RANDOM DIRICHLET FUNCTIONS: MULTIPLIERS AND SMOOTHNESS[1]

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Abstract. We show that if $f(z) = \sum a_n z^n$ is a holomorphic function in the Dirichlet space of the unit disk, then almost all of its randomizations $\sum \pm a_n z^n$ are multipliers of that space. This parallels a known result for lacunary power series, which also has a version for smoothness classes: every lacunary Dirichlet series lies in the Lipschitz class $\mathrm{Lip}_{1/2}$ of functions obeying a Lipschitz condition with exponent 1/2. However, unlike the lacunary situation, no corresponding "almost sure" Lipschitz result is possible for random series: we exhibit a Dirichlet function with *no* randomization in $\mathrm{Lip}_{1/2}$. We complement this result with a "best possible" sufficient condition for randomizations to belong almost surely to $\mathrm{Lip}_{1/2}$. Versions of our results hold for weighted Dirichlet spaces, and much of our work is carried out in this more general setting.

Introduction

The Dirichlet space $\mathcal{D}$ of the open unit disc U of the complex plane is the set of holomorphic functions on U for which

$$D(f) \equiv \int_U |f'(z)|^2 \, dA(z) < \infty,$$

where the measure dA is two-dimensional Lebesgue measure normalized so that $A(U) = 1$. If $f(z) = \sum a_n z^n$ then it is easily seen that $D(f) = \sum n |a_n|^2$. The Dirichlet space is a Hilbert space under the norm defined by: $\|f\|^2 = D(f) + |f(0)|^2$.

A *multiplier* of the Dirichlet space is a holomorphic function ϕ on U such that the pointwise product $\phi(z)f(z) \in \mathcal{D}$ whenever $f \in \mathcal{D}$. If ϕ is a multiplier of $\mathcal{D}$, then by the closed graph theorem the multiplication operator $M_\phi : f \mapsto \phi f$ is a bounded linear operator on $\mathcal{D}$. The study of such operators on $\mathcal{D}$ and its various weighted generalizations has been attracting considerable attention; see for example [2], [7], [11], [12], and [15].

[1]Research supported in part by the National Science Foundation

Since the constant function $f(z) \equiv 1$ is in $\mathcal{D}$, every multiplier of $\mathcal{D}$ must itself be in $\mathcal{D}$. An elementary argument using the continuity of point evaluation functionals shows that every multiplier must also be bounded on the unit disk ([6], Lemma 11, page 57). Beyond this, however, the multipliers of $\mathcal{D}$ defy simple description. For example, there are bounded functions in $\mathcal{D}$ that are *not* multipliers of $\mathcal{D}$. Indeed, there are functions in $\mathcal{D}$, continuous on $\overline{U}$, that are not multipliers [15, Theorem 9].

There are two concrete characterizations of the multipliers of $\mathcal{D}$. One, due to D. Stegenga [14], involves a Carleson-type condition expressed in terms of logarithmic capacity. The other, due to Kerman and Sawyer [9], uses a Carleson-type maximal function. However both characterizations can be difficult to use in practice. For example, it is not clear how one could use either criterion to decide if the magnitudes of the Taylor coefficients of a multiplier of $\mathcal{D}$ must satisfy a condition that is stronger than the one that defines $\mathcal{D}$: $\sum n|a_n|^2 < \infty$.

Our first result, Theorem 1 in §1 below, shows that no stronger condition is possible: if $f(z) = \sum a_n z^n$ is in the Dirichlet space, then the series $\sum \pm a_n z^n$ is a multiplier of $\mathcal{D}$ for almost every random choice of signs. In particular, *for every Dirichlet function there is a multiplier with same sequence of "Taylor coefficient magnitudes."* So Taylor coefficient magnitude sequences behave no better for multipliers than they do for arbitrary Dirichlet functions. Theorem 1 also holds for more general randomizations of the coefficients of f.

Brown and Shields [3, Proposition 20], recently showed that if $f(z) = \sum a_k z^{n_k}$ is a Hadamard lacunary power series in $\mathcal{D}$ (i.e.,$n_{k+1}/n_k \geq \lambda > 1$), then f is a multiplier of $\mathcal{D}$, and in addition $f \in \mathrm{lip}_{1/2}$, the "little-oh" Lipschitz class with exponent 1/2. Noting that properties of lacunary series often have analogues for Rademacher series, Allen Shields suggested to us that these results might have random versions. We show that this analogy is correct for multipliers, but breaks down dramatically for Lipschitz spaces.

In §2 we show that if $\sum a_n z^n \in \mathcal{D}$, then the "randomization" $f_\omega(z) = \sum \pm a_n z^n$ will almost surely belong to Lip_γ for all $\gamma < 1/2$, but need not a.s. be in $\mathrm{Lip}_{1/2}$. We show that if the function f obeys the stronger condition $\sum n(\log n)|a_n|^2 < \infty$, then $f_\omega \in \mathrm{lip}_{1/2}$ almost surely, and this result is "best possible" in a very strong sense. For this we adapt methods of W.T. Sledd [13], P.L. Duren [5, Theorem 1], and Anderson, Clunie and Pommerenke [1, Theorem 3.7], who obtained similar regularity theorems involving spaces which include BMO and the Bloch space. Results like this also follow from work of J. P. Kahane [8, Theorem 3 of Ch. 7, Theorem 2 of Ch. 8].

In §3 we show that the random/lacunary analogy for Lipschitz spaces breaks down in the worst possible way: there is a series $\sum a_n z^n$ in $\mathcal{D}$ such that $\sum \pm a_n z^n$ is in $\mathrm{Lip}_{1/2}$ for *no* choice of signs.

All of our work generalizes to a familiar class of "weighted Dirichlet spaces." We emphasize this point of view in sections 2 and 3, where it is of interest to note the connection between the strength of the weight and the Lipschitz smoothness of randomizations of functions in the space.

1 Random Dirichlet functions are multipliers

Let $\epsilon_n(\omega)$ be a Bernoulli sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$. This means that the sequence is independent, and each ϵ_n takes the values $+1$ and -1 with probability $1/2$ each. For a concrete example of such a sequence, take ϵ_n to be the n^{th} Rademacher function on the unit interval, with Lebesgue measure as the probability measure.

If $f(z) = \sum a_n z^n$ is analytic in the unit disk U, let

$$f_\omega(z) = f(z, \omega) \equiv \sum_n \epsilon_n(\omega) a_n z^n.$$

Since the series representing f converges absolutely in U, the function $f_\omega(\cdot)$ is analytic in U for each $\omega \in \Omega$. Furthermore if $f \in \mathcal{D}$ then $f_\omega \in \mathcal{D}$ for each $\omega \in \Omega$.

Theorem 1 *If $f \in \mathcal{D}$ then f_ω is a multiplier of $\mathcal{D}$ almost surely.*

The proof of this theorem uses Khintchine's inequality and a sufficient condition, due to Brown and Shields, for a function to be a multiplier of $\mathcal{D}$. We state these tools below as lemmas.

Lemma 1.1 (Khintchine's Inequality) [16, Thm.V.8.4]. *Let $\{a_n\}$ be a sequence of complex numbers and let $X(\omega) = \sum \epsilon_n(\omega) a_n$. Then for all $0 < p < \infty$,*

$$\|X\|_{L^p(\Omega)} \approx \|X\|_{L^2(\Omega)} = \left(\sum |a_n|^2\right)^{1/2}.$$

The last equality above holds because the ϵ_n's are orthonormal in $L^2(\Omega)$.

For the Brown - Shields sufficient condition, we introduce the following standard notation: If f is holomorphic in U, then for all $0 < p < \infty$ and all $0 \le r < 1$ let

$$M_p(f, r) \equiv \left(\int_{\partial U} \left|f\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi}\right)^{1/p}.$$

Similarly let

$$M_\infty(f,r) \equiv \max_\theta \left| f\left(re^{i\theta}\right)\right|.$$

LEMMA 1.2 [3, Proposition 19]. *Let $\phi(z)$ be a holomorphic function on U, and suppose that for some $2 < p \leq \infty$, $M_p(\phi', r) \in L^2([0,1], dr)$. Then ϕ is a multiplier of $\mathcal{D}$.*

The lemma fails when $p = 2$, since the condition that $M_2(\phi', r) \in L^2(dr)$ says nothing more than that ϕ is in the Dirichlet space.

Proof of Theorem 1. Let $f(z) = \sum a_n z^n$ be in the Dirichlet space, and let $f_\omega(z) = \sum \pm a_n z^n$ be the randomization of f. By the lemma above, it suffices to show that

$$\int_0^1 M_p^2(f'_w, r)\, dr < \infty \qquad \text{almost surely}$$

for some $p > 2$. In fact this is true for *all* $2 < p < \infty$. (The case $p = \infty$ is discussed in §2.) Fix $2 < p < \infty$. We will show that the expectation of $\int M_p^2(f'_\omega, r)\, dr$ is finite, from which the result will follow.

Using respectively: Fubini's theorem, Jensen's inequality, Fubini's theorem, and Khintchine's inequality, we obtain:

$$\begin{aligned}
\mathbf{E}\left(\int_0^1 M_p^2(f'_w, r)\, dr\right) &= \int_0^1 \int_\Omega \left[\int_0^{2\pi} \left|f'_\omega\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi}\right]^{2/p} dP\, dr \\
&\leq \int_0^1 \left[\int_\Omega \int_0^{2\pi} \left|f'_\omega\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi}\, dP\right]^{2/p} dr \\
&= \int_0^1 \left[\int_0^{2\pi} \left\|f'_\omega\left(re^{i\theta}\right)\right\|_{L^p(\Omega)}^p \frac{d\theta}{2\pi}\right]^{2/p} dr \\
&\leq C_p \int_0^1 \left[\int_0^{2\pi} \left\|f'_\omega\left(re^{i\theta}\right)\right\|_{L^2(\Omega)}^p \frac{d\theta}{2\pi}\right]^{2/p} dr \\
&= C_p \int_0^1 \left[\int_0^{2\pi} \left(\sum n^2 |a_n|^2 r^{2n-2}\right)^{p/2} \frac{d\theta}{2\pi}\right]^{2/p} dr \\
&= C_p \int_0^1 \left(\sum n^2 |a_n|^2 r^{2n-2}\right) dr
\end{aligned}$$

$$
\begin{aligned}
&= C_p \sum_{n=1}^{\infty} n^2 |a_n|^2 \frac{1}{2n-1} \\
&\le C_p \sum_{n=1}^{\infty} n |a_n|^2 < \infty . \qquad ///
\end{aligned}
$$

Theorem 1 can be generalized in two directions. One direction involves more general randomizations of the coefficients.

Corollary 1 . *Let $f(z) = \sum a_n z^n$ be in the Dirichlet space and let $f_\omega(z) = \sum X_n a_n z^n$ where $\{X_n(\omega)\}$ is a sequence of independent complex random variables satisfying*

- $\mathbf{E}\left(|X_n|^2\right) \le C < \infty$ *for all n;*
- $-X_n$ *has the same distribution as X_n.*

Then f_ω is a multiplier of $\mathcal{D}$ almost surely.

For example, these hypotheses are satisfied when X_n is uniformly distributed on the unit circle ∂U, or when X_n has a centered Gaussian distribution on the complex plane.

Proof of Corollary 1. The proof follows from what Kahane calls the "principle of reduction" [8, Sec. 1.7]. Let $\{X_n\}$ be a sequence of random variables as in the statement of the corollary and let $\{\epsilon_n\}$ be a Bernoulli sequence independent of the X_n's. Let $Y_n = \epsilon_n X_n$. By the independence, $\{Y_n\}$ can be realized on a product probability space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, P_1 \times P_2)$ such that the X_n's are defined on $(\Omega_1, \mathcal{A}_1, P_1)$ and the ϵ_n's are defined on $(\Omega_2, \mathcal{A}_2, P_2)$. Let $\mathbf{E}_1$ denote expectation with respect to P_1.

Let $f(z) = \sum a_n z^n$ be in the Dirichlet space. Then

$$\mathbf{E}_1 \left(\sum n |a_n X_n|^2\right) \le C \sum n |a_n|^2 < \infty,$$

so that

$$P_1 \left\{ \omega_1 : \sum n |a_n X_n(\omega_1)|^2 < \infty \right\} = 1.$$

Let Ω_1' denote the set in braces in the equation above. For each fixed $\omega_1 \in \Omega_1'$, Theorem 1 implies that

$$\sum \epsilon_n(\omega_2) X_n(\omega_1) a_n z^n$$

is a multiplier of $\mathcal{D}$ a.s.$[P_2]$. By Fubini's theorem,

$$P_2 \left\{ \omega_2 : \sum \epsilon_n(\omega_2) X_n(\omega_1) a_n z^n \text{ is a multiplier of } \mathcal{D} \text{ } a.s. \text{ } [P_1] \right\} = 1.$$

In particular there exists a choice of $\pm$ signs such that $\sum \pm X_n(\omega_1) a_n z^n$ is a multiplier a.s.$[P_1]$. Since $-X_n$ and X_n are identically distributed and the X_n's are independent, it follows that $\sum X_n(\omega_1) a_n z^n$ is a multiplier of $\mathcal{D}$ a.s.$[P_1]$. ///

The second direction of generalization involves weighted Dirichlet spaces. For each $0 \leq \alpha < \infty$, let $\mathcal{D}_\alpha$ denote the set of holomorphic functions f on U for which

$$D_\alpha(f) \equiv \sum_{n=1}^{\infty} n^\alpha |a_n|^2 < \infty.$$

A straightforward computation with power series, using Lemma 2.4 below, shows that if $f(z) = \sum a_n z^n$, then for $0 \leq \alpha < 2$,

$$D_\alpha(f) \approx \int_U |f'(z)|^2 (1 - |z|)^{1-\alpha} \, dA(z)$$

(see, for example, [15], Lemma 2). These weighted Dirichlet spaces $\mathcal{D}_\alpha$ form a monotonically decreasing family of Hilbert spaces, with $\mathcal{D}_1$ the original Dirichlet space $\mathcal{D}$, and $\mathcal{D}_0$ the Hardy space H^2. It is easy to see that the multipliers of the Hardy space are precisely the class of bounded holomorphic functions on U. However for the weighted Dirichlet spaces $\mathcal{D}_\alpha$ with $0 < \alpha < 1$ the situation is just as complicated as it is for $\mathcal{D} = \mathcal{D}_1$ (see [9], [14]).

For $\alpha > 1$ the situation reverts to simplicity. In this case the Hilbert space $\mathcal{D}_\alpha$ is *closed under pointwise multiplication* ([15], Theorem 7), so every function in the space is a multiplier (and, as for the unweighted Dirichlet space, it is easy to see that conversely, every multiplier is in the space). Theorem 1 can therefore be regarded as a random substitute for this phenomenon when $\alpha = 1$. We claim that the same kind of random substitute exists for each $0 < \alpha < 1$. For this we need a version of Lemma 1.2 for the spaces $\mathcal{D}_\alpha$.

LEMMA 1.3 *Let ϕ be holomorphic in U, $0 < \alpha \leq 1$, and suppose that for some $p > 2/\alpha$,*

$$\int_0^1 M_p^2(\phi', r)(1 - r)^{1-\alpha} \, dr < \infty.$$

Then ϕ is a multiplier of $\mathcal{D}_\alpha$.

The proof of this lemma is very similar to the proof of Proposition 19 in [3], and we omit the details, except to say that one needs to show that for $0 < \alpha < 1$ the space $\mathcal{D}_\alpha$ is contained in the Hardy space $H^{2/(1-\alpha)}$ (either by fractional integration or by interpolation; note that $\mathcal{D}_1 = \mathcal{D}$ is contained in *BMO* by Hardy's inequality), and then one uses Hölder's inequality as in [3].

Using this sufficient condition one arrives at the following theorem:

Theorem 2 *Suppose that f is in $\mathcal{D}_\alpha$ ($0 < \alpha \leq 1$) and f_ω is a randomization of the type considered in Corollary 1. Then f_ω is a multiplier of $\mathcal{D}_\alpha$ almost surely.*

Proof of Theorem 2. Follow the proof of Theorem 1 to show that

$$\mathbf{E}\left(\int_0^1 M_p^2\left(\phi', r\right)(1-r)^{1-\alpha}\, dr\right) \leq C_p \sum n^2 \left|a_n\right|^2 \int_0^1 r^{2n-2}(1-r)^{1-\alpha}\, dr.$$

The integral on the right hand side is the beta function $\beta(2n-1, 2-\alpha)$. From the beta function inequality (Lemma 2.4 of §2 below),

$$\beta\left(2n-1, 2-\alpha\right) \leq C_\alpha \frac{1}{(2n-1)^{2-\alpha}} \leq C_\alpha n^{\alpha-2}$$

for $n \geq 1$. So the expectation above is smaller than $C_{p,\alpha} \sum n^\alpha \left|a_n\right|^2$, which is finite since f is in $\mathcal{D}_\alpha$. Thus the sufficient condition in Lemma 1.3 is satisfied almost surely. ///

2 Smoothness of random Dirichlet functions

"Smoothness" in this section is expressed by Lipschitz conditions. For $0 < \gamma \leq 1$ we denote by Lip_γ the class of functions holomorphic on U for which there exists a constant $M < \infty$ such that $|f(z) - f(w)| \leq M|z-w|^\gamma$ for all $z, w \in U$. Clearly each such function extends continuously to the closed unit disc, on which it obeys the same modulus of continuity estimate. If, in addition, $\gamma < 1$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(w)| \leq \epsilon|z-w|^\gamma$ whenever $|z-w| < \delta$, (a condition that is non-trivial only as z and w approach the boundary), then we say $f \in \mathrm{lip}_\gamma$.

The connection between Lipschitz classes and the Dirichlet space comes from the following result of Hardy and Littlewood, which ties the boundary smoothness of a holomorphic function to the growth of its derivative.

Lemma 2.1 [4, Thm. 5.1, p. 74]. *Suppose f is holomorphic on U and $0 < \gamma < 1$. Then $f \in \mathrm{Lip}_\gamma$ if and only if*

$$M_\infty(f', r) = O\left(\frac{1}{1-r}\right)^{1-\gamma} \qquad (r \to 1-).$$

The corresponding result for lip_α holds as well: just replace "big-oh" by "little-oh" in the growth condition.

To observe this theorem in action, we turn to the work of Brown and Shields, who, in the course of proving that lacunary Dirichlet functions are multipliers, showed [3, Prop. 20] that each such function obeys the growth condition

$$\int_0^1 M_\infty^2(f', r)\, dr < \infty.$$

From this, and the fact that the sup-norm mean $M_\infty(f', r)$ increases with r, it follows easily that

$$M_\infty(f', r) = o\left(\frac{1}{1-r}\right)^{1/2}.$$

Along with Lemma 2.1, this last estimate allowed Brown and Shields to conclude that: *Every lacunary series in the Dirichlet space belongs to* $\mathrm{lip}_{1/2}$.

The goal of this section is twofold: We show that this last result does not have a "random" analogue, but that the slightly stronger hypothesis $\sum n \log n\, |a_n|^2 < \infty$ *does* imply that $f_\omega(z) \in \mathrm{lip}_{1/2}$ almost surely. Moreover, this sufficient condition is best possible in a very strong sense.

We present this material in the more general setting of weighted Dirichlet spaces in order to emphasize how containments between the original spaces control smoothness of the resulting randomizations. The statements above represent the special case $\alpha = 1$.

THEOREM 3 *Let $0 \le \alpha < 2$ and let $f(z,\omega) = \sum \epsilon_n(\omega) a_n z^n$, where $\{\epsilon_n\}$ is a Bernoulli sequence of random variables.*

(a) If $\sum n^\alpha \log n \, |a_n|^2 < \infty$,then almost surely

$$\int_0^1 M_\infty^2(f'_\omega, r)(1-r)^{1-\alpha} dr < \infty.$$

Consequently if $\alpha > 0$ then almost surely $f_\omega(z) \in \mathrm{lip}_{\alpha/2}$.

(b) On the other hand, given a sequence $c_n \searrow 0$, one can choose coefficients $a_n > 0$ such that $\sum c_n n^\alpha \log n \, |a_n|^2 < \infty$, but with

$$M_\infty\left(f'_\omega, r\right) \neq O\left(\frac{1}{1-r}\right)^{1-\alpha/2}, \qquad \text{almost surely.}$$

Consequently if $\alpha > 0$ then almost surely $f_\omega(z) \notin \mathrm{Lip}_{\alpha/2}$.

Note that part (a) of this theorem shows that if $f \in \mathcal{D}_\alpha$, then almost surely $f_\omega \in \mathrm{Lip}_\gamma$ whenever $\gamma < \alpha/2$, while part (b) shows that this conclusion does not extend to $\gamma = \alpha/2$.

The proof of part (a) of Theorem 3 follows closely that of the main theorem in Duren's paper [5], while the proof of part (b) follows that of Theorem 3.7 in Anderson, Clunie and Pommerenke [1]. For the sake of completeness, we present these arguments in some detail. Both rely upon the following fundamental estimate, due to Salem and Zygmund, of the size of the L^∞ norm of a random trigonometric polynomial:

LEMMA 2.2 [10, Theorems 4.3.1, 4.6.1, 4.9.9]. *For $\epsilon_k(\omega)$ a Bernoulli sequence of random variables, consider the random Fourier series $f_\omega(\theta) = \sum \epsilon_k(\omega) c_k e^{ik\theta}$, and let $s_n(\omega,\theta)$ be the n^{th} partial sum of f_ω. Let*

$$\sigma_n = \|s_n\|_2 = \left(\sum_{k=1}^n |c_k|^2\right)^{1/2}.$$

(a) There are constants C_ω, depending on ω but not on n, such that almost surely $\|s_n(\omega,\cdot)\|_\infty \le C_\omega (\log n)^{1/2} \sigma_n$.

(b) If in addition $\{c_k\}$ satisfies the regularity condition

$$\sum_{k=1}^{n} |c_k|^4 \leq \frac{const.}{n} \left(\sum_{k=1}^{n} |c_k|^2 \right)^2 ,$$

then the opposite inequality holds: almost surely there are positive constants C'_ω such that $\|s_n(\omega, \cdot)\|_\infty \geq C'_\omega (\log n)^{1/2} \sigma_n$ for n sufficiently large.

The proof given below of part (a) of Theorem 3 uses two other well-known inequalities which we list below for easy reference.

Lemma 2.3 (Hilbert's Inequality) . *For any sequence $\{b_j\}$ of complex numbers,*

$$\left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j+k} b_j \overline{b_k} \right| \leq \pi \sum_{j=1}^{\infty} |b_j|^2 .$$

Lemma 2.4 (Beta function inequality) .
For $x \geq 1$ and $\alpha > 0$, let

$$\beta(x, \alpha) \equiv \int_0^1 t^{x-1} (1-t)^{\alpha - 1} dt.$$

Then

$$\beta(x, \alpha) \leq C_\alpha \frac{1}{x^\alpha},$$

where C_α is a constant that depends only on α.

We omit the proof of Hilbert's inequality, and instead refer the reader to [4, Cor. to Thm. 3.14, p. 48]. As for the beta function, it is well-known that $\beta(x, \alpha) = \Gamma(x)\Gamma(\alpha)/\Gamma(x+\alpha)$, where $\Gamma(z)$ is the gamma function. The desired inequality now follows from Stirling's formula.

Proof of Theorem 3. (a) Let $f_\omega(z) = \sum \pm a_n z^n$ with

$$\sum n^\alpha \log n \, |a_n|^2 < \infty.$$

Notice that

$$z f'_\omega(z) = \sum_{1}^{\infty} \epsilon_n(\omega) n a_n r^n e^{in\theta},$$

where $z = re^{i\theta}$. Let

$$S_n(\omega, \theta) = \sum_{k=1}^{n} \epsilon_k(\omega) k a_k e^{ik\theta},$$

and let

$$\sigma_n^2 = \|S_n(\omega,\cdot)\|^2_{L^2[0,2\pi]} = \sum_{k=1}^{n} k^2 |a_k|^2 .$$

By Lemma 2.2, there is an exceptional subset $E \subset \Omega$, of probability zero, such that for every $\omega \in \Omega \backslash E$, there is a constant C_ω, depending on ω but not on n, such that

$$\max_\theta |S_n(\omega,\theta)| \le C_\omega \sigma_n \sqrt{\log n}.$$

For the rest of the proof of part (a) we fix a point $\omega \notin E$. For this particular ω we are going to show that $f_\omega \in \mathrm{lip}_{\alpha/2}$.

The first thing to note is that S_n is related to f'_w by the formula

$$z f'_\omega(z) = (1-r) \sum_{n=1}^{\infty} S_n(\omega,\theta)\, r^n.$$

So for each $\theta \in [0, 2\pi]$,

$$|z f'_\omega(z)| \le (1-r)\, C_\omega \sum_{n=1}^{\infty} \sigma_n (\log n)^{1/2}\, r^n,$$

and so

$$M_\infty(f'_\omega, r) \le C_\omega (1-r) \sum_{n=1}^{\infty} \sigma_n (\log n)^{1/2}\, r^{n-1}.$$

Now by the beta function inequality,

$$\begin{aligned}
&\int_0^1 (1-r)^{1-\alpha} \left[M_\infty(f'_\omega, r)\right]^2 dr \\
&\le C_\omega^2 \int_0^1 (1-r)^{3-\alpha} \left[\sum_{n=1}^{\infty} \sigma_n (\log n)^{1/2}\, r^{n-1}\right]^2 dr \\
&= C_\omega^2 \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(\sigma_j \sqrt{\log j}\right)\left(\sigma_k \sqrt{\log k}\right) \int_0^1 (1-r)^{3-\alpha}\, r^{j+k-2}\, dr. \\
&\le C_{\omega,\alpha} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(\sigma_j \sqrt{\log j}\right)\left(\sigma_k \sqrt{\log k}\right) \left(\frac{1}{j+k}\right)^{4-\alpha}.
\end{aligned}$$

Using respectively the fact that $j+k \geq 2\sqrt{j}\sqrt{k}$, and Lemma 2.3, we see that the expression in the last line above is less than or equal to

$$C_{\omega,\alpha} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j+k} \left(\frac{\sigma_j \sqrt{\log j}}{(\sqrt{j})^{3-\alpha}} \right) \left(\frac{\sigma_k \sqrt{\log k}}{\left(\sqrt{k}\right)^{3-\alpha}} \right)$$

$$\leq \pi C_{\omega,\alpha} \sum_{k=1}^{\infty} \frac{\sigma_k^2 \log k}{k^{3-\alpha}},$$

.

Recall that $\sigma_k^2 = \sum_1^k n^2 |a_n|^2$, so since $0 \leq \alpha < 2$,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\sigma_k^2 \log k}{k^{3-\alpha}} &= \sum_{k=1}^{\infty} \frac{\log k}{k^{3-\alpha}} \left(\sum_{n=1}^{k} n^2 |a_n|^2 \right) \\
&= \sum_{n=1}^{\infty} n^2 |a_n|^2 \left(\sum_{k=n}^{\infty} \frac{\log k}{k^{3-\alpha}} \right) \\
&\leq \sum_{n=1}^{\infty} n^2 |a_n|^2 \frac{C \log n}{n^{2-\alpha}} \\
&= C \sum n^{\alpha} \log n \, |a_n|^2 < \infty.
\end{aligned}$$

(In the second line of this display we estimate the sum that involves $(\log k)/k^{3-\alpha}$ by an integral.) We conclude that

$$\int_0^1 (1-r)^{1-\alpha} \left[M_\infty \left(f_\omega', r \right) \right]^2 dr < \infty$$

To show that $f_\omega \in \mathrm{lip}_{\alpha/2}$, notice that since $M_\infty(f', r)$ is an increasing function of r,

$$\begin{aligned}
\int_r^1 (1-s)^{1-\alpha} \left[M_\infty \left(f_\omega', s \right) \right]^2 ds &\geq \left[M_\infty \left(f_\omega', r \right) \right]^2 \int_r^1 (1-s)^{1-\alpha} \, ds \\
&= \left[M_\infty \left(f_\omega', r \right) \right]^2 \frac{(1-r)^{2-\alpha}}{2-\alpha}.
\end{aligned}$$

But the left-hand side tends to 0 as $r \to 1$, so

$$M_\infty \left(f_w', r \right) = o \left(\frac{1}{1-r} \right)^{1-\alpha/2}.$$

If $\alpha > 0$, then by Lemma 2.1, $f_\omega \in \mathrm{lip}_{\alpha/2}$, as promised.

Proof of (b). Let $\{c_n\}$ be a sequence of positive constants, decreasing monotonically to 0. Choose integers $\{n_k\}$ (as in [1, Thm.3.7]) which initially satisfy:

- $n_0 = 1$;
- $n_k > 2n_{k-1}$ for $k = 1, 2, \ldots$;
- $\sum (c_{n_k})^{1/2} < \infty$.

Define $a_n > 0$ by $a_1 = 1$ and

$$n^\alpha a_n^2 = \frac{1}{n_k \log n_k \sqrt{c_{n_{k-1}}}} \qquad (n_{k-1} < n \leq n_k).$$

Then

$$\sum_{n=1}^{\infty} c_n a_n^2 n^\alpha \log n \leq \sum_{k=1}^{\infty} c_{n_{k-1}} \log n_k \sum_{n=n_{k-1}+1}^{n_k} n^\alpha a_n^2 \leq \sum_{k=1}^{\infty} \sqrt{c_{n_{k-1}}} < \infty.$$

We will apply Lemma 2.2 to the Cesàro means of the partial sums of f'_ω. Let

$$s_{n_k}(\omega, \theta) = \sum_{j=1}^{n_k} \left(1 - \frac{j}{n_k}\right) j a_j \epsilon_j(\omega) e^{ij\theta}.$$

Let $\sigma_n = \|s_n\|_2$. Since $n_k > 2n_{k-1}$,

$$\begin{aligned}
\sigma_{n_k}^2 &\geq \sum_{\frac{n_k}{2} \leq j \leq \frac{3n_k}{4}} \left(1 - \frac{j}{n_k}\right)^2 j^2 a_j^2 \\
&\geq \frac{n_k}{4} \frac{1}{4^2} \left(\frac{n_k}{2}\right)^{2-\alpha} \cdot \frac{1}{n_k \log n_k \sqrt{c_{n_{k-1}}}}.
\end{aligned}$$

So

$$\sigma_{n_k} \sqrt{\log n_k} \geq C \lambda_k n_k^{(1-\alpha/2)},$$

where $\lambda_k = (c_{n_{k-1}})^{-1/4}$. Note that $\lambda_k \to \infty$ as $k \to \infty$.

Furthermore,

$$\begin{aligned}\sum_{j=n_{k-1}+1}^{n_k}\left(1-\frac{j}{n_k}\right)^4 j^4 a_j^4 &\le (n_k-n_{k-1})\, n_k^{4-2\alpha}\cdot\frac{1}{n_k^2\left(\log n_k\right)^2 c_{n_{k-1}}}\\ &\le C\frac{\sigma_{n_k}^4}{n_k}\ .\end{aligned}$$

In addition to the three original requirements placed on the sequence $\{n_k\}$, let us further demand that it grow fast enough so that

$$\sum_{j=n_{k-1}+1}^{n_k}\left(1-\frac{j}{n_k}\right)^4 j^4 a_j^4 \ge \sum_{j=1}^{n_{k-1}}\left(1-\frac{j}{n_k}\right)^4 j^4 a_j^4.$$

Thus

$$\sum_{j=1}^{n_k}\left(1-\frac{j}{n_k}\right)^4 j^4 a_j^4 \le 2C\frac{\sigma_{n_k}^4}{n_k}\ .$$

By Lemma 2.2 (see also [1, Lemma 3.3]), there are constants C_ω (independent of k) such that

$$\max_{\theta\in[0,2\pi]}\left|s_{n_k}(\omega,\theta)\right| > C_\omega\lambda_k n_k^{1-\alpha/2}$$

for k sufficiently large, almost surely. But since $\{s_{n_k}\}$ are Cesàro means of the partial sums of $f_\omega'(z)$,

$$\max_{|z|=1}\left|s_{n_k}(\omega,z)\right| \le 4\max_{|z|=1-1/n_k}\left|f_\omega'(z)\right|.$$

Letting $r_k = 1-1/n_k$, one has

$$M_\infty\left(f_\omega', r_k\right) > C_\omega\lambda_k\left(\frac{1}{1-r_k}\right)^{1-\alpha/2}.$$

Since $\lambda_k\to\infty$,

$$\left|f_\omega'(z)\right| \ne O\left(\frac{1}{1-|z|}\right)^{1-\alpha/2}$$

as $|z|\to 1$, almost surely. Again it follows from Lemma 2.1 that if $\alpha>0$ then $f_\omega\notin \mathrm{Lip}_{\alpha/2}$ almost surely. ///

3 $\sum \pm a_n z^n \in \mathrm{Lip}_{\alpha/2}$ FOR NO CHOICE OF SIGNS

According to the work of the previous section, there are functions $\sum a_n z^n \in \mathcal{D}_\alpha$ such that for almost every choice of signs, $\sum \pm a_n z^n \notin \mathrm{Lip}_{\alpha/2}$. In this section we show that more is true:

THEOREM 4 *Suppose* $0 < \alpha < 2$. *Then there exists a function* $\sum a_n z^n \in \mathcal{D}_\alpha$ *such that for* every *choice of signs,* $\sum \pm a_n z^n \notin \mathrm{Lip}_{\alpha/2}$.

Since every lacunary series in $\mathcal{D}_\alpha$ belongs to $\mathrm{Lip}_{\alpha/2}$, the functions of Theorem 4 cannot be lacunary (see Sec. 2). However the example produced in the proof is a "lacunary sum of lacunary polynomials."

Proof of Theorem 4. For any "sign sequence" $\epsilon = \epsilon_k$, each of whose terms is $+1$ or -1, and any holomorphic function $f(z) = \sum a_n z^n$, write $f_\epsilon(z) = \sum \epsilon_n a_n z^n$. Our example will be built from the functions

$$g_N(z) = N^{-1/2} 2^{-\alpha N/2} \sum_{j=1}^{N} z^{2^N + 2^j} \qquad (N = 1, 2, \ldots).$$

which are easily seen to form a bounded orthogonal sequence in $\mathcal{D}_\alpha$. Thus if $\{N_k\}$ is any strictly increasing sequence of positive integers, and $\{a_k\}$ any square summable sequence of complex numbers, the series $\sum a_k g_{N_k}$ will converge in $\mathcal{D}_\alpha$. In particular this is true of the function

$$f = \sum_{k=1}^{\infty} \frac{1}{k} g_{N_k},$$

where $N_k = 2^k$. This is the function that will occupy our attention for the rest of the section.

We are going to show that for any sign sequence ϵ, the function f_ϵ does not belong to $\mathrm{Lip}_{\alpha/2}$. To clarify the main points of the argument, we first consider only the "unperturbed" function f.

For this proof it is convenient to replace the Hardy-Littlewood inequality employed in the previous section (Lemma 2.1) by a similar result that uses Cesàro means in place of restrictions to circles (i.e., Abel means). For $f(z) = \sum a_n z^n$, let $\sigma_N(f)$ denote the N^{th} Cesàro mean of f:

$$\sigma_N(f) = \sum_{j=0}^{N} \left(1 - \frac{j}{N}\right) a_j z^j.$$

Then for $0 < \gamma < 1$,

$$f \in \mathrm{Lip}_\gamma \iff \|\sigma_N(f')\|_\infty = O\left(N^{1-\gamma}\right) \qquad (N \to \infty),$$

where $\| \ \|_\infty$ is the norm of $L^\infty(\partial U)$ ([16, Ch. VII, Ex. 14, p. 296] or see the paragraph at the end of the last section for the implication we are going to use).

In what follows, the symbol C, with or without subscripts, will denote a positive universal constant, which may nevertheless change from line to line. When divided by z^{2^N}, both g'_N and $\sigma_{3\cdot 2^N}(g'_N)$ become Hadamard lacunary polynomials, so by Sidon's theorem [16, Ch. VI, Th. 6.1, p. 247] the L^∞ norm of each is bounded below by a constant multiple of its ℓ^1 coefficient norm (the sum of the magnitudes of its Fourier coefficients). Since the ℓ^1 coefficient norm is clearly an upper bound for the L^∞ norm, we have

$$C_1\sqrt{N} \cdot 2^{(1-\alpha/2)N} \leq \|g'_N\|_\infty \leq C_2\sqrt{N} \cdot 2^{(1-\alpha/2)N}. \tag{1}$$

As for $\sigma_{3\cdot 2^N}(g'_N)$, observe that if the integer j is in the "Fourier support" of g_N, so that $j \leq 2^{N+1}$, then the j^{th} Cesàro coefficient multiplier for $\sigma_{3\cdot 2^N}$ is $1 - \frac{j}{3\cdot 2^N}$, which lies between 1 and 1/3. Thus

$$\frac{1}{3}\|g'_N\|_1 \leq \|\sigma_{3\cdot 2^N}(g'_N)\|_1 \leq \|g'_N\|_1,$$

where $\|\cdot\|_1$ denotes the ℓ^1 coefficient norm. Thus the reasoning of the last paragraph yields

$$C_1\sqrt{N} \cdot 2^{(1-\alpha/2)N} \leq \|\sigma_{3\cdot 2^N}(g'_N)\|_\infty \leq C_2\sqrt{N} \cdot 2^{(1-\alpha/2)N}. \tag{2}$$

Fix a positive integer K; for simplicity of notation let us temporarily write σ instead of $\sigma_{3\cdot 2^{N_K}}$. Since $2^{N_k} > 3 \cdot 2^{N_K}$ for $k > K$, (this is where the definition $N_k = 2^k$ first plays a role), we see that $\sigma(g_{N_k}) = 0$ for all $k > K$. Thus

$$\sigma(f) = \sum_{k=1}^{K} \frac{1}{k}\sigma(g_{N_k}),$$

from which follows

$$\|\sigma(f')\|_\infty \geq \frac{1}{K}\|\sigma(g'_{N_K})\|_\infty - \sum_{k=1}^{K-1}\frac{1}{k}\|\sigma(g'_{N_k})\|_\infty.$$

By (2) above, the first term on the right side of the last inequality is bounded below by a (universal) constant multiple of $K^{-1}\sqrt{N_K}\cdot 2^{(1-\alpha/2)N_K}$. An upper bound for each term of the second sum is its ℓ^1 coefficient norm, so

$$\|\sigma(g'_{N_k})\|_\infty \le C\sqrt{N}_k \cdot 2^{(1-\alpha/2)N_k} \qquad (1 \le k \le K).$$

Using this information in the least sophisticated way, we obtain

$$\begin{aligned}\|\sigma(f')\|_\infty &\ge C_1 K^{-1}\sqrt{N_K}\cdot 2^{(1-\alpha/2)N_K} - C_2(\log K)\sqrt{N_{K-1}}\cdot 2^{(1-\alpha/2)N_{K-1}} \\ &= K^{-1}\sqrt{N_K}\cdot 2^{(1-\alpha/2)N_K}\left(C_1 - C_2\cdot o(1)\right).\end{aligned}$$

Since $C_1 > 0$, this implies that $n^{-(1-\alpha/2)}\|\sigma_n(f')\|_\infty \to \infty$ as n runs through the subsequence $\{3\cdot 2^{N_K}\}$. The characterization of Lipschitz classes that was given above now shows that $f \notin \mathrm{Lip}_{\alpha/2}$.

What about the perturbations f_ϵ of f? The arguments just given depended only on the *magnitudes* of the Taylor coefficients of the polynomials g_N, so the estimates obtained for these polynomials hold as well for each of their perturbations $(g_N)_\epsilon$. Since different polynomials g_N involve distinct powers of z, we have $f_\epsilon = \sum k^{-1}(g_{N_k})_\epsilon$ for any sign sequence ϵ. Thus the proof that $f_\epsilon \notin \mathrm{Lip}_{\alpha/2}$ proceeds exactly as it did for f. ///

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Added in Proof. Professor J.M. Anderson has pointed out to us the following facts:

(a) A known result on fractional integrals asserts that $\sum a_n z^n$ belongs to $\mathrm{lip}_{1/2}$ if and only if $\sum n^{-1/2} a_n \in \lambda^*$, the "Little Zygmund Class." It follows that the question of whether or not random Dirichlet functions are a.s. in $\mathrm{lip}_{1/2}$ is equivalent to one raised in Anderson's paper [1] with Clunie and Pommerenke, and answered by them in Theorem 3.7 of that paper.

(b) This question also appears as Question 17 in A.L. Shields's survey article *Cyclic vectors in Banach spaces of analytic functions*, which appeared in the book *Operators and Function Theory*, S.C. Power, ed., Reidel 1985.

(c) Although we assert in §2 that the sufficient condition $\sum (n \log n)|a_n|^2 < \infty$ for a random power series $\sum \omega_n a_n z^n$ to a.s. belong to $\mathrm{lip}_{1/2}$ is best possible "in a very strong sense," there is in [1] a result equivalent to the fact that the stronger condition:

$$\sup_n \sum_{2^n \leq k < 2^{n+1}} (k \log k)|a_n|^2 < \infty$$

is also sufficient.