 224]. They ask whether the theorem is true when $p = 2$ and K is any kernel on T with $\hat{K}(n) > 0$ and $\hat{K}$ convex on $\{n \ge 0\}$. They prove it in the case $K = g_\alpha$, i.e., for the functions in the spaces D_α that consist of all h of the form

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F(t)\,dt}{(1 - e^{-it}z)^{1-\alpha}}$$

where $F \in L^2$.

UNIVERSITY OF WISCONSIN, MADISON
MICHIGAN STATE UNIVERSITY, EAST LANSING

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