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Putnam's Theorem, Alexander's Spectral Area Estimate, and VMO

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In this paper we show that if f is a bounded analytic function defined on the unit disk such that at each point of the unit circle the cluster set of f has area zero, then f has vanishing mean oscillation (see Sect. 1 for definitions). We discovered this result (Corollary 1.5) and its quantitative version (Theorem 1.4) using some techniques from operator theory. This proof is given in Sect. 1; the main tool is Putnam's Theorem (Lemma 1.1).

In the main result in Sect. 2 (Theorem 2.4) we extend the above result to the unit ball in $\mathbb{C}^n$, using commutative Banach algebra techniques rather than operator theory. Here the key tool is Alexander's spectral area estimate (Lemma 2.2).

Section 3 ties the techniques of Sects. 1 and 2 together by giving a simple proof of Putnam's Theorem for subnormal operators. Here the main tool (Lemma 3.2) is a quantitative version of the Hartogs-Rosenthal Theorem. Some concluding remarks indicate another method of combining the tools used in this paper.

Although there are obvious connections between the three sections of the paper, each section can be read independently. Throughout the paper, c denotes an absolute constant, independent of everything except the dimension n of $\mathbb{C}^n$ (in Sect. 2). However, the actual value of c may change; thus sometimes c^2 is replaced by c.

1. Putnam's Theorem and VMO

Let D denote the open unit disk in the complex plane. On the unit circle ∂D we put Lebesgue arc length measure, normalized so that the measure of the entire circle is one. For $I \subset \partial D$, we let $|I|$ denote the normalized measure of I. The usual Lebesgue spaces and Hardy spaces on the circle with respect to this measure are denoted L^p and H^p.

For $f \in L^1$ and I a subinterval of ∂D, let f_I denote the average of f over I, so $f_I = \int_I f/|I|$. Next define $\|f\|_{\mathrm{BMO}}$ to be the square root of the supremum (taken over

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all subintervals I of ∂D) of $\int_I |f-f_I|^2/|I|$. The space BMO (bounded mean oscillation) is the set of all functions $f \in L^2$ such that $\|f\|_{\mathrm{BMO}} < \infty$. A function $f \in \mathrm{BMO}$ is said to be in VMO (vanishing mean oscillation) if the limit as r decreases to zero of the supremum (taken over all subintervals I of ∂D with $|I| < r$) of $\int_I |f-f_I|^2/|I|$ equals zero. Good references for the basic facts about BMO and VMO on the circle are [5] and [24]. These spaces can also be defined on the line rather than the circle with almost identical theorems and proofs; see [14].

If f is in one of the Hardy spaces H^p, we also denote the usual analytic extension of f to D by f. Whether we mean the function defined on ∂D or the function defined on D will always be clear from the context.

If f is analytic on D and $\lambda \in \partial D$, then the cluster set of f at λ, denoted $\mathrm{cl}(f;\lambda)$, is defined to be the set of all complex numbers w such that there exists a sequence $\{z_n\} \subset D$ such that $z_n \to \lambda$ and $f(z_n) \to w$.

Our main results in this section (Theorem 1.4 and Corollary 1.5) relate the cluster set with VMO. Our main tool will be a version of Putnam's Theorem on hyponormal operators (Lemma 1.1). Throughout this section, C^*-algebra always means C^*-algebra with identity. We use $\mathrm{sp}(T)$ to denote the spectrum of T in the appropriate Banach algebra.

Lemma 1.1 (Putnam's Theorem). *Let B be a C^*-algebra. If $T \in B$ and $T^*T - TT^*$ is a positive element of the C^*-algebra B, then*

$$\|T^*T - TT^*\| \leqq (\text{area } \mathrm{sp}(T))/\pi .$$

Putnam's Theorem is usually stated and proved only for the C^*-algebra of bounded operators on a Hilbert space, but we will need it as stated above for more general C^*-algebras. For the original proof, see [19, Theorem 1]. A nice proof based on the Berger-Shaw Theorem can be found in [8, p. 294]. To prove the version stated above from the operator theory versions, simply use the representation theorem which states that every C^*-algebra can be thought of as a C^*-subalgebra of the algebra of all operators on some Hilbert space; see [10, Theorem 2.6.1]. This representation preserves all the C^* properties (norm, spectrum, adjoints, etc.), so Lemma 1.1 follows from the Hilbert space version.

Purely C^* theorems should have C^* proofs, not proofs that rely on Hilbert space, so the proof above of Lemma 1.1 is unsatisfying. One of our favorite examples of this principle is the following proposition: If B is a C^*-algebra and $T \in B$ is left invertible, then T^*T is invertible. This is fairly easy to prove for operators on Hilbert space (and thus we get a proof for arbitrary C^*-algebras as with Lemma 1.1 above). A purely C^* proof is more difficult (and also more interesting) to discover, but once you find the pure C^* proof you are likely to be convinced that it's the right proof. Thus we raise the problem of finding a C^* proof of the C^* version of Putnam's Theorem. No one has yet found such a proof; we suggest that Alexander's spectral area estimate (Lemma 2.2) or the quantitative Hartogs-Rosenthal Theorem (Lemma 3.2) may be useful.

Let P denote the orthogonal projection from L^2 onto H^2. For $f \in L^\infty$, the Toeplitz operator $T_f : H^2 \to H^2$ is defined by $T_f(g) = P(fg)$. It is easy to check that the adjoint of T_f is given by the formula $T_f^*(g)=P(\bar{f}g)$. If $f\in H^\infty$ the projection is unnecessary, so in that case we have $T_f(g)=fg$.

The orthogonal complement of H^2 in L^2 is denoted $(H^2)^\perp$. For $f\in L^\infty$ the Hankel operator $H_f: H^2\to(H^2)^\perp$ is defined by $H_f(g)=(1-P)(fg)$. The following lemma gives a useful relationship between Toeplitz and Hankel operators.

Lemma 1.2. *If $f\in H^\infty$, then $T_f^*T_f-T_fT_f^*=H_{\bar{f}}^*H_{\bar{f}}$.*

Proof. If $g\in H^2$, then

$$\begin{aligned}\langle (T_f^*T_f-T_fT_f^*)g,g\rangle &= \|T_fg\|^2-\|T_f^*g\|^2\\ &=\|fg\|^2-\|P(\bar{f}g)\|^2=\|\bar{f}g\|^2-\|P(\bar{f}g)\|^2\\ &=\|(1-P)(\bar{f}g)\|^2=\|H_{\bar{f}}g\|^2\\ &=\langle H_{\bar{f}}^*H_{\bar{f}}g,g\rangle .\end{aligned}$$

Since this equation holds for each $g\in H^2$, we have the desired result.

Let f be an analytic function defined on D. It is easy to prove that if the area of $f(D)$ counting multiplicities (so if a region is covered twice it is counted twice) is finite, then $f\in \mathrm{VMO}$; see the comments following the proof of Lemma 1.6. Results concerning the area of $f(D)$, not counting multiplicities, are more difficult. From now on, area $f(D)$ means the area of the set of $f(D)$; multiplicities are ignored. In [3, Theorem 1], Alexander et al. proved that if area $f(D)<\infty$, then $f\in H^2$. Hansen [15] improved this result by showing that if area $f(D)<\infty$, then $f\in H^p$ for all $p<\infty$. Finally, Stegenga [26, p. 428], noted that this work on geometric conditions for BMO implies that if area $f(D)<\infty$, then $f\in$BMO. Since BMO$\subset L^p$ for all $p<\infty$, Stegenga's result implies the earlier results.

In the following proposition we give a new easy proof of Stegenga's result using Putnam's Theorem. Proposition 1.3 should be interpreted to mean that if $f(D)$ has finite area, then $f\in H^2$ and the boundary value function of f is in BMO. In Sect. 2 we give another easy proof of Proposition 1.3.

Actually, we will not use Proposition 1.3 in the rest of the paper. We prove it because the main result in this section, Theorem 1.4, will be proved by localizing the proof given here of Proposition 1.3.

Proposition 1.3 (Stegenga). *There is a constant c such that*

$$\|f\|_{\mathrm{BMO}}\leqq c\sqrt{\text{area } f(D)}$$

for every function f analytic on D with area $f(D)<\infty$.

Proof. First, let $f\in H^\infty$. By Nehari's Theorem (see [24, p. 100], for a statement and proof with the same notation we are using), there is a function $h\in H^\infty$ such that $\|H_{\bar{f}}\|=\|\bar{f}+h\|_\infty$. Now $f=P(f)$ and $P(\bar{h})$ is a constant. In the following inequality, we do not use the full strength of the duality between H^1 and BMO; the constant c comes from the boundedness of P as an operator from L^p to BMO. We have

$$\|f\|_{\mathrm{BMO}}=\|P(f+\bar{h})\|_{\mathrm{BMO}}\leqq c\|f+\bar{h}\|_\infty=c\|H_{\bar{f}}\| .$$

Now

$$(\|f\|_{\mathrm{BMO}})^2 \leqq c^2\|H_{\bar f}\|^2 = c^2\|H_{\bar f}^* H_{\bar f}\|$$
$$= c^2\|T_f^* T_f - T_f T_f^*\| \qquad \text{(by Lemma 1.2)}$$
$$\leqq c(\text{area sp}(T_f) \qquad \text{(by Lemma 1.1)}.$$

However, it is easy to check that $\mathrm{sp}(T_f) \subset \overline{f(D)}$ (actually equality is true; see [11, Theorem 7.21]) so we get that

$$(\|f\|_{\mathrm{BMO}})^2 \leqq c(\text{area } \overline{f(D)}) \quad \text{for all} \quad f \in H^\infty, \tag{1}$$

where c is a constant independent of f.

Now suppose f is analytic on D, but not necessarily bounded. For $0<r<1$, let $f_r(z) = f(rz)$. Now as r increases to 1, we have that $\|f_r\|_{\mathrm{BMO}}$ tends to $\|f\|_{\mathrm{BMO}}$. It is also clear that

$$\text{area } \overline{f_r(D)} = \text{area } \overline{f(rD)} \leqq \text{area } f(D),$$

so the proof is completed by applying (1) to f_r and taking the limit as r increases to 1.

The following theorem is the main result of this section. To prove this theorem we will need three further lemmas, so the proof is deferred until after Lemma 1.8.

Theorem 1.4. *There is a constant c such that*

$$\mathrm{dist}_{\mathrm{BMO}}(f, \mathrm{VMO}) \leqq c \sup\{\sqrt{\text{area cl}(f; \lambda)} : \lambda \in \partial D\}$$

for all $f \in H^\infty$.

Since VMO is a closed subspace of BMÓ, the following corollary follows immediately from Theorem 1.4.

Corollary 1.5. *If* $f \in H^\infty$ *and* $\mathrm{cl}(f; \lambda)$ *has zero area for each* $\lambda \in \partial D$, *then* $f \in \mathrm{VMO}$.

It is somewhat surprising that in the corollary above, the condition that $f \in H^\infty$ cannot be replaced by the assumption that f is an analytic function on D which is in BMO. For example, consider the function f defined by $f(z) = \log(1-z)$. It is not hard to verify that $f \in \mathrm{BMO}$. As a function on ∂D, f is continuous at each point of ∂D except $z=1$. At $z=1$, the cluster set of f at 1 is the empty set (or perhaps the point at infinity if you want to change the definition). At any rate, the cluster set of f at each point of the circle contains at most one point and so has area zero. However, as a function on ∂D, the imaginary part of f has a jump discontinuity at $z=1$, so $f \notin \mathrm{VMO}$.

Let C denote the set of continuous complex valued functions defined on ∂D. Since $\mathrm{VMO} \cap H^\infty = \{f \in H^\infty : \bar f \in H^\infty + C\}$ (see [22, p. 398], or [24, p. 66]), in Corollary 1.5 we could replace the conclusion $f \in \mathrm{VMO}$ with $\bar f \in H^\infty + C$. With this observation, we note that the Lemma on p. 8 of [21] implies that if $f \in H^\infty$ and if at each point of ∂D the cluster set of f lies on a straight line, then $f \in \mathrm{VMO}$. However, the techniques used in [21] do not suffice to prove the following implication of Corollary 1.5: If at each point of ∂D the cluster set of f lies on a smooth curve, then $f \in \mathrm{VMO}$.

Lemmas 1.6 through 1.8 will be used in the proof of Theorem 1.4. If T is a bounded linear operator, then the essential norm of T, denoted $\|T\|_e$, is the distance from T to the set of compact operators.

Lemma 1.6. *There is a constant c such that*

$$\mathrm{dist}_{\mathrm{BMO}}(f, \mathrm{VMO}) \leqq c\|H_{\bar{f}}\|_e$$

for all $f \in H^\infty$.

Proof. Let $f \in H^\infty$. From [24, p. 101], we see that there is a function $h \in H^\infty$ and a function $g \in C$ such that $2\|H_{\bar{f}}\|_e \geqq \|\bar{f}+h+g\|_\infty$. Now $P(\bar{g}) \in \mathrm{VMO}$, so

$$\begin{aligned}\mathrm{dist}_{\mathrm{BMO}}(f, \mathrm{VMO}) &\leqq \|f+P(\bar{g})\|_{\mathrm{BMO}} = \|P(f+\bar{h}+\bar{g})\|_{\mathrm{BMO}} \\ &\leqq c\|f+\bar{h}+\bar{g}\|_\infty \\ &\leqq c\|H_{\bar{f}}\|_e\end{aligned}$$

as desired.

Lemma 1.6 provides one method of giving an easy proof that if f is analytic on D and the area of $f(D)$ counting multiplicities is finite, then $f \in \mathrm{VMO}$. Since we never need to use this fact, we just sketch the proof: Suppose the area of $f(D)$ counting multiplicities is finite. If $f(z) = \sum a_n z^n$, then the area of $f(D)$ counting multiplicities equals $\pi \sum n|a_n|^2$. Now examine the matrix of $H_{\bar{f}}$ with respect to the usual orthonormal bases, and note that the condition on the Taylor coefficients of f implies that the entries of the matrix are square summable. Thus $H_{\bar{f}}$ is a Hilbert-Schmidt operator, and hence $H_{\bar{f}}$ is compact. Lemma 1.6 now implies that f is in VMO.

Let $\mathscr{T}$ denote the norm closed algebra generated by $\{T_f : f \in L^\infty\}$. It turns out that $\mathscr{T}$ is also equal to the C^*-algebra generated by $\{T_f : f \in H^\infty\}$; see [11, Chap. 7], for more information about $\mathscr{T}$. For $\lambda \in \partial D$, let J_λ be the smallest norm closed two sided ideal of $\mathscr{T}$ containing $\{T_f : f \in C \text{ and } f(\lambda) = 0\}$. The canonical quotient map from $\mathscr{T}$ to $\mathscr{T}/J_\lambda$ will be denoted q_λ, so $q_\lambda(T) = T + J_\lambda$ for each $T \in \mathscr{T}$. For $f \in L^\infty$, $q_\lambda(T_f)$ is called a local Toeplitz operator; see [4, 6, 11–13] for more about this subject. In the following lemma we actually have equality but we don't need it.

Lemma 1.7. *If $f \in H^\infty$, then*

$$\mathrm{sp}(q_\lambda(T_f)) \subset \mathrm{cl}(f; \lambda) \quad \textit{for each} \quad \lambda \in \partial D.$$

Proof. Note that if $w \in C$, then

$$q_\lambda(T_w) = q_\lambda(T_{w-w(\lambda)} + w(\lambda)) = w(\lambda);$$

here we are letting $w(\lambda)$ sometimes denote $w(\lambda)$ times the identity of the appropriate Banach algebra.

Let $f \in H^\infty$. To prove Lemma 1.7 it suffices to prove that if $0 \notin \mathrm{cl}(f; \lambda)$, then $q_\lambda(T_f)$ is invertible. So suppose that $0 \notin \mathrm{cl}(f; \lambda)$. Write $f = bg$, where b is an inner function and g is an outer function. Since f is bounded away from 0 near λ, there is an open interval I of ∂D containing λ such that b is continuous (and unimodular) on I and g is bounded away from 0 on I. Since g is an outer function, there is a real valued function $u \in L^1$ such that $g = \exp(u + i\tilde{u})$. Let v equal $(-u)$ times the characteristic function of I, and let $h = \exp(v + i\tilde{v})$. On I, $u + v$ equals 0, so gh is continuous on I.

Let $w \in C$ be such that $w(\lambda) = 1$ and $w = 0$ on $\partial D \sim I$. As we saw in the first paragraph of this proof, $q_\lambda(T_w) = 1$, so

$$q_\lambda(T_f) q_\lambda(T_h) = q_\lambda(T_w) q_\lambda(T_{bg}) q_\lambda(T_h) = q_\lambda(T_{wb(gh)}) .$$

Since $wb(gh)$ is continuous on ∂D and nonzero at λ, the last quantity in the equation above is a nonzero scalar multiple of the identity. Since u is bounded away from $-\infty$ on I, the outer function h is invertible in H^∞, and so $q_\lambda(T_h)$ is invertible. Thus the equation above implies that $q_\lambda(T_f)$ is invertible, as desired.

The next lemma says that the essential norm of an operator in $\mathscr{T}$ depends only on the local behavior of the operator.

Lemma 1.8. *If* $S \in \mathscr{T}$, *then*

$$\|S\|_e = \sup\{\|q_\lambda(S)\| : \lambda \in \partial D\} .$$

Proof. Let $\oplus\{\mathscr{T}/J_\lambda : \lambda \in \partial D\}$ denote the C^* product of the C^*-algebras $\mathscr{T}/J_\lambda$; thus the norm of any element of the product is the supremum of the norms of the components. Let $\mathscr{K}$ denote the set of compact operators on H^2. Douglas [11, Theorem 7.49], has shown that the map from $\mathscr{T}/\mathscr{K}$ to $\oplus\{\mathscr{T}/J_\lambda : \lambda \in \partial D\}$ which sends $T + \mathscr{K}$ to $\oplus\{q_\lambda(T) : \lambda \in \partial D\}$ is injective. Since every injective homomorphism between C^*-algebras is an isometry [10, Corollary 1.83], we are done.

We have now assembled all the ingredients necessary to prove Theorem 1.4.

Proof of Theorem 1.4. Let $f \in H^\infty$. Then

$$\begin{aligned}
&[\mathrm{dist}_{\mathrm{BMO}}(f, \mathrm{VMO})]^2 \\
&\quad \leqq c\|H_{\bar{f}}\|_e^2 && \text{(by Lemma 1.6)} \\
&\quad = c\|H_{\bar{f}}^* H_{\bar{f}}\|_e \\
&\quad = c\|T_f^* T_f - T_f T_f^*\|_e && \text{(by Lemma 1.2)} \\
&\quad = c \sup\{\|q_\lambda(T_f)^* q_\lambda(T_f) - q_\lambda(T_f) q_\lambda(T_f)^*\| : \lambda \in \partial D\} && \text{(by Lemma 1.8)} \\
&\quad \leqq c \sup\{\text{area sp}(q_\lambda(T_f)) : \lambda \in \partial D\} && \text{(by Lemma 1.1)} \\
&\quad \leqq c \sup\{\text{area cl}(f; \lambda) : \lambda \in \partial D\} && \text{(by Lemma 1.7)},
\end{aligned}$$

and we are done.

In Sect. 2 we extend Theorem 1.4 to the unit ball in $\mathbb{C}^n$ for $n > 1$. A problem arises when trying to extend to several dimensions the operator theoretic proof given in this section. Specifically, it is unclear whether Lemma 1.6 holds when $n > 1$; see the discussion following the statement of Theorem 2.4.

The techniques used in this section can be used to prove a slightly stronger version of Theorem 1.4, although at the cost of losing the geometric flavor associated with the cluster sets; see the discussion following Proposition 2.10.

2. Alexander's Spectral Area Estimate and VMO

In this section we will use results from the theory of function algebras to obtain analogues of the distance estimate of Theorem 1.4 for bounded analytic functions on the unit ball of $\mathbb{C}^n$; see Theorem 2.4. In this context we use Alexander's spectral area estimate (Lemma 2.2) instead of Putnam's Theorem (Lemma 1.1), and we replace the BMO norm by Garsia's equivalent norm. In the next few paragraphs we define Garsia's norm in the setting of the unit circle, state Alexander's spectral area estimate, and illustrate our method by giving another proof of Stegenga's Theorem (Proposition 1.3). We then develop the major theme of this section – the generalization to $\mathbb{C}^n$ of Theorem 1.4.

For z in D let μ_z denote the Poisson measure (or harmonic measure) for z on ∂D:

$$d\mu_z(t) = (1-|z|^2)/|1-ze^{-it}|^2 dt/2\pi,$$

and for f in L^1 write

$$f(z) = \int_{\partial D} f d\mu_z \qquad (z \in D).$$

The function f defined on D by this formula is the Poisson integral of f; it is the natural harmonic extension of f from ∂D to D.

For f in L^2 we define the (possibly infinite) Garsia norm of f to be

$$\|f\|_G = \sup\left\{\left(\int_{\partial D} |f-f(z)|^2 d\mu_z\right)^{1/2} : z \in D\right\}.$$

Garsia proved that $f \in \mathrm{BMO}$ if and only if $\|f\|_G < \infty$, and that the seminorms $\|\ \|_G$ and $\|\ \|_{\mathrm{BMO}}$ are equivalent on BMO. A proof of the following lemma can be found in [24, pp. 36–37].

Lemma 2.1 (Garsia's Theorem). *There exists a constant c such that*

$$\|f\|_G/c \leqq \|f\|_{\mathrm{BMO}} \leqq c\|f\|_G$$

for every f in L^2.

In order to state Alexander's Theorem we need to recall some terminology. A function algebra (or uniform algebra) A on a compact Hausdorff space Q is a closed subalgebra of $C(Q)$ which contains the constant functions and separates the points of Q. The maximal ideal space M_A of A is the space of nonzero complex homomorphisms of A. With the weak-* topology, M_A is a compact Hausdorff space, in which Q is naturally embedded as a closed subspace. The Gelfand transform of a function f in A is the continuous function $\hat{f}$ on M_A defined by $\hat{f}(m) = m(f)$ for $m \in M_A$. The spectrum of f in A, denoted $\mathrm{sp}(f)$, is equal to $\hat{f}(M_A)$. The map $f \to \hat{f}$ is a Banach algebra isometry of A into $C(M_A)$.

If $m \in M_A$, then an A-representing measure for m is a Borel probability measure μ on M_A such that $\hat{f}(m) = \int_{M_A} \hat{f} d\mu$ for each f in A. We denote the collection of all such representing measures for m by $R_A(m)$. A standard argument involving the Hahn-Banach and Riesz Representation Theorems shows that $R_A(m)$ always contains a measure supported on Q.

Lemma 2.2 (Alexander's Spectral Area Estimate). *Suppose A is a function algebra. If $m \in M_A$, $\mu \in R_A(m)$, and $f \in A$, then*

$$\int_{M_A} |\hat{f} - \hat{f}(m)|^2 d\mu \leqq \text{area } \operatorname{sp}(f)/\pi .$$

This theorem was applied in [3, 1, 2] to derive lower bounds for the areas of projections of analytic varieties on the complex coordinate axes. Further applications are given in [2]. Alexander's proof of Lemma 2.2 depends on a quantitative version of the Hartogs-Rosenthal Theorem of function theory. In Sect. 3 we use this tool (Lemma 3.2) to give a new proof of Putnam's Theorem for subnormal operators. Right now we want to illustrate how Alexander's Theorem yields results about BMO. We begin by giving another easy proof of Proposition 1.3, which states that there is a constant c such that $\|f\|_{\mathrm{BMO}} \leqq c\sqrt{\text{area} f(D)}$ for every function f analytic on D.

Proof of Proposition 1.3. In Lemma 2.2, let A be the disc algebra; so A consists of all functions in $C(\bar{D})$ that are analytic on D. The maximal ideal space of A is $\bar{D}$. For $f \in A$ and $z \in D$, apply Lemma 2.2 with $\mu = \mu_z$, the Poisson measure. Taking the supremum of the left side of the resulting inequality over z in D we obtain

$$\|f\|_G^2 = \sup\left\{\int_{\partial D} |f - f(z)|^2 d\mu_z : z \in D\right\} \leqq \text{area}\, \overline{f(D)}/\pi .$$

More generally, if f is analytic in D with area $f(D) < \infty$, then as in the proof given in Sect. 1 of Proposition 1.3, we apply the above inequality to the dilate f_r for $0 < r < 1$ and let $r \to 1$. The result is

$$\|f\|_G \leqq \sqrt{\text{area} f(D)/\pi} .$$

This inequality and Garsia's Theorem (Lemma 2.1) complete the proof.

We now turn our attention to bounded analytic functions on the unit ball of $\mathbb{C}^n$ for $n \geqq 1$. Our goal is to prove an analogue of Theorem 1.4 by localizing the argument of the last paragraph to certain function algebras associated with points of the boundary of the ball. There is some question about what BMO should mean in this context. We will discuss this matter after setting out some notation and terminology.

Let B denote the open unit ball of $\mathbb{C}^n$, and let $\bar{B}$ denote its closure. Let σ denote surface area measure on the unit sphere ∂B, normalized so that the total mass is 1. Thus σ is the unique rotation-invariant Borel probability measure on ∂B. We write $L^p = L^p(\sigma) = L^p(\partial B, \sigma)$.

For $1 \leqq j \leqq n$, let u_j be the orthogonal projection of $\mathbb{C}^n$ onto the j^{th} complex coordinate axis; so $u_j(z) = z_j$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We denote the complex inner product on $\mathbb{C}^n$ by $\langle\,,\rangle$, where

$$\langle z, w\rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n .$$

The Euclidean norm is denoted $|\;|$ and is defined by $|z| = \sqrt{\langle z, z\rangle}$.

For $n > 1$ there are two natural generalizations of BMO from ∂D to ∂B. Each corresponds to a different class of spheres on ∂B, and the two spaces are different. For ξ in ∂B and $\delta > 0$ define

$$E_\delta(\xi) = \{z \in \partial B : |\xi - z| < \delta\} \tag{2}$$

and

$$N_\delta(\xi)=\{z\in\partial B: |1-\langle\xi,z\rangle|<\delta^2\}. \tag{3}$$

The Euclidean spheres $E_\delta(\xi)$ arise in connection with the study of harmonic functions on B. They model the singularity of the Poisson measures

$$d\mu_z(\xi)=(1-|z|^2)/|z-\xi|^{2n}d\sigma(\xi) \tag{4}$$

defined on ∂B for each z in B. If $f\in L^1(\sigma)$, then as in the case $n=1$, the Poisson integral

$$f(z)=\int_{\partial B} f d\mu_z \qquad (z\in B) \tag{5}$$

is the natural harmonic extension of f to B.

For f in $L^2(\sigma)$ we define $\|f\|_E$ as in Sect. 1, replacing ∂D by ∂B, normalized Lebesgue measure on ∂D by σ, and intervals I by spheres $E_\delta(\xi)$, and let BMO_E denote those f in $L^2(\sigma)$ for which $\|f\|_E<\infty$. There is also a Garsia norm equivalent to $\| \ \|_E$, given by the formula

$$\|f\|_{E,G}=\sup\left\{\left(\int_{\partial B}|f-f(z)|^2d\mu_z\right)^{1/2}: z\in B\right\}.$$

The non-isotopic spheres $N_\delta(\xi)$ are in many ways more closely connected with complex function theory on B than are the Euclidean spheres. They are associated with the Poisson-Szego measures

$$d\nu_z(\xi)=[(1-|z|^2)/|1-\langle z,\xi\rangle|^2]^n d\sigma(\xi) \tag{6}$$

defined on ∂B for each z in B. If $f\in L^1(\sigma)$, then the Poisson-Szego integral of f given by

$$f(z)=\int_{\partial B} f d\nu_z \qquad (z\in B) \tag{7}$$

is the natural $\mathscr{M}$-harmonic extension of f to B. We will not define this class of functions here – a detailed discussion can be found in [20, Chap. 4] – but will content ourselves with noting that $\mathscr{M}$-harmonicity is preserved under composition with analytic automorphisms of B (ordinary harmonicity is not), and that a real valued harmonic function in B is $\mathscr{M}$-harmonic if and only if it is the real part of an analytic function.

We define $\| \ \|_N$ and BMO_N in the obvious way relative to the spheres $N_\delta(\xi)$. We remark that Krantz's paper [17] contains a detailed discussion of BMO_E and BMO_N. In particular, if $n>1$ neither space contains the other, but the intersection of BMO_E with the $H^2(B)$ boundary functions lies in BMO_N. However, it is BMO_N, and not BMO_E, which is used to describe the dual space of the Hardy space $H^1(B)$.

We define the Garsia norm $\|f\|_{N,G}$ for $f\in L^2(\sigma)$ by

$$\|f\|_{N,G}=\sup\left\{\left(\int_{\partial B}|f-f(z)|^2d\nu_z\right)^{1/2}: z\in B\right\},$$

where $f(z)$ is now defined by (7) for $z\in B$. The same argument that proves Garsia's Theorem (Lemma 2.1) for ∂D yields the analogous inequalities relating $\| \ \|_E$ with $\| \ \|_{E,G}$, and $\| \ \|_N$ with $\| \ \|_{N,G}$. For completeness we prove the part of this result that we will need in the sequel for the N-norms.

Proposition 2.3. *There exists a constant c such that*

$$\|f\|_N \leqq c\|f\|_{N,G} \quad \text{for each } f \text{ in } L^2(\sigma).$$

Proof. Fix $\xi \in \partial B$ and $0<\delta \leqq \sqrt{2}$. Let $N=N_\delta(\xi)$ and let $f_N=\int_N f d\sigma/\sigma(N)$. In the Hilbert space $L^2(N, \sigma/\sigma(N))$ the constant f_N is the orthogonal projection of f onto the one dimensional subspace of constant functions, so for every $w \in \mathbb{C}$ we have

$$\int_N |f-f_N|^2 d\sigma/\sigma(N) \leqq \int_N |f-w|^2 d\sigma/\sigma(N).$$

Suppose $\delta \geqq 1$. Then since $\sigma = v_0$ we get

$$\begin{aligned}\int_N |f-f_N|^2 d\sigma(N)/\sigma(N) &\leqq \int_N |f-f(0)|^2 d\sigma/\sigma(N)\\ &\leqq \int_{\partial B} |f-f(0)|^2 d\sigma/\sigma(N)\\ &= \int_{\partial B} |f-f(0)|^2 dv_0/\sigma(N)\\ &\leqq \|f\|^2_{N,G}/\sigma(N).\end{aligned}$$

On the other hand, if $0<\delta<1$, let $r=1-\delta^2$. Then for z in $N=N_\delta(\xi)$ we have the following estimate on the kernel defining $v_{r\xi}$:

$$[(1-r^2)/|1-\langle r\xi, z\rangle|^2]^n \geqq 1/(4\delta^2)^n \geqq c/\sigma(N),$$

where the last inequality is in [20, Sect. 5.1.4]. Thus

$$\begin{aligned}\int_N |f-f_N|^2 d\sigma/\sigma(N) &\leqq \int_N |f-f(r\xi)|^2 d\sigma/\sigma(N)\\ &\leqq c\int_{\partial B} |f-f(r\xi)|^2 dv_{r\xi}\\ &\leqq c\|f\|^2_{N,G}.\end{aligned}$$

These inequalities show that $\|f\|_N \leqq c\|f\|_{N,G}$ for each f in $L^2(\sigma)$. This completes the proof.

Each of the BMO spaces defined above has a corresponding VMO subspace, denoted by VMO_E or VMO_N as the case may be. Each is a closed subspace of its respective BMO space, and each contains the continuous functions on ∂B [just as each BMO space contains $L^\infty(\sigma)$]. Let $\mathrm{dist}_N(f, \mathrm{VMO}_N)$ denote the distance in BMO_N from $f \in \mathrm{BMO}_N$ to VMO_N:

$$\mathrm{dist}_N(f, \mathrm{VMO}_N) = \inf\{\|f-g\|_N : g \in \mathrm{VMO}_N\}\,,$$

with a similar definition for $\mathrm{dist}_E(f, \mathrm{VMO}_E)$.

Let $H^\infty(B)$ denote the space of bounded analytic functions on B, endowed with the sup-norm. For each $f \in H^\infty(B)$, define a function on ∂B, also denoted f, by taking radial limits, which exist for almost every $(d\sigma)$ point of ∂B. The map that associates f with its radial limit function is an isometric algebra isomorphism taking $H^\infty(B)$ onto a closed subalgebra of $L^\infty(\sigma)$; see [20, Chap. 5]. We can now state the main result of this section. The cluster set of f at ξ, denoted $\mathrm{cl}(f;\xi)$, is defined exactly as in the unit disk – it is the set of w in $\mathbb{C}$ such that $w=\lim f(z_n)$ for some sequence $\{z_n\}$ in B with $\xi = \lim z_n$.

Theorem 2.4. *There is a constant c such that for each $f \in H^\infty(B)$, both* $\operatorname{dist}_E(f, \mathrm{VMO}_E)$ *and* $\operatorname{dist}_N(f, \mathrm{VMO}_N)$ *are less than or equal to*

$$c \sup\{\sqrt{\operatorname{area} \operatorname{cl}(f;\xi)} : \xi \in \partial B\}.$$

The proof of Theorem 2.4 will occupy the rest of this section, and will be divided into a number of lemmas and propositions. Of course, when $n=1$ Theorem 2.4 reduces to Theorem 1.4. We digress for a moment to discuss the problems which arise when attempting to use the techniques of Sect. 1 to give a proof of Theorem 2.4 for $n>1$.

To mimic the techniques of Sect. 1, we would define for f in $H^\infty(B)$ the Toeplitz operator $T_f: H^2(\partial B) \to H^2(\partial B)$ and the Hankel operator $H_{\bar f}: H^2(\partial B) \to (H^2(\partial B))^\perp$ by $T_f(h) = fh$ and $H_{\bar f}(h) = (1-P)(\bar f h)$, where P is the orthogonal projection of $L^2(\sigma)$ onto $H^2(\partial B)$. The main stumbling block that prevents the proof given in Sect. 1 from working when $n>1$ is that it is unclear whether Lemma 1.6 holds in this context. We raise the following question, whose answer would probably determine whether Lemma 1.6 holds for $n>1$: For which functions $f \in H^\infty(B)$ is $H_{\bar f}$ compact?

Readers familiar with a paper of Coifman, Rochberg, and Weiss may believe that this question has already been answered. Theorem VIII of [7] states that for $f \in H^\infty(B)$, the Hankel operator with symbol $\bar f$ is compact if and only if $f \in \mathrm{VMO}_N$. However, the Hankel operators appearing in [7] are not the same as the Hankel operators defined here. The Hankel operators used in [7] are (equivalent to) multiplication followed by projection onto

$$Y = \{\bar f : f \in H^2(\partial B) \text{ and } f(0) = 0\}.$$

Our Hankel operators are multiplication followed by projection onto $(H^2(\partial B))^\perp$. Of course, $Y \subset (H^2(\partial B))^\perp$ and when $n=1$ the two spaces are equal, but for $n>1$ the inclusion is strict. This distinction was overlooked in [9], which uses the same definition of Hankel operators as we do. In particular, the last paragraph of Sect. 3 of [9], where results from [7] are used, should be treated with caution. Perhaps the techniques of [7] can be used to answer the question raised in the above paragraph.

To conclude this discussion, we note that we cannot just adopt the definition of Hankel operator used in [7]. Our definition is the natural one when dealing with Toeplitz operators, because it provides the crucial link between Toeplitz and Hankel operators as expressed by Lemma 1.2, which holds even when $n>1$ with our definition.

From now on we denote the maximal ideal space of $H^\infty(B)$ by M, and use $\hat f$ to denote the Gelfand transform of $f \in H^\infty(B)$, so $\hat f$ is a continuous complex valued function on M. The fiber map $\pi: M \to \bar B$ is defined by

$$\pi(m) = (\hat u_1(m), \ldots, \hat u_n(m)) \qquad (m \in M),$$

where we recall that u_j is the j^{th} coordinate projection on $\mathbb{C}^n$. For $z \in \bar B$ the fiber of M over z is denoted M_z and is defined by $M_z = \pi^{-1}(z)$. Since π is continuous, M_z is closed in M. The fiber algebra A_z associated with z is the subalgebra of $C(M_z)$

which arises from restrictions of functions in $H^\infty(B)$:

$$A_z = \{\hat{f}|M_z : f \in H^\infty(B)\}.$$

Finally, if $m \in M$ we write $R_\infty(m)$ instead of $R_{H^\infty(B)}(m)$ for the set of $H^\infty(B)$-representing measures for m, viewing $H^\infty(B)$ as a function algebra on M.

We regard B as naturally embedded in M, and identify f with $\hat{f}$ on B. For z in B, the functions $u_j - z_j$ $[1 \leqq j \leqq n,\ z_j = u_j(z)]$ generate the maximal ideal consisting of all f in $H^\infty(B)$ which vanish at $z = (z_1, \ldots, z_n)$ (this is easy when $n = 1$, but difficult for $n > 1$; see [20, Sect. 6.6]). It follows that $M_z = \{z\}$ when $|z| < 1$, so M_z and A_z are nontrivial only if $z \in \partial B$. The following results are standard when $n = 1$ ([16, Chap. 10] and [25]), and known for $n > 1$.

Lemma 2.5. *Suppose $\xi \in \partial B$. Then*

(a) *$f \in H^\infty(B)$ extends continuously to $B \cup \{\xi\}$ if and only if $\hat{f}$ is constant on the fiber M_ξ.*

(b) *A_ξ is a closed subalgebra of $C(M_\xi)$, and its maximal ideal space is M_ξ.*

(c) *If $m \in M_\xi$ and $\mu \in R_\infty(m)$, then support $\mu \subset M_\xi$.*

(d) *For each $f \in H^\infty(B)$ we have* $\mathrm{cl}(f; \xi) = \hat{f}(M_\xi)$.

The proofs of (a) and (b) follow exactly as in the case $n = 1$ from the existence of a function h continuous on $\bar{B}$, analytic on B, with $|h| < 1$ on $\bar{B} \sim \{\xi\}$ and $h(\xi) = 1$. For example take $h(z) = (1 + \langle z, \xi \rangle)/2$. See [16, p. 161], for the details.

Part (c) is an immediate consequence of (a). For (a) implies that $(h)^n$ tends pointwise to the characteristic function of M_ξ, so by dominated convergence

$$1 = h(m)^n = \int_M (h)^n d\mu \to \mu(M_\xi)$$

as $n \to \infty$. This shows that μ is concentrated on M_ξ.

The proof of (d) is not an immediate generalization of the case $n = 1$, and we will not give it here. For the details see McDonald's paper [18].

Before proceeding further, we pause to note explicitly a convenient way of rewriting the left hand side of Alexander's spectral area estimate (Lemma 2.2).

Lemma 2.6. *Suppose μ is any probability measure, $f \in L^2(\mu)$, and $w = \int f d\mu$. Then*

$$\int |f - w|^2 d\mu = \int |f|^2 d\mu - |w|^2.$$

Proof.

$$\begin{aligned}\int |f - w|^2 d\mu &= \int |f|^2 d\mu - 2\,\mathrm{Re}(\bar{w} \textstyle\int f d\mu) + |w|^2 \\ &= \int |f|^2 d\mu - 2|w|^2 + |w|^2,\end{aligned}$$

as desired.

We can now give our main application of Alexander's spectral area estimate to $H^\infty(B)$. This result provides the crucial step in the proof of Theorem 2.4.

Theorem 2.7. *If $f \in H^\infty(B)$ and $\xi \in \partial B$, then*

$$\limsup_{z \in B,\, z \to \xi} \left[\sup \left\{ \int_M |\hat{f} - f(z)|^2 d\mu : \mu \in R_\infty(z) \right\} \right] \leqq (\text{area } \mathrm{cl}(f; \xi))/\pi.$$

Proof. Fix f in $H^\infty(B)$, and let w denote the lim sup on the left side of the inequality above, and choose sequences $\{z_j\}$ in B and μ_j in $R_\infty(z_j)$ such that $z_j \to \xi$ and

$$\int_M |\hat{f} - f(z_j)|^2 d\mu_j \to w\,.$$

Since the measures $\{\mu_j\}$ are probability measures we may – by passing to a subnet if necessary – assume that $\{\mu_j\}$ converges to a Borel measure μ on M in the weak-* topology induced by $C(M)$. It is easy to check that μ is a probability measure that is multiplicative on $H^\infty(B)$:

$$\int_M \hat{g}\hat{h} d\mu = \left(\int_M \hat{g} d\mu\right)\left(\int_M \hat{h} d\mu\right) \qquad [g, h \in H^\infty(B)]\,,$$

so $\mu \in R_\infty(m)$ for some m in M. Now the weak-* convergence of μ_j to μ yields for each g in $H^\infty(B)$ [since $g \in C(M)$]:

$$\hat{g}(m) = \int_M \hat{g} d\mu = \lim \int_M \hat{g} d\mu_j = \lim g(z_j)\,,$$

hence $z_j \to m$ in M. In particular,

$$\pi(m) = \lim \pi(z_j) = \lim z_j = \xi\,,$$

so $m \in M_\xi$, and hence by Lemma 2.5(c), we have support $\mu \subset M_\xi$.

We claim that

$$\int_M |\hat{f} - \hat{f}(m)|^2 d\mu = w\,. \tag{8}$$

This follows from Lemma 2.6 and weak-* convergence:

$$\begin{aligned} w &= \lim \int_M |\hat{f} - f(z_j)|^2 d\mu_j && \text{(definition)} \\ &= \lim \int_M |\hat{f}|^2 d\mu_j - |f(z_j)|^2 && \text{(Lemma 2.6)} \\ &= \int_M |\hat{f}|^2 d\mu - |\hat{f}(m)|^2 && \text{(weak-* convergence)} \\ &= \int_M |\hat{f} - \hat{f}(m)|^2 d\mu && \text{(Lemma 2.6)}\,. \end{aligned}$$

Since support $\mu \subset M_\xi$, we know that μ is an A_ξ-representing measure for the restriction to A_ξ of the complex homomorphism m. Thus in view of Lemma 2.5 we can apply Alexander's spectral area estimate (Lemma 2.2) to μ and $\hat{f}|M_\xi$ with $A = A_\xi$ and $M_A = M_\xi$. The result is

$$\begin{aligned} \int_{M_\xi} |\hat{f} - \hat{f}(m)|^2 d\mu &\leqq \operatorname{area} \hat{f}(M_\xi)/\pi \\ &= \operatorname{area} \operatorname{cl}(f;\xi)/\pi \quad [\text{Lemma 2.5(d)}]\,. \end{aligned}$$

This inequality and (8) complete the proof.

In order to use Theorem 2.7 to prove Theorem 2.4, we have to make some connections between $H^\infty(B)$-representing measures, which live on M, and more concrete measures on $\bar{B}$. The ball algebra $A(B)$ is the subalgebra of $C(\bar{B})$ consisting of functions analytic on B. By identifying each point z of $\bar{B}$ with the linear functional of point evaluation at z, we identify $\bar{B}$ with the maximal ideal space of $A(B)$. For z in B, let $R_*(z)$ denote the collection of $A(B)$-representing measures μ

for point evaluation at z such that for each f in $H^\infty(B)$, the radial limit

$$\lim f(r\xi) \qquad (r\to 1-)$$

exists for almost all $(d\mu)\xi$ in ∂B.

We stress that although we are allowing support μ to intersect B, the measures of most interest are actually supported on ∂B. The surface area measure σ is in $R_*(0)$, and so is normalized arc length measure on $L\cap\partial B$ for any complex line L through the origin. The Poisson measure μ_z and the Poisson-Szego measure v_z given by (4) and (6) for z in B both lie in $R_*(z)$, since they are in $R_{A(B)}(z)$ [20, Chap. 3] and are absolutely continuous with respect to σ. It is apparently not known whether $R_*(z)=R_{A(B)}(z)$ [20, Sect. 11.3.5]. The next result shows that each $\mu\in R_*(z)$ can be canonically associated with an $H^\infty(B)$-representing measure for z. Recall that π is the canonical map from M onto $\bar{B}$.

Proposition 2.8. *If $z\in B$ and $\mu\in R_*(z)$, then there exists a measure $\hat{\mu}\in R_\infty(z)$ such that*

(i) $$\hat{\mu}\pi^{-1}=\mu$$

and such that if $\Phi:\mathbb{C}\to\mathbb{C}$ is continuous and $f\in H^\infty(B)$, then

(ii) $$\int_M \Phi\circ\hat{f}d\hat{\mu}=\int_{\bar{B}}\Phi\circ f d\mu\,.$$

Remarks. Recall that for $\xi\in\partial B$ we are denoting the radial limit of f at ξ, if it exists, by $f(\xi)$. Thus the integrand on the right side of (ii) exists almost everywhere $(d\mu)$ because $\mu\in R_*(z)$. In our application of Proposition 2.8 we will not need (i) and will need (ii) only for the two functions $\Phi(w)=w$, and $\Phi(w)=|w|^2$ $(w\in\mathbb{C})$.

Proof of Proposition 2.8. Suppose $\mu\in R_*(z)$: We introduce some notation for concepts involving $L^\infty(\mu)$. If $f\in H^\infty(B)$, let $[f]$ denote the μ-equivalence class of f. Let X_μ denote the maximal ideal space of $L^\infty(\mu)$. For $m\in X_\mu$, let $\varphi(m)$ be the complex homomorphism of $H^\infty(B)$ defined by

$$\varphi(m)(f)=m([f])\,.$$

Let $\tilde{g}$ be the Gelfand transform of $g\in L^\infty(\mu)$. Then the definition of $\varphi(m)$ can be rephrased:

$$\hat{f}(\varphi(m))=[f]^{\sim}(m) \qquad [f\in H^\infty(B)]\,. \tag{9}$$

It is easy to check that φ is a continuous map from X_μ to M. Since $C(X_\mu)$ is precisely the set of all Gelfand transforms $\tilde{g}$ for $g\in L^\infty(\mu)$, we can define a Borel probability measure $\tilde{\mu}$ on X_μ by

$$\int_{X_\mu}\tilde{g}d\tilde{\mu}=\int_{\bar{B}} g d\mu \qquad [g\in L^\infty(\mu)]\,. \tag{10}$$

We are going to show that $\hat{\mu}=\tilde{\mu}\varphi^{-1}$ is the measure we seek.

Suppose $\Phi:\mathbb{C}\to\mathbb{C}$ is continuous. Then because the Gelfand transform is a C^*-algebra isometry of $L^\infty(\mu)$ onto $C(X_\mu)$, it is easy to check that $(\Phi\circ g)^{\sim}=\Phi\circ\tilde{g}$ for each g in $L^\infty(\mu)$ (for example, begin with Φ a polynomial in z and $\bar{z}$, then take uniform limits on the essential range of g). Using the definitions of μ and φ, the change of variable formula, and the above fact, in that order, we obtain for $f \in H^\infty(B)$

$$\begin{aligned}\int_M \Phi \circ \hat{f} d\hat{\mu} &= \int_M \Phi \circ \hat{f} d(\tilde{\mu}\varphi^{-1}) \\ &= \int_{X_\mu} \Phi(\hat{f}(\varphi(m))) d\tilde{\mu}(m) \quad \text{(change of variable)} \\ &= \int_{X_\mu} \Phi \circ [f]^{\sim} d\tilde{\mu} \quad \text{[by (9)]} \\ &= \int_{X_\mu} [\Phi \circ f]^{\sim} d\tilde{\mu} \\ &= \int_{\bar{B}} \Phi \circ f d\mu \quad \text{[by (10)]},\end{aligned}$$

which proves (ii).

To show that $\hat{\mu}\pi^{-1} = \mu$, we fix f and g in $H^\infty(B)$ and apply (ii) to $f+g$ and $f-g$ with $\Phi(w) = |w|^2$ ($w \in C$). Upon subtracting the resulting equations we obtain

$$\operatorname{Re} \int_M \hat{f}\bar{\hat{g}} d\hat{\mu} = \operatorname{Re} \int_{\bar{B}} f\bar{g} d\mu .$$

Replacing g by ig and repeating this argument, we obtain the same result for imaginary parts. So

$$\int_M \hat{f}\bar{\hat{g}} d\hat{\mu} = \int_{\bar{B}} f\bar{g} d\mu \tag{11}$$

for every $f, g \in H^\infty(B)$. Now let W denote the collection of all finite sums of products $f\bar{g}$ with f and g in $A(B)$. Since the $H^\infty(B)$ Gelfand transform $\hat{f}$ of a function f in $A(B)$ is constant on each fiber M_ξ ($\xi \in \partial B$), we have $f \circ \pi = \hat{f}$ on M. So if h is the finite sum $h = \sum f_j \bar{g}_j$ with f_j, g_j in $A(B)$, then

$$\begin{aligned}\int_{\bar{B}} hd(\hat{\mu}\pi^{-1}) &= \int_M h \circ \pi d\hat{\mu} \\ &= \sum \int_M (f_j \circ \pi)(g_j \circ \pi)^{-} d\hat{\mu} \\ &= \sum \int_M \hat{f}_j \bar{\hat{g}}_j d\hat{\mu} \\ &= \sum \int_{\bar{B}} f_j \bar{g}_j d\mu \quad \text{[by (11)]} \\ &= \int_{\bar{B}} hd\mu .\end{aligned}$$

By the Stone-Weierstrass Theorem W is uniformly dense in $C(B)$. Thus we have $\int_{\bar{B}} hd(\hat{\mu}\pi^{-1}) = \int_{\bar{B}} hd\mu$ for every $h \in C(\bar{B})$, and hence $\hat{\mu}\pi^{-1} = \mu$.

It remains to check that $\hat{\mu}$ is an $H^\infty(B)$-representing measure for z. Fix $f \in H^\infty(B)$ and let f_r ($0 \leq r < 1$) denote the dilate of f by r, as in Sect. 1 [$f_r(\xi) = f(r\xi)$ for $\xi \in \bar{B}$]. Then $f_r \in A(B)$ and the Lebesgue Dominated Convergence Theorem yields (as $r \to 1-$)

$$\int_{\bar{B}} fd\mu = \lim \int_{\bar{B}} f_r d\mu .$$

But $\int_B f_r d\mu = f(rz)$ since $\mu \in R_{A(B)}(z)$, and $f(rz) \to f(z)$ as $r \to 1-$. So for all f in $H^\infty(B)$ we have

$$\int_B f d\mu = f(z). \tag{12}$$

Apply (ii) with Φ equal to the identity map on $\mathbb{C}$, and use (12), getting

$$\int_M \hat{f} d\hat{\mu} = \int_B f d\mu = f(z) \qquad [f \in H^\infty(B)],$$

and so $\hat{\mu} \in R_\infty(z)$. The proof is complete.

Corollary 2.9. *If $f \in H^\infty(B)$ and $\xi \in \partial B$, then*

$$\limsup_{z \in B,\, z \to \xi} \left[\sup \left\{ \int_B |f - f(z)|^2 d\mu : \mu \in R_*(z) \right\} \right] \leqq (\text{area cl}(f; \xi))/\pi.$$

Proof. Fix $z \in B$ and $\mu \in R_*(z)$ for a moment. Let $\hat{\mu}$ be the measure promised by the last proposition, and apply (ii) to $f - f(z)$ with $\Phi(w) = |w|^2$. The result is

$$\int_B |f - f(z)|^2 d\mu = \int_M |\hat{f} - f(z)|^2 d\hat{\mu}.$$

Since $\hat{\mu} \in R_\infty(z)$, the corollary now follows immediately from Theorem 2.7.

We can now give the proof of Theorem 2.4. It consists of showing that Corollary 2.9 is in fact a stronger result.

Proof of Theorem 2.4. For f in $H^\infty(B)$ and $0 \leqq r < 1$, let

$$G(f; r) = \sup \left\{ \left(\int_B |f - f(z)|^2 d\mu \right)^{1/2} : \mu \in R_*(z),\ r \leqq |z| < 1 \right\}.$$

Clearly $G(f; r)$ decreases as $r \to 1-$. An immediate consequence of Corollary 2.9 is that

$$\lim_{r \to 1} G(f; r) \leqq \sup \{ \sqrt{\text{area cl}(f; \xi)} : \xi \in \partial B \} / \sqrt{\pi}, \tag{13}$$

so to finish the proof we need to show that both $\text{dist}_E(f, \text{VMO}_E)$ and $\text{dist}_N(f, \text{VMO}_N)$ are controlled by $\lim G(f; r)$. The key to this argument is the following inequality:

$$G(f_s; r) \leqq G(f; rs) \qquad [0 \leqq r, s < 1,\ f \in H^\infty(B)]. \tag{14}$$

Here, as usual, f_s is the dilate of f by s.

To prove (14), fix z in B with $r \leqq |z| < 1$, and take μ in $R_*(z)$. For ξ in ∂B let $\nu_{s\xi}$ be the Poisson-Szego representing measure given by (6), which we write as

$$d\nu_{s\xi}(\eta) = P(s\xi, \eta) d\sigma(\eta) \qquad (\eta \in \partial B),$$

where

$$P(\lambda, \eta) = [(1 - |\lambda|^2)/|1 - \langle \lambda, \eta \rangle|^2]^n.$$

(Actually, we could as well use the Poisson measures $d\mu_{s\xi}$ here.) Now using the Cauchy-Schwarz inequality on the probability measure $\nu_{s\xi}$, we get

$$|f(s\xi)|^2 = \left| \int_{\partial B} f d\nu_{s\xi} \right|^2 \leqq \int_{\partial B} |f|^2 d\nu_{s\xi},$$

so by Fubini's Theorem

$$\int_{\bar{B}} |f(s\xi)|^2 d\mu(\xi) \leqq \int_{\partial B} |f(\eta)|^2 \left(\int_{\bar{B}} P(s\xi,\eta)d\mu(\xi)\right) d\sigma(\eta)$$
$$= \int_{\partial B} |f|^2 dv, \tag{15}$$

where

$$dv(\eta) = \left(\int_{\bar{B}} P(s\xi,\eta)d\mu(\xi)\right) d\sigma(\eta).$$

We claim that $v \in R_*(sz)$. Since v is absolutely continuous with respect to σ, it is enough to show that $v \in R_{A(B)}(sz)$. It is easy to check that v is a probability measure on ∂B. Suppose $g \in A(B)$. Then

$$\begin{aligned}
\int_{\partial B} g\,dv &= \int_{\bar{B}} \left(\int_{\partial B} g(\eta)P(s\xi,\eta)d\sigma(\eta)\right) d\mu(\xi) \quad &&\text{(Fubini)}\\
&= \int_{\bar{B}} g(s\xi)d\mu(\xi)\\
&= \int_{\bar{B}} g_s(\xi)d\mu(\xi)\\
&= g_s(z) &&[\mu \in R_{A(B)}(z)]\\
&= g(sz).
\end{aligned}$$

Thus $v \in R_*(sz)$, as desired. From this fact, Lemma 2.6, and inequality (15) we obtain

$$\begin{aligned}
&\int_{\bar{B}} |f_s - f_s(z)|^2 d\mu\\
&= \int_{\bar{B}} |f_s|^2 d\mu - |f_s(z)|^2 \quad &&\text{(Lemma 2.6)}\\
&\leqq \int_{\partial B} |f|^2 dv - |f(sz)|^2 &&[\text{by (15)}]\\
&= \int_{\partial B} |f - f(sz)|^2 dv &&[\text{Lemma 2.6, since } v \in R_*(sz)]\\
&\leqq [G(f; rs)]^2,
\end{aligned}$$

and (14) follows because $\mu \in R_*(z)$ and $r \leqq |z| < 1$ were chosen arbitrarily.

To finish the proof of Theorem 2.4, suppose $\varepsilon > 0$ is given. Fix $0 < r < 1$. Since $f_s \to f$ in $L^2(\sigma)$ as $s \to 1-$, we can choose $1 > s \geqq r$ so that

$$\int_{\partial B} |f - f_s|^2 d\sigma < \varepsilon(1-r)^n.$$

Thus if $|z| \leqq r$, Lemma 2.6 yields

$$\begin{aligned}
&\int_{\partial B} |(f - f_s) - (f - f_s)(z)|^2 dv_z\\
&= \int_{\partial B} |f(\eta) - f_s(\eta)|^2 P(z,\eta)d\sigma(\eta) - |(f - f_s)(z)|^2\\
&< 2^n \varepsilon.
\end{aligned}$$

From Proposition 2.3 we obtain

$$\begin{aligned}
\|f - f_s\|_N^2 &\leqq c \sup\left\{\int_{\partial B} |(f - f_s) - (f - f_s)(z)|^2 dv_z : z \in B\right\}\\
&\leqq c[2^n \varepsilon + G(f - f_s; r)^2]
\end{aligned}$$

by (16) and the definition of G. Since $s \geqq r$, inequality (14) yields

$$\begin{aligned} G(f-f_s; r) &\leqq G(f; r) + G(f; sr) \\ &\leqq G(f; r) + G(f; r^2) \\ &\leqq 2G(f; r^2). \end{aligned}$$

But f_s is continuous on ∂B and hence is in VMO_N. So by the last two inequalities we have

$$\operatorname{dist}_N(f, \mathrm{VMO}_N) \leqq c[\sqrt{\varepsilon} + G(f; r^2)] \tag{17}$$

for each $0 \leqq r < 1$. Let $r \to 1-$ and then let $\varepsilon \to 0+$ in (17). We obtain

$$\operatorname{dist}_N(f, \mathrm{VMO}_N) \leqq c \lim G(f; r),$$

which, along with (13), completes the proof of the "N" part of Theorem 2.4. The "E" part is entirely similar, so we are done.

For the rest of this section we return to the case $n = 1$, where Theorem 2.4 can be somewhat strengthened. As in Sect. 1, H^∞ and L^∞ denote the usual Hardy and Lebesgue spaces on the unit circle. The space of multiplicative linear functionals on H^∞ is still denoted by M, and we let X denote the maximal ideal space of L^∞. For convenience we now write f rather than $\hat{f}$ for the Gelfand transform of f. Each $m \in M$ has a unique extension to a linear functional of norm 1 on L^∞, and thus we can consider X to be a subset of M. For $m \in M$, let μ_m be the unique Borel probability measure on X such that

$$m(f) = \int_X f d\mu_m \quad \text{for every} \quad f \in H^\infty.$$

The closed support of the measure μ_m is denoted $\operatorname{supp} m$.

Instead of localizing to the fiber algebras as in Lemma 2.5, we now localize to smaller sets. For $m \in M$, let A_m be the closed subalgebra of $C(\operatorname{supp} m)$ defined by

$$A_m = \{f|(\operatorname{supp} m) : f \in H^\infty\}.$$

It is easy to see that the maximal ideal space of A_m can be identified with M_m, where

$$M_m = \{v \in M : \operatorname{supp} v \subset \operatorname{supp} m\}.$$

For every $m \in M \sim D$, $\operatorname{supp} m$ is considerably smaller than any of the fibers M_z ($z \in \partial D$), so the following proposition is stronger than Theorem 1.4.

Proposition 2.10. *There is a constant c such that* $\operatorname{dist}_{\mathrm{BMO}}(f, \mathrm{VMO}) \leqq c \sup \{\sqrt{\operatorname{area} f(M_m)} : m \in M \sim D\}$ *for all* $f \in H^\infty$.

The proof of Proposition 2.10 consists of simply carefully examining the proof of Theorem 2.4. A change needs to be made in only one place – in the proof of Theorem 2.7. In that proof, consider only representing measures supported on X (rather than on M), and when Alexander's spectral area estimate (Lemma 2.2) is used, apply it to the algebra A_m (with maximal ideal space equal to M_m) rather than to the fiber algebras.

In addition to supports of representing measures, maximal anti-symmetric sets play an important role in the study of function algebras. Consider the algebra $H^\infty + C$, whose maximal ideal space is equal to $M \sim D$. For E a subset of X which is a maximal anti-symmetric set for $H^\infty + C$ (from now on we call such sets just maximal anti-symmetric sets), define A_E and M_E by

$$A_E = \{f|E : f \in H^\infty\},$$
$$M_E = \{m \in M : \operatorname{supp} m \subset E\}.$$

The operator theory techniques of Sect. 1 can be coupled with the methods of [4, Sect. 7], (which require transfinite induction) to localize to the maximal anti-symmetric sets, which are much smaller than fibers. More precisely, the right hand side of the inequality in Proposition 2.10 could be replaced by

$$c \sup\{\sqrt{\operatorname{area} f(M_E)} : E \text{ is a maximal anti-symmetric set}\}.$$

For each $m \in M \sim D$, there exists a maximal anti-symmetric set E such that $M_m \subset M_E$, so the estimate given by the paragraph above is not stronger than Proposition 2.10. It is reasonable to believe that the techniques of Sect. 1 should lead to the same results as the techniques of Sect. 2, so perhaps this adds a slight bit of evidence for an affirmative answer to the following question: For every maximal anti-symmetric set E, does there exist $m \in M \sim D$ such that $E = \operatorname{supp} m$? This question has been raised before; see the discussion at the end of [23] for some comments about its connection with the Chang-Marshall Theorem.

3. Putnam's Theorem and the Quantitative Hartogs-Rosenthal Theorem

Let H be a Hilbert space and let $\mathscr{B}(H)$ denote the set of bounded linear operators from H to H. An operator $S \in \mathscr{B}(H)$ is called subnormal if there exist a Hilbert space X containing H and a normal operator $N \in \mathscr{B}(X)$ such that $N(H) \subset H$ and $N|H = S$. Our goal in this section is to give an easy proof to the following theorem.

Theorem 3.1 (Putnam's Theorem for Subnormal Operators). *Let S be a subnormal operator. Then*

$$\|S^*S - SS^*\| \leqq (\operatorname{area} \operatorname{sp}(S))/\pi.$$

An operator $S \in \mathscr{B}(H)$ is called hyponormal if $S^*S - SS^*$ is a positive operator. The class of subnormal operators is strictly contained in the class of hyponormal operators, so of course Theorem 3.1 follows immediately from Putnam's Theorem (Lemma 1.1). The easiest proof of Putnam's Theorem [8, p. 294] relies heavily on the Berger-Shaw Theorem for hyponormal operators. Recently Hadwin and Nordgren have simplified the proof of the Berger-Shaw Theorem which appears in [8], but even with these improvements the Berger-Shaw Theorem still has a difficult proof.

Mathematicians working with subnormal operators have suspected that the hypothesis of subnormality (in place of hyponormality) should lead to easier proofs. In his book on subnormal operators, Conway [8] makes the following statement in the introduction to the chapter on the Berger-Shaw Theorem and Putnam's Theorem: "It would seem that if subnormality were assumed instead of hyponormality, easier proofs would be achievable. Unfortunately, only the proofs for the hyponormal case exist." In this section we partly remedy this situation by providing a proof of Putnam's Theorem for subnormal operators which is easier than the proofs for hyponormal operators. We thank Jim Dudziak for a suggestion which slightly shortened our original proof.

For K a nonempty compact subset of the complex plane, let $R(K)$ denote the closure in $C(K)$ of the set of rational functions with poles off K. The Hartogs-Rosenthal Theorem states that if K has area zero, then $R(K)=C(K)$. Note that by the Stone-Weierstrass Theorem, $R(K)=C(K)$ if and only if $\bar{z}\in R(K)$, so the following lemma can be called a quantitative version of the Hartogs-Rosenthal Theorem.

Lemma 3.2 (Alexander). *Let K be a nonempty compact subset of the complex plane. Then*

$$\operatorname{dist}_{C(K)}(\bar{z}, R(K)) \leqq \sqrt{(\text{area } K)/\pi}\,.$$

Alexander used Lemma 3.2 as one of the main tools in proving his spectral area estimate (Lemma 2.2). We will use Lemma 3.2 to prove Putnam's Theorem for subnormal operators. Proofs of Lemma 3.2 can be found in Alexander's papers [1, Lemma 2], and [2, p. 5]. We believe that the proof of Lemma 3.2 is considerably easier than the proof of the Berger-Shaw Theorem (which we will not use). Furthermore, using Lemma 3.2 rather than the Berger-Shaw Theorem seems to give some insight as to why Putnam's Theorem is true, at least for subnormal operators.

For K a nonempty compact subset of the complex plane and μ a positive finite Borel measure on K, let $R^2(K,\mu)$ denote the closure of $R(K)$ in $L^2(K,\mu)$.

Proof of Theorem 3.1. First consider the case where S is a subnormal operator on a Hilbert space H with a rationally cyclic vector. This means there is a vector $f\in H$ such that

$$\{r(S)f : r \text{ is a rational function with poles off } \operatorname{sp}(S)\}$$

is dense in H. By a standard representation theorem for subnormal operators with a rationally cyclic vector (see [8, Theorem III.5.2]), there is a positive finite Borel measure μ on $\operatorname{sp}(S)$ such that S is unitarily equivalent to the operator of multiplication by z on $R^2(\operatorname{sp}(S),\mu)$. So we can assume that $H=R^2(\operatorname{sp}(S),\mu)$ and that $Sg=zg$ for all $g\in R^2(\operatorname{sp}(S),\mu)$.

Let P denote the orthogonal projection of $L^2(\operatorname{sp}(S),\mu)$ onto $R^2(\operatorname{sp}(S),\mu)$. For $g,h\in R^2(\operatorname{sp}(S),\mu)$ we have

$$\langle S^*g,h\rangle = \langle g,Sh\rangle = \langle g,zh\rangle = \langle \bar{z}g,h\rangle = \langle P(\bar{z}g),h\rangle\,,$$

so $S^*g=P(\bar{z}g)$ for all $g\in R^2(\operatorname{sp}(S),\mu)$.

Consider $g\in R^2(\operatorname{sp}(S),\mu)$ with $\|g\|_2=1$. Then

$$\begin{aligned}
\langle (S^*S-SS^*)g,g\rangle &= \|Sg\|^2-\|S^*g\|^2\\
&= \|zg\|^2-\|P(\bar{z}g)\|^2\\
&= \|\bar{z}g\|^2-\|P(\bar{z}g)\|^2 = \|(1-P)(\bar{z}g)\|^2\\
&= [\operatorname{dist}_{L^2(\operatorname{sp}(S),\mu)}(\bar{z}g, R^2(\operatorname{sp}(S),\mu))]^2\\
&\leqq [\inf\{\|\bar{z}g-hg\|_2 : h\in R(\operatorname{sp}(S))\}]^2\\
&\leqq [\inf\{\|\bar{z}-h\|_\infty : h\in R(\operatorname{sp}(S))\}]^2\\
&\leqq (\text{area } \operatorname{sp}(S))/\pi \text{ (by Lemma 3.2)}.
\end{aligned}$$

Since the above inequality holds for each unit vector $g \in R^2(\mathrm{sp}(S), \mu)$, we have $\|S^*S - SS^*\| \leqq (\text{area } \mathrm{sp}(S)/\pi)$, completing the proof in the case where S has a rationally cyclic vector.

Now suppose S is an arbitrary subnormal operator on a Hilbert space H, not necessarily with a rationally cyclic vector. Fix $f \in H$ with $\|f\| = 1$. Let Y denote the closure of

$$\{r(S)f : r \text{ is a rational function with poles off } \mathrm{sp}(S)\}$$

in H. Obviously $S(Y) \subset Y$. Define $T\colon Y \to Y$ by $T = S|Y$. It is clear from the definition of T that $\mathrm{sp}(T) \subset \mathrm{sp}(S)$ and that f is a rationally cyclic vector for T. Let Q denote the orthogonal projection of H onto Y. For all $g \in Y$ we have

$$\begin{aligned}\langle T^* f, g\rangle &= \langle f, Tg\rangle = \langle f, Sg\rangle = \langle S^* f, g\rangle \\ &= \langle Q(S^* f), g\rangle,\end{aligned}$$

so $T^* f = QS^* f$. Now

$$\begin{aligned}\langle (S^*S - SS^*)f, f\rangle &= \|Sf\|^2 - \|S^* f\|^2 \\ &\leqq \|Sf\|^2 - \|QS^* f\|^2 \\ &= \|Tf\|^2 - \|T^* f\|^2 \\ &= \langle (T^*T - TT^*)f, f\rangle \\ &\leqq \|T^*T - TT^*\| \\ &\leqq (\text{area } \mathrm{sp}(T))/\pi \text{ (by the first case)} \\ &\leqq (\text{area } \mathrm{sp}(S))/\pi .\end{aligned}$$

Since the above inequality holds for each unit vector f in H, we have $\|S^*S - SS^*\| \leqq (\text{area } \mathrm{sp}(S))/\pi$, and we are done.

Of course, it would also be nice to have a proof of the Berger-Shaw Theorem for subnormal operators which is easier than the proof for hyponormal operators. It seems to us that Alexander's spectral area estimate (Lemma 2.2) and the quantitative Hartogs-Rosenthal Theorem (Lemma 3.2) are excellent candidates for the tools which might be used in such a proof.

Additional Remarks

T. Gamelin and the referee have pointed out that the ingredients we have assembled can be used to give a more efficient proof of Theorem 1.4 and the non-isotropic case of Theorem 2.4. We now sketch these ideas, returning to the notation of Sect. 2.

The proof of Lemma 1.6 contains the inequality

$$\mathrm{dist}_{\mathrm{BMO}_N}(f, \mathrm{VMO}_N) \leqq c\, \mathrm{dist}_{L^\infty}(\bar{f}, H^\infty(B) + C(\partial B)) \tag{18}$$

for $f \in H^\infty(B)$. (As in the case of the unit disk, the projection $P : L^2(\partial B) \to H^2(\partial B)$ restricts to a bounded operator from $L^\infty(\partial B)$ to BMO_N on the n-sphere. This fails for the isotropic BMO_E; see [17].)

Next, we need the equality

$$\operatorname{dist}_{L^\infty}(\bar{f}, H^\infty(B)+C(\partial B)) = \max\{\operatorname{dist}(\bar{f}|M_\xi, A_\xi) : \xi \in \partial B\}, \tag{19}$$

where we recall that A_ξ is the fiber algebra $\{f|M_\xi : f \in H^\infty(B)\}$ and the distance on the right is computed in $C(M_\xi)$. For $n=1$, this equality was proved by Sarason in Sect. 5 of [21]. Since $H^\infty(B)+C(\partial B)$ is also an algebra for $n>1$ (see [20, Theorem 6.5.5]), Sarason's proof also works for the case, where $n>1$.

Now suppose $f \in H^\infty(B)$ and $\xi \in \partial B$. Let $K_\xi = f(M_\xi)$. Alexander's use of the functional calculus [2, proof of Theorem 1, p. 6] can be applied to the function algebra A_ξ to show that

$$\operatorname{dist}(\bar{f}|M_\xi, A_\xi) \leqq \operatorname{dist}(\bar{z}, R(K_\xi)).$$

By applying Lemma 3.2 (Alexander's quantitative Hartogs-Rosenthal Theorem) and Lemma 2.5(d) to the right hand side of the above inequality, we obtain

$$\operatorname{dist}(\bar{f}|M_\xi, A_\xi) \leqq (\operatorname{area}\operatorname{cl}(f;\xi)/\pi)^{1/2}. \tag{20}$$

Combining (18), (19), and (20) gives the desired result.

T. Gamelin (private communication) has extended these results to the context of smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$. He has also observed that Putnam's Theorem can be used to give a proof of the quantitative Hartogs-Rosenthal Theorem.

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Note added in proof: The above-mentioned comments of Gamelin have been incorporated in his recent paper *On an Estimate of Axler and Shapiro* (preprint). Charles S. Stanton of the University of California, Riverside, has communicated to us an entirely function-theoretic proof of Corollary 1.5. The main ingredient of Stanton's proof is a generalization due to him of Lehto's principle of majorization for counting functions (Thesis, University of Wisconsin-Madison, 1982). Stanton points out that his argument works as well for the unit ball in higher dimensions, and that his result is actually more general than our corollary 1.5: it does not require the H^2 function f with BMO boundary function to be bounded.