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SUBSPACES OF $L^p(G)$ SPANNED BY CHARACTERS: $0 < p < 1$

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ABSTRACT

Let G be an infinite compact abelian group, μ a Borel measure on G with spectrum E, and $0 < p < 1$. We show that if μ is not absolutely continuous with respect to Haar measure, then $L^p_E(G)$, the closure in $L^p(G)$ of the E-trigonometric polynomials, does not have enough continuous linear functionals to separate points. If μ is actually singular, then $L^p_E(G)$ does not have any nontrivial continuous linear functionals at all. Our methods recover the classical F. and M. Riesz theorem, and a related several variable result of Bochner; they reveal the existence of small sets of characters that span $L^p(\mathbf{T})$, where $\mathbf{T}$ is the unit circle; and they show that the H^p spaces of the "big disc algebra" have one-dimensional dual.

Introduction

If G is an infinite compact abelian group and $0 < p < 1$, then the linear metric space $L^p(G)$ has no nontrivial continuous linear functionals. This suggests the following problem in harmonic analysis: *for which subsets E of the dual group of G does $L^p_E(G)$, the closure in $L^p(G)$ of the E-trigonometric polynomials on G, fail to have enough continuous linear functionals to separate points*? In this paper we show that it is sufficient for E to be the spectrum of a measure on G that is not absolutely continuous with respect to Haar measure. Equivalently: *if $L^p_E(G)$ has enough continuous linear functionals to separate points for some $0 < p < 1$, then every E-spectral measure on G is absolutely continuous* (here "measure" means "finite, regular Borel measure").

This exhibits the classical F. and M. Riesz theorem, which states that every measure on the circle with spectrum contained in the positive integers must be absolutely continuous, as a consequence of the fact that when $0 < p < 1$ the L^p-closure of the space of trigonometric polynomials with positive frequencies

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still has enough continuous linear functionals to separate points. In fact our methods will provide an even simpler proof, using ideas which lead to a related theorem of Bochner about measures on the torus.

Along the way it is necessary to prove the following result, which is actually our main theorem: *if E is the spectrum of a measure on G that is singular with respect to Haar measure, then $L^p_E(G)$ has no nontrivial continuous linear functional for $0<p<1$.* This result is of independent interest. It yields the existence of sets of integers having density zero for which the corresponding exponentials span a subset of $L^p(\mathbf{T})$ with trivial dual; it shows that the H^p spaces of the "big disc algebra" have one dimensional dual when $0<p<1$, and it explains a recent theorem of de Leeuw [4, theor. 3.1, p. 113] which asserts that if E is the spectrum of a measure on G that is singular with respect to Haar measure, then every co-finite subset of E spans a dense subspace of $L^p_E(G)$ for $0<p<1$. We use de Leeuw's result in turn to show that the "modification sets" introduced by Rudin [13], [14] span dense subspaces of $L^p(G)$, and this provides sets of integers having density zero for which the corresponding characters (i.e. exponentials) span a dense subspace of $L^p(\mathbf{T})$ for all $0<p<1$. In fact de Leeuw's paper [4] was the starting point of this investigation, and his methods play an important role in what follows.

Of course none of this makes sense for $p \geqq 1$, and all of it would be trivial if there were no infinite sets E for which $L^p_E(G)$ had enough continuous linear functionals to separate points. Fortunately, however, every infinite compact abelian group has such sets E in its dual. For example, when $G=\mathbf{T}$ (the circle group) it is well known that if $E=(\pm n_k)$ where (n_k) is a sequence of positive integers with $n_{k+1}/n_k > q > 1$, then on the E-trigonometric polynomials the L^2 and L^p topologies coincide [15, Vol. I, theor. 8.4, p. 213]. More generally the same is true when G is a compact abelian group and E is a Sidon set in its dual group [12, theor. 5.7.7, p. 128]. Since every compact abelian group has an infinite Sidon set in its dual [12, sec. 5.7.6, p. 126], it follows that unless G is finite there will always be infinite sets E for which the L^p-topology on $L^p_E(G)$ is locally convex for $0<p<1$, hence $L^p_E(G)$ will have enough continuous linear functionals to separate points.

The paper is organized into five sections. Basic notations, definitions, and preliminary results occupy the first, singular measures and trivial duals the second, and H^p spaces of the big disc algebra the third. The fourth section contains de Leeuw's result, along with its application to modification sets; and the paper concludes with our generalization of the F. and M. Riesz theorem.

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1. Preliminaries

In this paper G is always a compact abelian group with normalized Haar measure m and dual group Γ. We generally follow the notation and terminology of Rudin's book [12]; in particular we write (x, α) instead of $\alpha(x)$ when $x \in G$ and $\alpha \in \Gamma$; dx instead of $dm(x)$; and $L^p(G)$ instead of $L^p(m)$. In contrast to [12], however, *the index p always lies strictly between* 0 *and* 1.

The space of finite, regular, complex valued Borel measures on G is denoted by $M(G)$, and its elements simply called "measures". The subspace consisting of measures that are singular with respect to m is denoted by $M_s(G)$. Henceforth the terms "absolutely continuous" and "singular" are always intended with respect to m.

If $\mu \in M(G)$ then $\hat{\mu}$ denotes the Fourier–Stieltjes transform of μ:

$$\hat{\mu}(\alpha) = \int (-x, \alpha)\, d\mu(x) \quad (\alpha \text{ in } \Gamma),$$

and the *spectrum* of μ is the support of $\hat{\mu}$:

$$\operatorname{spec} \mu = \{\alpha \in \Gamma : \hat{\mu}(\alpha) \neq 0\}.$$

If E is a subset of Γ and $\operatorname{spec} \mu \subset E$ we say μ is an *E-spectral measure.* These notions are transferred to functions in $L^1(G)$ by identifying the function f with the measure $f(x)dx$.

We will be particularly concerned with finite linear combinations of characters, usually called *trigonometric polynomials* on G. A finite linear combination of characters taken from a fixed subset E of Γ is called an *E-trigonometric polynomial*, and the collection of all E-trigonometric polynomials is denoted by $T_E(G)$. In other words, $T_E(G)$ is the linear span of E.

$L^p(G)$ is given its usual topology here: the one induced by the metric $d(f, g) = \|f - g\|_p^p$ where

$$\|f\|_p^p = \int |f(x)|^p\, dx.$$

This metric turns $L^p(G)$ into a complete linear topological space which, unless G is finite, is *not* locally convex, and in fact has no nontrivial continuous linear functionals [3].

For E a subset of Γ we denote the closure of $T_E(G)$ in $L^p(G)$ by $L^p_E(G)$. So $L^p_E(G)$ is the closed linear subspace of $L^p(G)$ spanned by the characters E.

It will be convenient to use the following terminology from functional analysis. If X is a linear topological space then the *dual of* X is the collection of continuous linear functionals on X. It is denoted by X', *and should not be confused with the dual group of* G. We say a subset S of X' *separates the points of* X if for every nonzero x in X there exists λ in S with $\lambda(x) \neq 0$. If X' separates the points of X we say X has *separating dual.* If $X' = \{0\}$ we say X has *trivial dual.*

Finally we require the following two results. The first is a generalization due to Jean Boclé of a standard differentiation theorem, while the second is a well known result. For completeness we include proofs of both.

LEMMA 1.1 [1, theor. II, p. 17]. *Let $\mathcal{S}$ denote the collection of symmetric neighborhoods of* 0 *in* G, *directed downward by inclusion. For* V *in* $\mathcal{S}$ *let* $k_V = I_V/m(V)$, *where* I_V *is the characteristic function of* $V(=1$ *on* V *and* 0 *off* $V)$. *If* $\mu \in M_s(G)$ *then the net* $(k_V * \mu : V \in \mathcal{S})$ *converges in Haar measure to zero.*

PROOF. Since $|k_V * \mu| \leqq k_V * |\mu|$ we may without loss of generality take μ to be a positive measure. Suppose ε and a are >0. By the regularity and singularity of μ there exist sets $K \subset U \subset G$ with K compact, U open, and

$$\mu(U) = \mu(G) = \|\mu\|,$$

$$\mu(U \setminus K) < \varepsilon a/2,$$

$$m(U) < \varepsilon/2.$$

Define λ in $M(G)$ by $\lambda(B) = \mu(B \cap K)$ for B a Borel subset of G. Then $\mu = \lambda + \theta$ where λ is concentrated on K and $\theta(G) < \varepsilon a/2$.

Choose V_0 in $\mathcal{S}$ such that $K + V_0 \subset U$. The symmetry of $V_0(V_0 = -V_0)$ implies that $(V_0 + t) \cap K = \varnothing$ whenever $t \notin U$, hence

$$k_V * \lambda(t) = \frac{\lambda(V+t)}{m(V+t)} = \frac{\mu(\{V+t\} \cap K)}{m(V+t)}$$

vanishes off U whenever $V \subset V_0$. So for $V \subset V_0$ we have $k_V * \mu = k_V * \theta$ off U, hence

$$\begin{aligned} \int_{G \setminus U} k_V * \mu(x)\, dx &= \int_{G \setminus U} k_V * \theta(x)\, dx \\ &\leqq \|k_V\|_1 \|\theta\| \\ &\leqq \varepsilon a/2. \end{aligned}$$

By Chebyshev's inequality:

$$m\{\{k_V * \mu > a\} \cap (G \setminus U)\} \leqq \varepsilon/2.$$

So given ε, $a > 0$ there exists V_0 in $\mathcal{S}$ such that

$$m\{k_V * \mu > a\} \leqq \varepsilon/2 + m(U) < \varepsilon$$

whenever $V \subset V_0$. This completes the proof.

LEMMA 1.2 (cf. [10, p. 30]). *If V is a closed, translation invariant subspace of $L^1(G)$ and $h \in L^1(G)$, then $h * f \in V$ for every f in V. In particular* spec $f \subset V$ *for every f in V.*

PROOF. By the Hahn–Banach and Riesz representation theorems it is enough to show that if g in $L^\infty(G)$ annihilates V, then it annihilates $h * V$. So suppose $g \in L^\infty(G)$ and

$$\int f(x) g(x)\, dx = 0$$

for all f in V. Then using the translation invariance of V and Fubini's Theorem:

$$\begin{aligned} 0 &= \int \left\{ \int f(x-y) g(x)\, dx \right\} h(y)\, dy \\ &= \int \left\{ \int f(x-y) h(y)\, dy \right\} g(x)\, dx \\ &= \int f * h(x) g(x)\, dx \end{aligned}$$

which is the desired result.

In particular, taking $h(x) = (x, \alpha)$ for $\alpha \in \Gamma$ we have

$$\hat{f}(\alpha)\alpha = h * f \in V,$$

so if $\alpha \in \operatorname{spec} f$, then $\alpha \in V$. This completes the proof.

2. Singular measures and trivial duals

Recall that G is a compact abelian group, and that $0 < p < 1$. This section is devoted to the proof and first consequences of the following result.

THEOREM 2.1. *If $\mu \in M_s(G)$, $E = \operatorname{spec} \mu$, and $0 < p < 1$, then $L^p_E(G)$ has trivial dual.*

PROOF. Suppose Φ is a continuous linear functional on $L^p_E(G)$. Then Φ is L^1-continuous on the dense subspace $T_E(G)$ since the L^1-topology is stronger thereon than the L^p-topology. By the Riesz and Hahn–Banach theorems there exists ϕ in $L^\infty(G)$ such that for each f in $T_E(G)$:

$$\Phi(f) = \int f(x)\phi(-x)\,dx = \sum \hat{f}(\alpha)\hat{\phi}(\alpha). \tag{2.1}$$

So in order to show that Φ vanishes identically on $L^p_E(G)$ it is enough to check that $\hat{\phi}(\alpha) = 0$ for each α in E.

This will be accomplished by applying Φ to a net of test functions obtained by convolving the singular measure μ with the approximate identity $(k_V : V \in \mathcal{S})$ that was considered in Lemma 1.1. Let $f_V = k_V * \mu$. Then $f_V \in L^1_E(G) \subset L^p_E(G)$,

$$\|f_V\|_1 \leqq \|k_V\|_1 \|\mu\| = \|\mu\| \quad (V \in \mathcal{S}), \tag{2.2}$$

and $f_V \to 0$ in Haar measure by Lemma 1.1.

Now fix α in E and recall that our goal is to show that $\hat{\phi}(\alpha) = 0$. Since the net of measures $(k_V dm : V \in \mathcal{S})$ converges in the weak star topology of $M(G)$ to the unit mass at the identity of G, we have $\hat{k}_V(\alpha) \to 1$. Since both the topology of the complex plane and the topology of convergence in Haar measure on G are metrizable, we can choose a sequence (V_n) from $\mathcal{S}$ so that simultaneously $\hat{k}_{V_n}(\alpha) \to 1$ and $g_n = f_{V_n} \to 0$ in measure. Now according to (2.2) the sequence $(|g_n|^p)$ is uniformly integrable, and we have just seen that it converges to zero in measure. So Vitali's Convergence Theorem [7, theor. C, p. 108] shows that (g_n) converges to zero in $L^p(G)$ (cf. [4, theor. 3.1, p. 113] where this is also the crucial step in the proof). The translation invariance of Haar measure insures that the same is true for the sequence of translates $(g_n^{(t)})$ for each t in G, where

$$g_n^{(t)}(x) = g_n(x+t) \quad (x \text{ in } G).$$

We finish the proof by analysing the functions

$$F_n(t) = \Phi(g_n^{(t)}) \quad (t \text{ in } G),$$

recalling that since $g_n \in L^p_E(G)$, the same is true for $g_n^{(t)}$. First of all, each F_n is continuous on G, since translation of L^p functions takes G continuously into $L^p(G)$, and Φ is continuous. Next, for t in G:

$$|F_n(t)| \leqq \|\Phi\| \; \|g_n^{(t)}\|_p \leqq \|\Phi\| \; \|g_n\|_1 \leqq \|\Phi\| \; \|\mu\|$$

for each n, so the sequence (F_n) is uniformly bounded on G. Finally, $F_n(t) \to 0$ for each t in G since, as we have just noted, $g_n^{(t)} \to 0$ in $L^p(G)$ for each t. Thus (F_n) is a uniformly bounded sequence of continuous functions on G that converges pointwise to zero, so by the bounded convergence theorem we have $\hat{F}_n(\alpha) \to 0$. Now a quick calculation shows that

$$F_n = k_{V_n} * \mu * \phi$$

so recalling that $\hat{k}_{V_n}(\alpha) \to 1$:

$$0 = \lim_n \hat{k}_{V_n}(\alpha)\hat{\mu}(\alpha)\hat{\phi}(\alpha) = \hat{\mu}(\alpha)\hat{\phi}(\alpha)$$

which implies that $\hat{\phi}(\alpha) = 0$, since $\alpha \in E = \operatorname{spec} \mu$. This completes the proof.

Note that when μ is the unit mass at the identity of G, then spec $\mu = \Gamma$ and we recover the fact mentioned in the Introduction that $L^p(G)' = \{0\}$ whenever G is infinite. This result was first proved for the L^p spaces of general non-atomic measures by M. M. Day [3].

We next illustrate how Theorem 2.1 can be used to produce examples of rather "thin" sets E for which $L^p_E(G)$ has trivial dual. Recall that $\mathbf{T}$ denotes the circle group, and that the dual of $\mathbf{T}$ is identified with the group $\mathbf{Z}$ of integers by associating $n \in \mathbf{Z}$ with the character $\omega \to \omega^n$ $(\omega \in \mathbf{T})$.

Corollary 2.2. *There exist subsets E of $\mathbf{Z}$ having density zero for which $L^p_E(\mathbf{T})$ has trivial dual* $(0 < p < 1)$.

Proof. We use Riesz products to produce measures $\mu \in M_s(\mathbf{T})$ for which $E = \operatorname{spec} \mu$ has density zero. Suppose (n_k) is a sequence of positive integers with $\inf n_{k+1}/n_k > 3$. Then it is well known [15, Vol. I, ch. 5, sec. 7] that the measures

$$d\mu_n(t) = \prod_{k=1}^{n} (1 + \cos n_k t)\, dt/2\pi$$

converge in the weak star topology of $M(\mathbf{T})$ to a singular measure μ whose spectrum consists of all finite sums $\Sigma\, \theta_k n_k$ where $\theta_k = 0$, 1, or -1. It is easy to check that this set has density zero, so the proof is complete.

For the problems we are considering a measure of thinness that is perhaps better than "density zero" is $\alpha_E(N) =$ the maximum number of elements E has in common with an arithmetic progression of length N. Clearly $\alpha_E(N) \leqq N$; and E has density zero if $\alpha_E(N) = o(N)$, but not conversely. Rudin [11, theor. 3.8, p. 215] shows that given any function $w(N) \uparrow \infty$, the sequence (n_k) in the proof of Corollary 2.2 can be chosen so that $\alpha_E(N) \leqq w(N)$. Thus the statement of Corollary 2.2 can be refined to read: *given $w(N) \uparrow \infty$ there is a subset E of $\mathbf{Z}$ such that $\alpha_E(N) \leqq w(N)$, yet $L^p_E(\mathbf{T})$ has trivial dual for all $0 < p < 1$.*

3. H^p spaces of the big disc algebra

In this section we apply Theorem 2.1 to a situation that arises in the theory of uniform algebras. We continue to follow the notation and terminology of Rudin

[12,ch. 8]. In particular recall that an abelian group Γ is *ordered* if there is a linear order $\leqq$ on it such that $\alpha \leqq \beta$ implies $\alpha+\lambda \leqq \beta+\lambda$ for all λ in Γ [12, sec. 8.1, p. 193]. The collection of non-negative elements of Γ will be denoted by Γ^+. If G is a compact abelian group whose dual Γ is ordered, then we say the Γ^+-trigonometric polynomials are of *analytic type*, and we write $H^p(G)$ instead of $L^p_{\Gamma^+}(G)$. Note that these definitions depend in an essential way on the order chosen for Γ, and that this is suppressed in the notation. If we take $G = \mathbf{T}$ and order its dual group $\mathbf{Z}$ in the usual way, then $H^p(G)$ is the space of boundary functions of the usual Hardy space H^p, and is in fact isometrically isomorphic to H^p [5, theor. 3.3, p. 36].

We will need the following result from the theory of uniform algebras.

LEMMA 3.1 [12, theor. 8.4, p. 206]. *If G is a compact abelian group with ordered dual, and f is a trigonometric polynomial on G of analytic type, then*

$$|\hat{f}(0)| \leqq \exp \int \log |f(x)|\, dx.$$

This lemma and the arithmetic-geometric mean inequality yield:

COROLLARY 3.2 [6, theor. 3.1, p. 124]. *If G and f are as above, then*

$$|\hat{f}(0)| \leqq \|f\|_p$$

for all $p > 0$.

For the remainder of this section we concentrate on the following special case, which is described in detail in [6, ch. VII]. We take Γ to be a dense subgroup of the real line R, let Γ_d denote Γ in the discrete topology, and let G be the dual of Γ_d. So G is a compact, connected abelian group with character group Γ_d. We give Γ_d the order it inherits from R. The uniform closure of the trigonometric polynomials of analytic type on G is an important function algebra called the "big disc algebra", and $H^p(G)$ can be regarded as the L^p-closure of the big disc algebra. We are going to show that the dual space of $H^p(G)$ is one dimensional, in sharp contrast to the case $G = \mathbf{T}$. This follows from Corollary 3.2 and the following result.

THEOREM 3.3. *If E is a subset of Γ that is relatively open in the topology of the real line, then $L^p_E(G)$ has trivial dual for $0<p<1$.*

PROOF. According to Theorem 2.1 it is enough to find $\mu \in M_s(G)$ with $\operatorname{spec} \mu = E$. This construction is completely standard, but to keep things reasonably self-contained we will give it in detail.

The idea is to transfer measures from R to G by means of the following homomorphism. For s in R define e_s in G (= the dual of Γ_d, recall) by

$$(\alpha, e_s) = e^{i\alpha s} \quad (\alpha \text{ in } \Gamma), \tag{3.1}$$

and define $h : R \to G$ by

$$h(s) = e_s \quad (s \text{ in } R). \tag{3.2}$$

Clearly h is a continuous homomorphism of R into G, and is one-to-one because Γ is dense in R; but h is *not* bicontinuous as a map from R onto $h(R)$. Nevertheless $h(R)$ is sigma-compact, hence a Borel subset of G.

If μ is a finite Borel measure on R, let $\mu_G = \mu h^{-1}$, that is,

$$\mu_G(B) = \mu\{h^{-1}(B)\}$$

for each Borel subset B of G. Then $\mu_G \in M(G)$ and a straightforward calculation employing (3.1), (3.2), and the change of variable formula [7, theor. C, p. 163] yields

$$\hat{\mu}_G(\alpha) = \hat{\mu}(\alpha) \quad (\alpha \text{ in } \Gamma), \tag{3.3}$$

where on the left side of (3.3) we are viewing α as a character on G, and on the right side as a real number.

The assumption on E is that $E = \Gamma \cap U$ where U is an open subset of R. Choose any finite Borel measure μ on R with

$$\operatorname{spec} \mu = \{t \text{ in } R : \hat{\mu}(t) \neq 0\} = U.$$

Then spec $\mu_G = E$ by (3.3), so we will be done if we can show that μ_G is singular with respect to the Haar measure m on G. Since μ_G is concentrated on $h(R)$ it is enough to show that $m\{h(R)\} = 0$.

This is easy. If K_n is the closed interval in R between n and $n+1$, then $h(K_n)$ is a compact subset of G, and

$$h(R) = \bigcup_{n \in Z} h(K_n).$$

So we need only show that $m\{h(K_n)\} = 0$ for each n. Suppose not. Then since h is a homomorphism and the K_n's are all translates of each other, so are the $h(K_n)$'s, hence they all have the same positive Haar measure. But the $h(K_n)$'s are pairwise disjoint; because the K_n's are, and h is one-to-one. So $m\{h(R)\} = \infty$, which contradicts the fact that $m(G) = 1$. This completes the proof.

COROLLARY 3.4 *For $0<p<1$ the dual space of $H^p(G)$ has dimension* 1.

PROOF. It is enough to prove the result with $T^+ = T_{\Gamma^+}(G)$ in place of $H^p(G)$. Let

$$\lambda_0(f) = \hat{f}(0) \quad (f \text{ in } T^+).$$

By Corollary 3.2 the linear functional λ_0 is L^p-continuous on T^+. Suppose λ is *any* L^p-continuous linear functional on T^+. Since $E = \Gamma^+ \setminus \{0\}$ is relatively open in Γ, Theorem 3.3 asserts that $L^p_E(G)$ has trivial dual: in particular λ must vanish on the dense subspace $T_E(G)$. But $T_E(G)$ is the null space of λ_0, so λ is a scalar multiple of λ_0, and the proof is complete.

A similar result holds when Γ^+ is replaced by a closed interval in Γ, and suggests an interesting problem.

COROLLARY 3.5. *Suppose $a, b \in \Gamma$ with $a<b$. Let $E = \Gamma \cap [a,b]$. Then the dual space of $L^p_E(G)$ has dimension two when $0<p<1$.*

PROOF. For f in $T_E(G)$ let

$$\lambda_a(f) = \hat{f}(a) \quad \text{and} \quad \lambda_b(f) = \hat{f}(b).$$

We claim that both these linear functionals are L^p-continuous. For given f in $T_E(G)$ define F in $T_{\Gamma^+}(G)$ by

$$F(x) = (x,b)\overline{f(x)} \quad (x \text{ in } G),$$

where b is now viewed as a character on G. Using Corollary 3.2 on the analytic polynomial F:

$$|\lambda_b(f)| = |\hat{f}(b)| = |\hat{F}(0)| \leqq \|F\|_p = \|f\|_p$$

which establishes the continuity of λ_b. A similar argument works for λ_a.

Let $E_0 = E \setminus \{a,b\}$. Then E_0 is relatively open in Γ, so Theorem 3.3 implies that

$$T_{E_0}(G) = \ker \lambda_a \cap \ker \lambda_b$$

has trivial dual in the L^p-topology. Thus any L^p-continuous linear functional on $T_E(G)$ must vanish on the null spaces of both λ_a and λ_b, and must therefore be a linear combination of these two functionals. This completes the proof.

It would be of interest to know if a similar result holds when E is a finite disjoint union of closed intervals in Γ. If E_0 is the interior of such an E (relative to Γ), then Theorem 3.3 insures that $L^p_{E_0}$ has trivial dual, so the argument given above shows that the dimension of $[L^p_E(G)]'$ is $\leqq$ twice the number of these intervals. However it is not clear that the endpoints of these intervals induce continuous linear functionals on $L^p_E(G)$, even when there are only two intervals.

4. Modification sets span $L^p(G)$

In this section we use Theorem 2.1 to prove the following theorem of de Leeuw, and we use de Leeuw's result to show that rather thin sets of characters can span dense linear subspaces of $L^p(G)$. Here G is any compact abelian group.

THEOREM 4.1 [4, theor. 3.1, p. 113]. *Suppose $\mu \in M_s(G)$, $E = \operatorname{spec}\mu$, and F is a finite subset of E. Then $E \setminus F$ spans a dense linear subspace of $L^p_E(G)$ for all $0 < p < 1$.*

PROOF. Theorem 2.1 asserts that $T_E(G)$ has no nontrivial L^p-continuous linear functional, so the same is true of the quotient space $T_E(G)/X$, where X is the L^p-closure of $T_{E \setminus F}(G)$ in $T_E(G)$. We want to show that $X = T_E(G)$. In any case X has finite codimension in $T_E(G)$, so if it is not the whole space, then $T_E(G)/X$ is a nontrivial finite dimensional Hausdorff linear topological space, hence is isomorphic to the complex Euclidean space C^n for some $n > 0$ [8, theor. 7.3, p. 59], and therefore has a nontrivial continuous linear functional. But this is impossible, so $X = T_E(G)$, and the proof is complete.

A subset M of Γ is called a *modification set* if for every f in $L^1(G)$ there exists μ in $M_s(G)$ such that $\hat{\mu} = \hat{f}$ off M. That is, M is a modification set if every f in $L^1(G)$ can be converted into a singular measure by modifying its Fourier transform only on M. Rudin [13], [14] has proved that rather small modification sets exist in many groups. In particular he has found modification sets in $\mathbf{Z}$ of density zero [14].

COROLLARY 4.2. *If $M \subset \Gamma$ is a modification set then M spans a dense linear subspace of $L^p(G)$ for all $0 < p < 1$.*

PROOF. It is enough to show that each character not already in M belongs to $L^p_M(G)$. Fix α in $\Gamma \setminus M$ and choose μ in $M_s(G)$ so that $\hat{\mu} = \hat{\alpha}$ off M; hence $\alpha \in \operatorname{spec}\mu \subset M \cup \{\alpha\}$. Letting $E = \operatorname{spec}\mu$ we have from Theorem 4.1:

$$\alpha \in L^p_E(G) = L^p_{E \setminus \{\alpha\}}(G) \subset L^p_M(G),$$

which completes the proof.

This result, along with Rudin's construction of modification sets in $\mathbf{Z}$ of density zero yields:

Corollary 4.3. *There exist subsets $E=\{n_k\}$ of $\mathbf{Z}$ having density zero for which the exponentials $\{e^{in_k\theta}\}$ span a dense linear subspace of $L^p(\mathbf{T})$ for all $0<p<1$.*

Remarks. (a) This last corollary, in some ways a disturbing complement to Corollary 2.2, shows that $L^p_E(G)$ may have trivial dual simply because it coincides with $L^p(G)$, even if E is rather small. This raises a general question: *given a set E of characters, what is $\Gamma\cap L^p_E(G)$?* We will say more about this problem in the next section. In general it appears to be quite difficult.

(b) There is also a related question: *if E is the spectrum of a singular measure, is $L^p_E(G)$ linearly homeomorphic with $L^p(G)$?*

5. Separating duals and absolutely continuous measures

In this section we use Theorem 2.1 and de Leeuw's original proof of Theorem 4.1 to generalize the F. and M. Riesz theorem. The following notation will be convenient: for $E\subset\Gamma$ and $0<p<1$, let $[E]_p=L^p_E(G)\cap\Gamma$. Note that:

$$\text{(5.1)}\qquad \begin{aligned}&\text{(a)}\quad E\subset[E]_p.\\ &\text{(b)}\quad \text{If } F=[E]_p \text{ then } L^p_F(G)=L^p_E(G).\\ &\text{(c)}\quad [E_1\cap E_2]_p\subset[E_1]_p\cap[E_2]_p.\end{aligned}$$

In this notation Theorem 4.1 asserts that $[E]_p=[E\setminus F]_p$ whenever E is the spectrum of a singular measure and F is a finite subset of E; and Corollary 4.2 shows that $[M]_p=\Gamma$ for every modification set M.

For μ in $M(G)$ let μ_a and μ_s be respectively the absolutely continuous and singular parts of μ with respect to the Haar measure m. The main result of this section is:

Theorem 5.1. *If $\mu\in M(G)$ then* $[\operatorname{spec}\mu]_p$ *contains both* $\operatorname{spec}\mu_a$ *and* $\operatorname{spec}\mu_s$ *for all $0<p<1$.*

Proof. (cf. de Leeuw [4, theor. 3.1, p. 113]. We use the approximate identity $(k_V : V\in\mathcal{S})$ that appeared in Lemma 1.1 and in the proof of Theorem 2.1. A routine argument shows that $k_V*f\to f$ in $L^1(G)$ for each f in $L^1(G)$.

Let $E=\operatorname{spec}\mu$ and write $d\mu_a(x)=f(x)\,dx$ where $f\in L^1(G)$. Then as in the proof of Theorem 2.1 there is a sequence (V_n) in $\mathcal{S}$ such that

$$\text{(5.2)}\qquad \|k_{V_n}*f-f\|_1\to 0$$

and

$$\| k_{V_n} * \mu_s \|_p \to 0 . \tag{5.3}$$

Thus the sequence $k_{V_n} * \mu$ converges to f in $L^p(G)$. Since each $k_{V_n} * \mu$ belongs to $L^1_E(G)$, it also belongs to $L^p_E(G)$, hence so does f.

Let $\langle f \rangle$ denote the closure in $L^1(G)$ of the linear span of the translates of f. Then $\langle f \rangle$ is a closed, translation invariant subspace of $L^1(G)$ which contains f, so by Lemma 1.2 it also contains $\operatorname{spec} f = \operatorname{spec} \mu_a$. Since $f \in L^p(G)$ so is every translate of f, hence $\langle f \rangle \subset L^p_E(G)$. Thus $\operatorname{spec} \mu_a$ lies in $L^p_E(G)$, and therefore in $[E]_p$. Since both $\hat{\mu}$ and $\hat{\mu}_a$ vanish outside $[E]_p$ so does $\hat{\mu}_s = \hat{\mu} - \hat{\mu}_a$, and this completes the proof.

An immediate consequence of this result and Theorem 2.1 is the following, which is the "generalized F. and M. Riesz theorem" mentioned in the Introduction.

Corollary 5.2. *Suppose $E \subset \Gamma$, $0 < p < 1$, and $F = [E]_p$. If $T_F(G)$ has enough L^p-continuous linear functionals to separate points, then every E-spectral measure on G is absolutely continuous.*

Proof. If μ is an E-spectral measure on G that is *not* absolutely continuous, then by Theorem 5.1

$$S = \operatorname{spec} \mu_s \subset [\operatorname{spec} \mu]_p \subset [E]_p = F$$

so $T_S(G) \subset T_F(G)$. By Theorem 2.1 the space $T_S(G)$ has no non-trivial L^p-continuous linear functionals, so each L^p-continuous linear functional on $T_F(G)$ must vanish on $T_S(G)$. Since the latter space is non-trivial, it follows that $T_F(G)$ does not have enough L^p-continuous linear functionals to separate points. This completes the proof.

Remarks. (a) Corollary 5.2 clearly implies that *if $L^p_E(G)$ has separating dual for some $0 < p < 1$, then every E-spectral measure on G is absolutely continuous.* For example when $G = \mathbf{T}$ and E is the non-negative integers, we noted in section 3 that $L^p_E(\mathbf{T})$ is isometrically isomorphic to the Hardy space H^p of the open unit disc. Since H^p has separating dual [5, ch. 7, p. 118], so does $L^p_E(\mathbf{T})$, and we have a proof of the F. and M. Riesz theorem. In a few moments we will give a simpler proof in which $T_E(\mathbf{T})$ is shown directly to have enough L^p-continuous linear functionals to separate points, and the existence of H^p is completely ignored.

(b) We do not know if $L^p_E(G)$ must have separating dual whenever $T_E(G)$, taken in the L^p-topology, does. More generally we do not know if the dual of a topological vector space must separate points whenever it separates the points of a dense subspace.

Corollary 5.2 raises once again the problem mentioned in Remark (a) of section 4. Restated in the notation of this section it is: *given* $E \subset \Gamma$ *and* $0<p<1$, *find* $[E]_p$. Not much seems to be known about this problem other than the few results we have already mentioned, and the following one which we need to efficiently recover the F. and M. Riesz, and Bochner theorems.

LEMMA 5.3. *If G is a compact abelian group with ordered dual Γ, then* $[\Gamma^+]_p = \Gamma^+$ *for every* $0<p<1$.

PROOF. Suppose $\alpha < 0$ and f is a trigonometric polynomial on G of analytic type. Then

$$F(x) = (x, -\alpha) f(x) - 1$$

is also a trigonometric polynomial of analytic type, with $\hat{F}(0) = -1$ and $|F| = |f - \alpha|$ on G. These observations, along with Corollary 3.2, yield

$$\|\alpha - f\|_p = \|F\|_p \geqq |\hat{F}(0)| = 1$$

so $\operatorname{dist}(\alpha, H^p(G)) = 1$. In particular,

$$\alpha \notin H^p(G) \cap \Gamma = [\Gamma^+]_p$$

and the proof is complete.

We next give a sufficient condition for $T_E(G)$ to have enough L^p-continuous linear functionals to separate points. In what follows, $E - \alpha$ is the translate of the set $E \subset \Gamma$ by the character α, *not* the set-theoretic difference.

LEMMA 5.4. *Suppose G is a compact abelian group with ordered dual Γ, and E is a subset of Γ^+ such that $E - \alpha$ has at most finitely many negative elements for each α in E. Then $T_E(G)$ has enough L^p-continuous linear functionals to separate points for each* $0<p<1$.

PROOF. According to Corollary 3.2 the linear functional

$$\lambda_0(f) = \hat{f}(0) \quad (f \text{ in } T_\Gamma(G))$$

is L^p-continuous on $T_E(G)$, so it is also L^p-continuous on $T_{E \cup F}(G)$ for each finite subset F of Γ, since the latter space contains the former as a subspace of finite codimension. In particular λ_0 is L^p-continuous on $T_{E-\alpha}(G)$, say with norm M_α. Suppose $f \in T_E(G)$. Then

$$F(x) = (x, -\alpha) f(x) \quad (x \text{ in } G)$$

is an $(E-\alpha)$-trigonometric polynomial, hence

$$|\hat{f}(\alpha)| = |\hat{F}(0)| \leqq M_\alpha \|F\|_p = M_\alpha \|f\|_p,$$

which shows that the linear functional

$$\lambda_\alpha(f) = \hat{f}(\alpha) \quad (f \text{ in } T_\Gamma(G))$$

is L^p-continuous on $T_E(G)$ for each α in E. Since these functionals separate the points of $T_E(G)$, the proof is complete.

The F. and M. Riesz theorem now follows immediately from the last three results.

Corollary 5.5 (F. and M. Riesz [9], Rudin [12, theor. 8.2.1, p. 198], Duren [5, theor. 3.8, p. 41]). *If $\mu \in M(\mathbf{T})$ and $\hat{\mu}(n) = 0$ for all $n < 0$, then μ is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$.*

Proof. Take $G = \mathbf{T}$, m = normalized Lebesgue measure on $\mathbf{T}$, $\Gamma = \mathbf{Z}$, and $E = \Gamma^+ = \mathbf{Z}^+$ in Lemmas 5.3 and 5.4. It follows from these lemmas that $[E]_p = E$ and $T_E(\mathbf{T})$ has enough L^p-continuous linear functionals to separate points for each $0 < p < 1$. This, along with Corollary 5.2, completes the proof.

Remark. In addition to Theorem 5.1 the main element in this proof of the F. and M. Riesz theorem is Lemma 3.1, which is a non-trivial result in the theory of function algebras. However for the special case just considered it is an immediate consequence of the subharmonicity of $|f(z)|^p$ where f is a polynomial in the complex variable z.

These ideas also provide another proof of a theorem of Bochner. This time the generality of Lemma 3.1 is used in a more essential way. In what follows the ordered pair $(m, n) \in \mathbf{Z}^2$ is identified with the character $(\xi, \eta) \to \xi^m \eta^n$ on $\mathbf{T}^2$.

Corollary 5.6 [2], [12, theor. 8.2.5, p. 201]. *Suppose S is a plane sector of angular opening less than π radians. If $\mu \in M(\mathbf{T}^2)$ and $\operatorname{spec} \mu \subset S$, then μ is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}^2$.*

Proof. Without loss of generality we may assume that S has vertex at the origin. Since the sides of S make an angle of less than π radians we have $S = \Pi_\alpha \cap \Pi_\beta$ where Π_α and Π_β are half-planes bounded by the lines containing the sides of S. Let E, Γ_α, and Γ_β be the intersections with $\mathbf{Z}^2$ of S, Π_α, and Π_β respectively. Then Γ_α is the set of positive elements for an ordering of $\mathbf{Z}^2$ (see

[12, sec. 8.1]), so Lemma 5.3 insures that $[\Gamma_\alpha]_p = \Gamma_\alpha$. The same is true for Γ_β, and since $E = \Gamma_\alpha \cap \Gamma_\beta$ it follows from (5.1) that

$$E \subset [E]_p \subset [\Gamma_\alpha]_p \cap [\Gamma_\beta]_p = \Gamma_\alpha \cap \Gamma_\beta = E$$

that is, $[E]_p = E$.

So we need only show that $T_E(\mathbf{T}^2)$ has enough L^p-continuous linear functionals to separate points, by the above results and Corollary 5.2. To see this, consider $\mathbf{Z}^2$ in the ordering induced by any half-plane which contains $S \setminus \{0\}$ in its interior and whose boundary is a line of irrational slope through the origin. It is geometrically clear that for such an ordering the set E obeys the hypotheses of Lemma 5.4, and this completes the proof.

Remarks. (a) In [11, theor. 5.7] Rudin generalizes the F. and M. Riesz theorem as follows: *if E is the union of a $\Lambda(1)$ set with the positive integers, then every E-spectral measure on the circle is absolutely continuous.* We do not know if this result can be obtained from Corollary 5.2. That is, we do not know if the space of $[E]_p$-trigonometric polynomials has enough L^p-continuous linear functionals to separate points.

(b) Finally it should be observed that the results of this paper extend to certain Orlicz—type spaces, with exactly the same proofs. More precisely, suppose ϕ is a non-negative, continuous, strictly increasing concave function on $[0,\infty)$ which vanishes only at the origin. Then ϕ is automatically subadditive, and the space $L^\phi(G)$ consisting of (equivalence classes of) m-measurable complex valued functions f on G with

$$\|f\| = \int \phi(|f|)\, dm < \infty$$

is a complete linear topological space in the metric

$$d(f,g) = \|f-g\|.$$

Of course $L^p(G)$ is just the case $\phi(t) = t^p$ $(0<p<1)$. Suppose ϕ is *strongly concave*, that is, $\phi(t)/t \to 0$ as $t \to \infty$. *Then Theorems* 2.1, 3.3, 4.1, *and Corollary* 4.2 *all hold with $L^\phi_E(G)$ replacing $L^p_E(G)$; while Theorem* 5.1 *and Corollary* 5.2 *hold with $[E]_\phi = L^\phi_E(G) \cap \Gamma$ replacing $[E]_p$*. The proofs are the same once we observe that if (f_n) is a norm bounded sequence in $L^1(G)$, then $(\phi(|f_n|))$ is uniformly integrable.

The results that deal with existence of continuous linear functionals—Corollaries 3.2, 3.4, and 3.5—also generalize immediately to this situation if, in addition to being strongly concave,

$$\phi(t) = \psi(\log t)$$

where ψ is convex on the real line. Then Jensen's convexity theorem can be used in place of the arithmetic-geometric mean inequality to prove an analogue of Corollary 3.2, and the other results follow as before.

Added in Proof. Regarding Remark (b) following Corollary 5.2: it has been pointed out to us by Professor N. T. Peck that V. Klee has given an example of a metrizable linear topological space whose dual separates points, but does not separate points of the completion (*Exotic topologies for linear spaces*, Proceedings of Symposium on General Topology and its Relations to Modern Analysis, Prague, 1961, pp. 238–249). It is not known, however, if this can happen for $T_E(G)$ in the L^p topology $(0 < p < 1)$.

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