ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1985

LINEAR TOPOLOGICAL PROPERTIES OF THE HARMONIC HARDY SPACES h^p FOR $0 < p < 1$

BY

JOEL H. SHAPIRO[1]

Dedicated to the memory of David L. Williams

1. Introduction

Let Δ denote the open unit disc of the complex plane. In this paper we study, for $0 < p < 1$, the space h^p of complex valued functions u harmonic on Δ, for which

$$\|u\|_p^p = \sup_{0 \le r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta/2\pi < \infty. \tag{1}$$

If $p \ge 1$ then the functional $\|\cdot\|_p$ is a norm which makes h^p into a Banach space. But if $0 < p < 1$ it is instead the *p-norm* $\|\cdot\|_p^p$ which is subadditive, and used to induce the translation-invariant metric. In either case metric convergence implies uniform convergence on compact subsets of Δ, so even if $0 < p < 1$, the space h^p is complete, has enough continuous linear functionals to separate points; and its topology is "natural" for harmonic functions.

For $p \ge 1$ the h^p spaces are well known objects with many desirable properties. For example [5; Chapters 2, 3, and 4]:

(i) The Poisson integral establishes an isometric isomorphism between h^p and a classical Banach space: $L^p(\partial\Delta)$ if $p > 1$, and the space of complex Borel measures on $\partial\Delta$ if $p = 1$.

(ii) Each function in h^1 has a finite non-tangential limit at almost every point of $\partial\Delta$.

(iii) The conjugate function operator $u \to \tilde{u}$ is well behaved. If $1 < p < \infty$, the M. Riesz theorem asserts that h^p is "self-conjugate", that is, if u is in h^p, then so is its harmonic conjugate $\tilde{u}$. This is not true for h^1, but here Kolmogorov's theorem provides a substitute: if $u \in h^1$ then $\tilde{u} \in h^p$ for all $p < 1$.

Received February 28, 1983.

[1]Research supported in part by the National Science Foundation.

The spaces h^p for $0 < p < 1$ have also been in the literature for a long time, used mostly to provide counterexamples to possible extensions of the results mentioned above. In fact the best known feature of these spaces seems to be the following result of Hardy and Littlewood ([19], [9; Section 4.3, page 471], [10; Chapter 10, Section 6]): *there exists u in $\cap\{h^p\colon 0 < p < 1\}$ for which $\tilde{u}$ belongs to no h^p.*

The results of this paper show that when $0 < p < 1$, this sort of behavior is forced on h^p by linear topological phenomena having no counterpart in either h^p for $p \geq 1$, or in the holomorphic Hardy spaces H^p when $p > 0$. Our work continues in the general direction inspired by the paper [6] of Duren, Romberg, and Shields, who discovered unusual linear topological properties of H^p when $0 < p < 1$, and related these properties to interesting questions about analytic functions. Subsequently Duren and Shields [7] noted some of the basic properties of h^p $(0 < p < 1)$ and suggested a study of these spaces modelled after their earlier work on H^p. We will see here that, just as with H^p, such a study involves an interesting interplay between classical and functional analysis. But the main point is that when $0 < p < 1$ the spaces h^p are of interest not because their linear topological properties resemble those of H^p, but because they are in fact completely different.

We begin to pursue this theme seriously in Section 3, after devoting a section to preliminary definitions and results. In Section 3 we explore some consequences of the most immediate difference between h^p and H^p: when $0 < p < 1$ the space H^p is separable, but h^p is not. This result, already implicit in the work of Hardy and Littlewood [9], focuses attention on a natural separable subspace: the closure in h^p of the harmonic polynomials. Although this subspace, which we denote by $h^p(\mathscr{P})$, contains many familiar closed subspaces —H^p and l^2, for example—it also contains a strikingly different one: the rotation-invariant subspace

$$h_0^p = \left\{u \in h^p\colon \lim_{r \to 1-} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta = 0\right\}.$$

This subspace, which is nontrivial *only* when $0 < p < 1$—which we assume from now on without further comment—is in fact infinite dimensional: it contains all Poisson integrals of singular measures on $\partial\Delta$. We show in Section 3 that, except for the trivial subspace $\{0\}$, no rotation invariant subspace of h_0^p can be locally convex. Nevertheless we will see that local convexity shows up everywhere in h_0^p: every infinite dimensional closed subspace of h_0^p has a further subspace isomorphic to the sequence space c_0. We use this result to prove that no infinite dimensional closed subspace of h_0^p can be mapped into h^p by the conjugate function operator, and we obtain some further results on the pathology of conjugate functions for subspaces of h_0^p.

We close Section 3 by observing that h_0^p is *weakly dense* in $h^p(\mathscr{P})$; that is, among the continuous linear functionals on $h^p(\mathscr{P})$, only the zero functional annihilates h_0^p. Such failures of the Hahn-Banach theorem play an essential role in the theory of F-spaces (complete, metrizable topological vector spaces) which are not locally convex. This phenomenon was first observed in l^p by N.T. Peck [20]; in H^p by Duren, Romberg, and Shields [6; Theorem 14, page 56]; and in other settings by Shapiro [25], [26] and Kalton [11], [13]. In particular Kalton [11; Corollary 5.3, page 162] showed that the qualitative extension form of the Hahn-Banach theorem must fail in *every* F-space which is not locally convex, and he later showed that whenever such a space obeys a mild (but necessary) additional hypothesis then it also contains a proper, closed, weakly dense subspace [13]. The survey paper [28] contains an exposition of some of these matters.

Since h_0^p is weakly dense in $h^p(\mathscr{P})$, the quotient space $h^p(\mathscr{P})/h_0^p$ has trivial dual. In Section 4 we identify this quotient as the space $L^p(\partial\Delta)$. We use the method of A.B. Aleksandrov [1] to construct the isomorphism from the Poisson integral. The method also yields an approximation theorem: h_0^p is the closure in h^p of the linear span of the rotates of the Poisson kernel; and the theorem itself yields additional linear topological information about the relationship between h_0^p and $h^p(\mathscr{P})$.

I wish to thank Professor Nigel Kalton for some helpful conversations about the material of this paper.

2. Preliminaries

Unless otherwise noted we always take $0 < p < 1$. In this section we establish our notation and state some basic results about h^p. We show that h^p is a complete metric space, and that h_0^p is a closed subspace of $h^p(\mathscr{P})$.

2.1 *Notation.* Recall that Δ denotes the open unit disc. We write T for the unit circle, and $\bar{\Delta}$ for the closed unit disc. Normalized Lebesgue measure on T will be denoted by m. If u is a continuous complex valued function on Δ, then for each $w \in \bar{\Delta}$ we define the dilate u_w by

$$u_w(z) = u(wz) \quad (|z| < 1/|w|). \tag{1}$$

Note that if $u \in h^p$ then $u_w \in h^p(\mathscr{P})$, and $\|u_w\|_p \le \|u\|_p$.

If $p > 0$ and $0 \le r < 1$, for u harmonic in Δ we write

$$M_p^p(u; r) = \int_T |u_r|^p \, dm. \tag{2}$$

Thus h^p is the space of complex harmonic functions u on Δ for which

$$\|u\|_p = \sup\{M_p(u; r) \colon 0 \le r < 1\} < \infty,$$

and h_0^p is the collection of u in h^p for which

$$\lim M_p(u; r) = 0 \quad (r \to 1-).$$

The Hardy space H^p is the closed subspace of h^p consisting of functions holomorphic on Δ.

Each function u harmonic in Δ has a series expansion

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} \hat{u}(n) r^{|n|} e^{in\theta} \tag{3}$$

which converges absolutely and uniformly on compact subsets of Δ. For u as above we define $\tilde{u}$ to be the harmonic conjugate of u which vanishes at the origin. That is,

$$\tilde{u}(re^{i\theta}) = \sum_{-\infty}^{\infty} \omega_n \hat{u}(n) r^{|n|} e^{in\theta} \tag{4}$$

where $\omega_n = -i$ if $n > 0$, 0 if $n = 0$, and $+i$ if $n < 0$; so $u + i\tilde{u}$ is holomorphic in Δ.

2.2 *Growth estimates.* There exists a positive constant C, which depends only on p, such that for every u in h^p,

(a) $|u(z)| \leq C\|u\|_p(1 - |z|)^{-1/p}$ (z in Δ)

and

(b) $|\hat{u}(n)| \leq C\|u\|_p(|n| + 1)^{(1/p)-1}$ ($n \in \mathbf{Z}$).

These estimates were first obtained for $0 < p < 1$ by Hardy and Littlewood [9; Theorem 1, page 410] who actually obtained them for the holomorphic completion of u, which need not lie in h^p. An easier proof was recently provided by Fefferman and Stein [8; Lemma 2, page 172]. Growth estimate 2.2 (a) shows that for each z in Δ, the evaluation functional $u \to u(z)$ is continuous on h^p, so h^p has enough continuous linear functionals to separate points. It also shows that h^p convergence implies uniform convergence on compact subsets of Δ, which immediately implies the next result, whose proof we omit.

2.3 PROPOSITION. *h^p is complete in the metric*

$$d(u, v) = \|u - v\|_p^p \quad (u, v, \in h^p).$$

2.4 PROPOSITION. *h_0^p is closed in h^p.*

Proof. Consider the functional N defined on h^p by

$$N(u) = \limsup_{r \to 1-} M_p^p(u; r) \quad (u \in h^p).$$

Then N is sub-additive, and $\leq \|\cdot\|_p^p$ on h^p, so it is continuous on h^p. Therefore $h_0^p = N^{-1}\{0\}$ is closed in h^p.

The next result shows that h_0^p is infinite dimensional.

2.5 PROPOSITION. *If μ is a complex Borel measure on T, singular with respect to m, then the Poisson integral of μ belongs to h_0^p for every $0 < p < 1$.*

Proof. Let $u = P[\mu]$ be the Poisson integral of μ. For $0 \leq r < 1$ we regard the dilate u_r as a function defined only on T. Then the family $\{u_r: 0 \leq r < 1\}$ is bounded in $L^1(T)$, and

$$\lim_{r \to 1-} u_r = 0 \quad \text{a.e. } [m]$$

[21; Chapter 11]. Thus the functions $|u_r|^p$ $(0 \leq r < 1)$ form a uniformly integrable family which tends a.e. $[m]$ to zero as $r \to 1-$; so by Vitali's theorem [21; Chapter 6, problems 10 and 11]

$$\lim_{r \to 1-} \int_T |u_r|^p \, dm = 0,$$

which shows that $u \in h_0^p$, as desired.

2.6 PROPOSITION. *h_0^p is contained in $h^p(\mathscr{P})$, the closure in h^p of the harmonic polynomials.*

Proof. Fix u in h_0^p and $\varepsilon > 0$. We must find a harmonic polynomial v which approximates u within ε. Choose $\rho(\varepsilon) > 0$ so that

$$M_p^p(u; \rho) < \varepsilon/4$$

whenever $\rho(\varepsilon) < \rho < 1$. Since $u_\rho \to u$ uniformly on compact subsets of Δ as $\rho \to 1-$, we can choose $\sigma(\varepsilon) \geq \rho(\varepsilon)^{1/2}$ such that whenever $\sigma(\varepsilon) < \rho < 1$ we have

$$|u_\rho(z) - u(z)| < (\varepsilon/2)^{1/p}$$

for all $|z| \leq \rho(\varepsilon)^{1/2}$. We claim that

$$\|u - u_\rho\|_p^p < \varepsilon/2 \tag{1}$$

whenever $\sigma(\varepsilon) < \rho < 1$. To see this, fix such a ρ, and consider two cases. First suppose $\rho(\varepsilon)^{1/2} < r < 1$. Then $r\rho > \sqrt{\rho(\varepsilon)\sigma(\varepsilon)} \geq \rho(\varepsilon)$, so

$$\begin{aligned} M_p^p(u_\rho - u; r) &\leq M_p^p(u_\rho; r) + M_p^p(u; r) \\ &= M_p^p(u; r\rho) + M_p^p(u; r) \\ &< \varepsilon/4 + \varepsilon/4 \\ &= \varepsilon/2. \end{aligned}$$

The only other possibility is $0 \leq r < \rho(\varepsilon)^{1/2}$, in which case our choice of $\sigma(\varepsilon)$ guarantees that $M_p^p(u_\rho - u; r) < \varepsilon/2$. This completes the proof of (1).

Since u_ρ is harmonic in a neighborhood of $\bar{\Delta}$, its series expansion 2.1 (3) converges to u_ρ uniformly on $\bar{\Delta}$, and it therefore converges to u_ρ in h^p. Let v be a symmetric partial sum of this series, chosen so that $\|u_\rho - v\|_p^p < \varepsilon/2$. Then v is a harmonic polynomial and $\|u - v\|_p^p < \varepsilon$, as desired.

2.7 PROPOSITION. *If $u \in h^p(\mathscr{P})$ then the map $w \to u_w$ takes Δ continuously into $h^p(\mathscr{P})$.*

Proof. The result is true if u is a harmonic polynomial. For general u in $h^p(\mathscr{P})$ it follows from this special case by a standard $\varepsilon/3$-argument and the fact that dilation is linear and "norm"-decreasing on h^p. We omit the details.

2.8 c_0 *and* l^∞. We denote by l^∞ the Banach space of bounded complex sequences, taken in the supremum norm:

$$\|\alpha\|_\infty = \sup_n |\alpha_n|, \quad \alpha = (\alpha_n)_1^\infty \in l^\infty.$$

c_0 is the closed subspace of l^∞ consisting of sequences which converge to zero.

2.9 *Isomorphism.* We call two F-spaces *isomorphic* if there is a linear homeomorphism taking one onto the other. A linear homeomorphism is called an *isomorphism*.

2.10 *p-norms and quasinorms.* Suppose X is a real or complex vector space and $\|\cdot\|$ is a non-negative function on X which vanishes precisely at 0. We call this functional a *p-norm* if it is sub-additive and p-homogeneous:

$$\|ax\| = |a|^p\|x\|$$

for every $x \in X$ and scalar a. If, on the other hand the functional is (1 −) homogeneous and *quasi*-sub*additive*,

$$\|x + y\| \leq C(\|x\| + \|y\|) \quad (x, y \in X)$$

where $C < \infty$ is independent of x and y, then we call it a *quasinorm*. For $X = h^p$, the functional $\|\cdot\|_p$ is a quasinorm, while $\|\cdot\|_p^p$ is a p-norm.

3. h_0^p and some of its subspaces

We begin this section by proving that $\{0\}$ is the only rotation-invariant locally convex subspace of h_0^p. However if the requirement of rotation-invariance is dropped, then matters are different: every infinite dimensional closed subspace of h_0^p contains a further closed subspace isomorphic to c_0. Our proof yields in addition the existence of l^∞ subspaces in h^p, and hence the nonseparability of h^p. We show how the c_0 subspaces force the conjugate function operator to behave badly; and we close the section by showing that h_0^p is weakly dense in $h^p(\mathscr{P})$.

3.1 THEOREM. *The zero subspace is the only rotation-invariant linear subspace of h_0^p that is locally convex.*

Proof. Suppose that $X \neq \{0\}$ is a rotation-invariant linear subspace of $h^p(\mathscr{P})$ which is locally convex. We will show that X cannot be contained in h_0^p. There is no loss of generality in assuming that X is closed: for the closure of X is a rotation-invariant locally convex subspace of $h^p(\mathscr{P})$ which would be contained in h_0^p (since h_0^p is closed) if this were true of X.

Choose u in X, $u \not\equiv 0$. Then there exists an integer n such that $\hat{u}(n) \neq 0$. Let

$$e_n(re^{i\theta}) = r^{|n|}e^{in\theta}. \tag{1}$$

For each ω in T the rotate u_ω belongs to X, and Proposition 2.7 guarantees that the map $\omega \to u_\omega$ takes T continuously into X. Since X is locally convex, it is isomorphic to a Banach space, and since $X \subset h^p(\mathscr{P})$, it is separable. Thus the right side of the formula

$$\hat{u}(n)e_n(z) = \int_T u_\omega(z)\bar{\omega}^n\,dm(\omega), \tag{2}$$

valid for all z in Δ, can be regarded as the Bochner integral of the X-valued continuous function $\omega \to u_\omega\bar{\omega}^n$ $(\omega \in T)$ [30; Chapter 5, Sections 4 and 5]. With this interpretation, formula (2) above can be rewritten as

$$\hat{u}(n)e_n = \int_T u_\omega\bar{\omega}^n\,dm(\omega),$$

where the right-hand side belongs to X. Since $\hat{u}(n) \neq 0$ this implies that $e_n \in X$, so $X \not\subset h_0^p$, and the proof is complete.

3.2 *Remarks.* (a) Theorem 3.1 shows, in particular, that every non-trivial rotation-invariant subspace of h_0^p is infinite dimensional. To get examples different from h_0^p, choose a complex Borel measure μ on T, singular with respect to m, whose Fourier-Stieltjes transform has support $E \neq \mathbf{Z}$, and let X be the closure in h^p of the linear span of the rotates of $P[\mu]$ (the Poisson integral of μ). By Propositions 2.4 and 2.5, $X \subset h_0^p$. Growth estimate 2.2 (b) shows that the coefficient functionals

$$u \to \hat{u}(n) \quad (u \in h^p)$$

are all continuous on h^p, so if $u \in X$ and $n \notin E$, then $\hat{u}(n) = 0$. Thus $X \neq h_0^p$, as desired. Note that Riesz products [31; Vol. I, Chapter 5, Section 7] provide examples of singular measures μ with rather thin spectra: E can even be a set of density zero.

(b) In contrast to h_0^p, the larger space $h^p(\mathscr{P})$ has many locally convex rotation-invariant subspaces, both finite and infinite dimensional. For example, let E be any subset of $\mathbf{Z}$, and let X_E denote the closure in h^p of the linear span of the functions $\{e_n: n \in E\}$, where e_n is defined by equation 3.1 (1). Then X_E is a closed, rotation-invariant subspace of $h^p(\mathscr{P})$ which is finite dimensional if and only if E is a finite set. If E is infinite, but "thin", then X_E can still be locally convex. For example if E is an infinite Hadamard sequence of positive integers ($E = \{n_k\}_1^\infty$ where $\inf n_{k+1}/n_k > 1$) then it is a standard fact of Fourier analysis [31; Vol. I, Chapter 5, Theorem 8.20, page 215] that there is a positive constant $c = c(p, E)$ such that for each u in X_E and $0 \leq r < 1$,

$$M_2(u; r) \leq cM_p(u; r),$$

so $\|u\|_2 \leq c\|u\|_p$ for every u in X. Since $\|u\|_p \leq \|u\|_2$ for every harmonic function, it follows that X is isomorphic to a closed subspace of the Hilbert space h^2; hence X is itself isomorphic to a Hilbert space.

(c) The proof of Theorem 3.2 shows that if X is a closed, rotation-invariant subspace of $h^p(\mathscr{P})$ which is locally convex, and if $E = \{n \in \mathbf{Z}: \hat{u}(n) \neq 0$ for some $u \in X\}$, then $X = X_E$. It would be interesting to characterize those sets $E \subset \mathbf{Z}$ for which X_E is locally convex. For example, (a) above shows that such an E cannot contain the spectrum of a singular measure.

We turn next to the construction of c_0 and l^∞ subspaces in h_0^p. The next lemma provides the crucial step.

3.3 LEMMA. *Suppose $(u_n)_1^\infty$ is a sequence in h_0^p which converges to zero uniformly on compact subsets of Δ, and suppose that $\|u_n\|_p = 1$ for all n. Then for each $\varepsilon > 0$ there is a subsequence*

$$v_k = u_{n_k}, \quad n_k \uparrow \infty,$$

such that for every $\alpha = (\alpha_k)_1^\infty$ *in* l^∞, *the series* $\sum \alpha_k v_k$ *converges uniformly on compact subsets of* Δ *to a function* $u_\alpha \in h^p$, *and*

$$(1) \qquad (1-\varepsilon)\|\alpha\|_\infty \le \|u_\alpha\|_p \le (1+\varepsilon)\|\alpha\|_\infty.$$

Proof. By induction choose positive integers $n_1 < n_2 < \cdots$ and numbers $0 = r_0 < r_1 < r_2 \cdots \to 1$ such that if $v_k = u_{n_k}$, then

$$M_p^p(v_k; r) < \varepsilon/2^k$$

for $r \in [0,1)\setminus[r_{k-1}, r_k)$. Suppose $\alpha = (\alpha_k) \in l^\infty$ and fix $0 \le r < 1$. Let j be the unique index for which $r_{j-1} \le r < r_j$. Then

$$\begin{aligned}\sum_k M_p^p(\alpha_k v_k; r) &= \sum_k |\alpha_k|^p M_p^p(v_k; r)\\ &\le |\alpha_j|^p M_p^p(v_j; r) + \sum_{k \ne j} |\alpha_k|^p \varepsilon/2\\ &\le (1+\varepsilon)\|\alpha\|_\infty^p.\end{aligned}$$

It follows from this inequality and growth estimate 2.2 (a) that the partial sums of the series $\sum \alpha_k v_k$ form a sequence that is Cauchy uniformly on compact subsets of the open disc $\{|z| < r\}$. Since r is arbitrary, the series converges uniformly on compact subsets of Δ to a harmonic function u_α, and the last inequality implies also that $\|u_\alpha\|_p^p \le (1+\varepsilon)\|\alpha\|_\infty^p$.

In the other direction, suppose $\lambda > 0$ with $\varepsilon < \lambda^p < 1$; and choose an index j so that $|\alpha_j| > \lambda\|\alpha\|_\infty$. By the choice of u_j, there exists $r \in [r_{j-1}, r_j)$ such that $M_p(v_j; r) = 1$. Then

$$\begin{aligned}\|u_\alpha\|_p^p &\ge M_p^p(u_\alpha; r)\\ &\ge |\alpha_j|^p M_p^p(v_j; r) - \sum_{k \ne j} |\alpha_k|^p M_p^p(v_k; r)\\ &\ge |\alpha_j|^p - \|\alpha\|_\infty \sum_{k=1}^\infty \varepsilon/2^k\\ &\ge (\lambda^p - \varepsilon)\|\alpha\|_\infty^p.\end{aligned}$$

Letting $\lambda \to 1$ and re-adjusting ε properly we get the desired result.

3.4 THEOREM. *Every infinite dimensional closed subspace of* h_0^p *contains a further closed subspace isomorphic to* c_0.

Proof. Suppose X is an infinite dimensional closed subspace of h_0^p. Let κ denote the restriction to X of the topology of uniform convergence on compact subsets of Δ. Growth estimate 2.2 (a) shows that κ is weaker than the h^p-topology on X. In fact it is *strictly* weaker, since otherwise the closed unit ball of X, which is a normal family, and therefore κ-sequentially relatively compact, would also be κ-closed and therefore compact in the h^p topology (which we are assuming equals the κ-topology on X). But this forces X to be finite dimensional [22; Theorem 1.22, page 17], which it is not.

Thus there exists a sequence $(u_n)_1^\infty$ in X which is both bounded and bounded away from zero in the h^p-topology, but which converges to zero uniformly on compact subsets of Δ. By replacing u_n with $u_n/\|u_n\|_p$ we may assume $\|u_n\|_p = 1$ for all n. Fix $\varepsilon > 0$ and apply Lemma 3.3 to get the sequence (v_k). Define the linear operator S: $l^\infty \to h^p$ by

$$S\alpha = \sum \alpha_k v_k.$$

By Lemma 3.3 (inequality (1)), S is an isomorphism taking l^∞ into h^p. We will be done if we show that $S(c_0) \subset h_0^p$.

Fix α in c_0. By the right side of inequality (1) of Lemma 3.3 we have

$$\left\| u_\alpha - \sum_{k=1}^{n} \alpha_k v_k \right\|_p = \left\| \sum_{k=n+1}^{\infty} \alpha_k v_h \right\|_p \le (1+\varepsilon) \sup_{k>n} |\alpha_h|,$$

and the last expression tends to zero as $n \to \infty$, since $\alpha \in c_0$. Thus the series $\Sigma \alpha_k v_k$ converges in h_0^p, so $u_\alpha \in c_0$ and the proof is complete.

In the course of the proof we also obtained the following result, essentially due to Hardy and Littlewood [9; sec. 4.4, page 417].

3.5 THEOREM. *h^p contains closed subspaces isomorphic to l^∞. In particular, h^p is not separable.*

Results similar to Theorems 3.4 and 3.5 have also been obtained by Alec Matheson (private communication) for spaces of harmonic functions satisfying a growth restriction, and by Walter Rudin [23] for the Lumer-Hardy spaces in the unit ball of C^n, $n > 1$.

To get a feeling for the next result, observe that $h^p(\mathscr{P})$ contains infinite dimensional subspaces isomorphic to closed subspaces of Lebesgue spaces. For example it contains H^p, which is isometrically embedded in $L^p(T)$ by the radial limit map [5; Chapters 2 and 3]; and by Remark 3.2 (b) it contains a closed subspace isomorphic to separable Hilbert space, hence to $L^2(T)$. Theorem 3.4 implies that nothing like this can happen in h_0^p.

3.6 THEOREM. *Suppose μ is a positive measure and $0 < r < \infty$. Then no infinite dimensional closed subspace of h_0^p is isomorphic to a closed subspace of $L^r(\mu)$.*

Proof. By Theorem 3.4 it is enough to prove that $L^r(\mu)$ contains no closed subspace isomorphic to c_0. This is a well known fact, but for completeness we present a proof. Suppose X is a closed subspace of $L^r(\mu)$ isomorphic to c_0. Then there is a sequence $(x_n)_1^\infty$ in X and a constant $c > 0$ such that for every α in c_0 the series $\Sigma\alpha_n X_n$ converges in X, and

$$(1) \qquad c^{-1}\|\alpha\|_\infty \leq \|\sum \alpha_n x_n\|_r \leq c\|\alpha\|_\infty,$$

where here the symbol $\|\cdot\|_r$ denotes the L^r quasinorm. Let $(\varepsilon_n)_1^\infty$ denote the sequence of Rademacher functions on $[0,1]$. Then by successively applying (1), Fubini's theorem, Kintchine's inequality [31; Vol. I, Chapter 5, Theorem 8.4, page 213], and the orthonormality of the Rademacher functions on $[0,1]$, we obtain

$$\begin{aligned} C\|\alpha\|_\infty &\geq \int_0^1\left\{\int\left|\sum \varepsilon_n(t)\alpha_n x_n\right|^r d\mu\right\} dt \\ &= \int\left\{\int_0^1\left|\sum \varepsilon_n(t)\alpha_n x_n\right|^r dt\right\} d\mu \\ &\geq A\int\left\{\int_0^1\left|\sum \varepsilon_n(t)\alpha_n x_n\right|^2 dt\right\}^{r/2} d\mu \\ &= A\int\left\{\sum|\alpha_n|^2|x_n|^2\right\}^{r/2} d\mu, \end{aligned}$$

where A is the constant in Kintchine's inequality; $A = A(r)$ independent of α and (x_n). Applying this inequality to α consisting of 1's in the first N positions and zero elsewhere, and then letting $N \to \infty$, we see that

$$\int\left(\sum|x_n|^2\right)^{r/2} d\mu < \infty.$$

Thus the series $\Sigma|x_n|^2$ converges in $L^{r/2}(\mu)$, so in particular $x_n \to 0$ in $L^r(\mu)$. But this contradicts the left side of inequality (1), so X cannot be isomorphic to c_0. The proof is complete.

The next two results use Theorem 3.6 to "explain" why the conjugate function operator behaves so badly on h^p.

3.7. THEOREM. *Every closed infinite dimensional subspace of h_0^p contains a function whose harmonic conjugate does not belong to h^p.*

Proof. Suppose X is a closed subspace of h_0^p mapped into h^p by the conjugate function operator. A routine argument employing the Closed Graph

Theorem and the continuity of the coefficient functionals shows that this operator takes X *continuously* into h^p. It follows that the map R^+ defined by

$$R^+ = (u + i\tilde{u})/2 \quad (u \text{ in } X)$$

is a continuous linear operator taking X into the holomorphic Hardy space H^p. Similarly the linear map R^- defined by

$$R^- = (u - i\tilde{u})/2$$

takes X continuously into $\overline{H^p}$ (the closed subspace of h^p consisting of complex conjugates of H^p functions). We define a map T from X into the direct sum $H^p \oplus \overline{H^p}$ by

$$Tu = (R^+u, R^-u) \quad (u \text{ in } X).$$

Then T is a bounded linear operator, and if we denote the quasinorm in $H^p \oplus \overline{H^p}$ by $|||\cdot|||_p$, then for each u in X we have

$$\begin{aligned} |||Tu|||_p^p &= \|R^+u\|_p^p + \|R^-u\|_p^p \quad \text{(by definition)} \\ &\geq \|R^+u + R^-u\|_p^p \\ &= \|u\|_p^p, \end{aligned}$$

so T is an isomorphism onto its range. Since the radial limit map takes H^p isometrically onto a closed subspace of $L^p(T)$ [5; Chapters 2 and 3], we see that $H^p \oplus \overline{H^p}$ is isomorphic to a closed subspace of $L^p(T) \oplus L^p(T)$, which is in turn isometrically isomorphic to $L^p(\mu)$, where μ is normalized Lebesgue measure on the disjoint union of two circles. Thus X is isomorphic to a closed subspace of $L^p(\mu)$, so by Theorem 3.6, X is finite dimensional. The proof is complete.

Our next goal is to refine the classical result, noted in the Introduction, which states that there exist functions in

$$\bigcap \{h^p : 0 < p < 1\}$$

whose harmonic conjugates lie in no h^p space.

3.8. Definition. Denote by h^{1-} the space $\bigcap\{h^p\colon 0 < p < 1\}$, equipped with its natural metric topology, in which a sequence converges to zero if and only if it converges to zero in each h^p $(0 < p < 1)$. Let $h_0^{1-} = \bigcap\{h_0^p\colon 0 < p < 1\}$.

Thus h_0^{1-} is a closed subspace of h^{1-}, and the conclusion of Theorem 3.1 holds here also, with essentially the same proof. To motivate the next definition note that a subspace of h^{1-} which is closed in h^p for *some* $0 < p < 1$ is then closed in h^{1-}

3.9. DEFINITION. A linear subspace of h^{1-} is *universally* closed (u.c. for short) if it is closed in h^p for *every* $0 < p < 1$.

Clearly every finite dimensional subspace of h^{1-} is u.c., and we will see before long that even h_0^{1-} contains infinite dimensional u.c. subspaces. The Open Mapping Theorem shows that a closed subspace of h^{1-} is universally closed if and only if the restrictions to the subspace of the h^p topologies coincide for all $0 < p < 1$. In particular, every closed subspace of a u.c. subspace is again u.c..

Our main result about u.c. subspaces is the following.

3.10. THEOREM. *Every infinite dimensional universally closed subspace of h_0^{1-} contains a function whose harmonic conjugate belongs to no h^p space.*

Proof. Suppose X is an infinite dimensional u.c. subspace of h_0^{1-}. Fix a sequence $1 > p_n \downarrow 0$, and let

$$X_n = \{u \in X \colon \tilde{u} \in h^{p_n}\}.$$

Since X_n is Borel measurable and, by Theorem 3.7, not equal to X, it is of first category in X [3; Chapitre III, Théoreme 1, page 36]. Thus $\bigcup_n X_n$ is also of first category, and is therefore not all of X. The proof is complete.

The next result shows that the hypothesis of Theorem 3.10 is not vacuous.

3.11. PROPOSITION. *h_0^{1-} contains infinite dimensional universally closed subspaces.*

Proof. Fix u in h_0^{1-} with $u(0) = 0$, but u not identically zero. Define a sequence $(u_n)_1^\infty$ in h_0^{1-} by $u_n(z) = u(z^n)$ (z in Δ). Then $u_n \to 0$ uniformly on compact subsets of Δ, and for $0 < p < 1$ we have $\|u_n\|_p = \|u\|_p$. By Lemma 3.3 and a diagonal argument there exists a subsequence $v_k = u_{n_k}$ ($n_k \uparrow \infty$) such that for each α in l^∞ the series $\Sigma \alpha_k v_k$ converges uniformly on compact subsets of Δ to a harmonic function u_α, and for each $0 < p < 1$ there exist constants A_p and $B_p > 0$, independent of α, such that

$$A_p \|\alpha\|_\infty \le \|u_\alpha\|_p \le B_p \|\alpha\|_\infty.$$

This shows that the map $S\colon \alpha \to u_\alpha$ takes c_0 isomorphically into h_0^p for all $0 < p < 1$. Thus $S(c_0)$ is contained in h_0^{1-}, and closed in h^p for all $0 < p < 1$. In other words, $S(c_0)$ is an infinite dimensional u.c. subspace of h_0^{1-}. The proof is complete.

Our proof has also shown that S maps l^∞ isomorphically into h^{1-}. Thus h^{1-} is not separable.

The next result shows that Theorem 3.7 is in some sense optimal.

3.12. THEOREM. *If $n = 2, 3, 4, \ldots,$ then $h^{1/n}$ has a closed subspace X which is mapped by the conjugate function operator into $\bigcap\{h^p\colon p < 1/n\}$.*

Proof. We phrase this argument in the language of Schwartz distributions on T; see [18; Chapter 1, sec. 7, problem 5, page 43] for the relevant definitions. In particular, the series expansion 2.1 (3) and growth estimate 2.2 (b) show that each u in $h^{1/n}$ is the Poisson integral of a distribution μ_u on T of order at most $n-1$.

Fix a compact set $E \subset T$ with infinitely many points, such that $m(E) = 0$, and for each $p < 1/n$ the distance function

$$\rho_E(z) = \inf\{|z - \tau|\colon \tau \in E\} = \operatorname{dist}(z, E),$$

defined for $z \in \overline{\Delta}$, satisfies

$$\int_T \rho_E(\omega)^{-np}\, dm(\omega) < \infty. \tag{1}$$

We will see in a moment that such sets E exist. Granting this, let h^p_E denote the collection of functions u in h^p whose representing distribution μ_u has its support contained in E. Since E is infinite, h^p_E contains infinitely many rotates of the Poisson kernel, so its intersection with h^p_0 is infinite dimensional. Let $X = h^p_0 \cap h^p_E$. Growth estimate 2.2 (b) shows that if $u_n \to u$ in h^p, then $\mu_{u_n} \to \mu_u$ in the sense of distributions, so h^p_E is closed in h^p. Thus X is an infinite dimensional closed subspace of h^p_0.

We claim that the conjugate function operator takes X into h^p for every $p < 1/n$. To see this, fix such a p, and fix u in X. Let μ be the representing distribution for u, so μ has order $\le n-1$, support $\mu \subset E$, and $u = P[\mu]$. Then $u + i\tilde{u} = C[\mu]$, the *Cauchy transform* of μ, and by [29; Lemma A, page 342] we know that there exists a positive constant A_μ such that for each z in Δ,

$$|C[\mu](z)| \le A_\mu \rho_E(z)^{-n}. \tag{2}$$

Writing $z = re^{i\theta}$ and using the standard estimate

$$\rho_E(re^{i\theta}) \ge r^{1/2} \rho_E(e^{i\theta})$$

([29; page 348], for example) we obtain from (2) the inequality

$$|C[\mu](re^{i\theta})| \le A_\mu r^{-1/2} \rho_E(e^{i\theta})^{-n}, \tag{3}$$

valid for each $re^{i\theta}$ in Δ. It follows from (1) and (3) that $C[\mu] \in H^p$, so $\tilde{u} \in h^p$ as desired.

We now prove that the sets E with the desired properties exist. If E is any compact subset of T and (I_n) is the collection of disjoint open arcs whose union is $\mathbf{T} \setminus E$, then a straightforward calculation shows that the conditions $m(E) = 0$ and (1) are equivalent to

$$\sum m(I_n) = 1 \tag{4}$$

and

$$\sum m(I_n)^{1-np} < \infty \quad \text{for all } 0 < p < 1/n. \tag{5}$$

Thus we can even realize E as a countable set with exactly one limit point; we leave the details to the reader. The proof is now complete.

We close this section with a result that sets the stage for the work of the next section, and shows how the Hahn-Banach theorem can fail in an F-space which is not locally convex.

3.13 Proposition. *h_0^p is weakly dense in $h^p(\mathscr{P})$.*

Proof. Suppose λ is a continuous linear functional on $h^p(\mathscr{P})$ which vanishes on h_0^p. We must show that λ is zero on all of $h^p(\mathscr{P})$. It is enough to show that $\lambda(e_n) = 0$ for all $n \in \mathbf{Z}$, where e_n is given by equation 3.1 (1). Let u denote the Poisson kernel, so for $z = re^{i\theta} \in \Delta$,

$$u(z) = \operatorname{Re}\frac{1+z}{1-z} = \sum_{-\infty}^{\infty} r^{|n|}e^{in\theta}.$$

By Proposition 2.5, $u \in h_0^p$, so Propositions 2.6 and 2.7 guarantee that the dilation map $w \to u_w$ takes $\bar{\Delta}$ continuously into $h^p(\mathscr{P})$. Thus the complex function Λ defined by

$$\Lambda(w) = \lambda(u_w) \quad (w \in \bar{\Delta})$$

is continuous on $\bar{\Delta}$. Since $u_w \in h_0^p$ for every $w \in T$, we see that Λ vanishes identically on T.

Now if $w = \rho e^{i\theta} \in \Delta$, then the series expansion

$$u_w = \sum_{-\infty}^{\infty} \rho^{|n|}e^{in\theta}e_n$$

converges uniformly on $\bar{\Delta}$, hence in $h^p(\mathscr{P})$, so

$$\Lambda(w) = \lambda(u_w) = \sum \rho^{|n|}e^{in\theta}\lambda(e_n),$$

and *this* series converges uniformly on compact subsets of Δ. Thus Λ is harmonic in Δ, and since it is continuous on $\bar{\Delta}$ and $\equiv 0$ on T we must have $\Lambda \equiv 0$ on Δ. Thus for each $n \in \mathbf{Z}$,

$$0 = \hat{\Lambda}(n) = \lambda(e_n)$$

which shows that λ vanishes on $h^p(\mathscr{P})$. The proof is now complete.

As we noted in the introduction, this result shows that the quotient space $h^p(\mathscr{P})/h_0^p$ has trivial dual. In the next section we prove a sharper result: the quotient space is isomorphic to $L^p(T)$.

4. $h^p(\mathscr{P})/h_0^p = L^p$

The main result of this section, Theorem 4.7, assets that when $0 < p < 1$ the Poisson integral can be used to establish an isomorphism between $L^p(T)$ and the quotient space $h^p(\mathscr{P})/h_0^p$. This result should be viewed as the analogue for the Poisson integral of a similar one recently proved for the conjugate function by A.B. Aleksandrov [1; Theorem 1, page 303]. In fact our proof is an adaptation of Aleksandrov's and it is possible to derive our result from his. However, there are several reasons for giving an independent proof. First, we wish to keep the paper reasonably self-contained, and our proof is slightly easier than Aleksandrov's since it deals with the Poisson integral rather than the Hilbert transform.[2] Next, our proof generalizes quickly to the h^p spaces of the unit ball in R^n for $n > 2$ (Section 4.8 contains a discussion of this), where generalized harmonic conjugates do not behave properly for all values of $p < 1$. Finally, we need some of the estimates involved in our proof to show that the rotates of the Poisson kernel span a dense linear subspace of h_0^p (Theorem 4.10). Before proving this last result we discuss some linear topological consequences of Theorem 4.7 which follow from recent work of Kalton and Peck.

4.1 *Notation.* (a) *Revised notation for quasinorms.* For this section only we let $\|\cdot\|_p$ denote the L^p-quasinorm, and $\|\cdot\|_{h_p}$ the h^p-quasinorm. We also write L^p for $L^p(T)$.

(b) *Characteristic functions.* If E is a subset of T, then χ_E denotes the characteristic function of T; that is, $\chi_E = 1$ on E and $= 0$ off E.

(c) *Quotient space.* We will be dealing with the quotient space $h^p(\mathscr{P})/h_0^p$, which is by definition the space of cosets

$$u + h_0^p = \{u + u_0 \colon u_0 \in h_0^p\}$$

[2] However see Remarks 4.12 for a simpler proof suggested by the referee, based on a more function theoretic idea of Aleksandrov.

where u runs through $h(\mathscr{P})$. Since h_0^p is a closed subspace of $h^p(\mathscr{P})$ (Theorems 2.4 and 2.6), the functional

$$|||u + h_0^p|||_p = \inf\left\{\|u + u_0\|_{h_p} : u_0 \in h_0^p\right\}$$

is a quasinorm which makes $h^p(\mathscr{P})/h_0^p$ into an F-space, and induces the quotient topology.

(d) *Poisson kernel.* For $z = re^{i\theta} \in \Delta$ we write

$$P(z) = P(r,\theta) = \operatorname{Re}\frac{1+z}{1-z} = \frac{1-r^2}{1 - 2r\cos\theta + r^2}$$

for the Poisson kernel, and we denote the n^{th} derivative of $P(r,\theta)$ with respect to θ by $P^{(n)}(r,\theta) = P^{(n)}(z)$. For t real and $m = 1, 2, \ldots,$ we write

$$Q_m(z,t) = \sum_{n=0}^{m-1} P^{(n)}(z)t^n/n!,$$

where $P^{(0)} = P$. Note that $Q_m(re^{i\theta}, t)$ is a Taylor polynomial for $P(r, \theta - t)$, where t is viewed as the variable and the center of the expansion is $t = 0$.

For $f \in L^1$ define $P[f]$ to be the Poisson integral of f, and

$$Q_m[f](z) = \int_{-\pi}^{\pi} Q_m(z,t) f(t)\, dt/2\pi$$

$$= \sum_{n=0}^{m-1} \frac{P^{(n)}(z)}{n!} \int_{-\pi}^{\pi} t^n f(e^{it})\, dt/2\pi.$$

All the functions $P^{(n)}$, Q_m are harmonic on Δ.

(e) *Positive constants.* Throughout the proofs of this section we let C denote a positive number which may vary from line to line, but which depends at each occurrence only on the index p.

We require some standard integral estimates, whose proofs we omit.

4.2 Proposition. *For each $0 < \alpha < \infty$ there is a positive constant C_α such that*

(a) $\int_{-\pi}^{\pi} |1 - re^{i\theta}|^{-\alpha}\, d\theta \le C_\alpha (1-r)^{1-\alpha}$ *if* $\alpha > 1$ *and* $0 \le r < 1$ [5; *page* 65]; *and*

(b) $\int_{|\theta| \le \delta} |1 - re^{i\theta}|^{-\alpha}\, d\theta \le C_\alpha \delta^{1-\alpha}$ *if* $0 < \delta \le \pi$ *and* $0 < \alpha < 1$.

We also need a pointwise estimate on the derivatives $P^{(n)}(z)$.

4.3 Lemma. *For $n = 0, 1, 2, \ldots$ there exists a constant $C_n < \infty$ such that for each z in Δ with $|z| \ge 1/2$, and each real t we have*

$$|P^{(n)}(z)| \le C_n(1 - |z|)/|1 - z|^{n+2}.$$

Proof. Write $g(\theta) = 1 - 2r\cos\theta + r^2$. Then an induction argument shows that the n^{th} derivative of $1/g$ has the form

$$(1/g)^{(n)}(\theta) = \sum_{k=1}^{n} a_k F_k / g(\theta)^{k+1}$$

where each a_k is real and F_k is a sum of products of derivatives of g with each such product containing $g'(\theta)$ to the power

$$(2k - n)^+ = \max(2k - n, 0).$$

This shows that

$$P^{(n)}(re^{i\theta}) = \sum_{k=1}^{n} a_k \frac{(1 - r^2) F_k(r,\theta)}{(1 - 2r\cos\theta + r^2)^{k+1}} \tag{1}$$

where $F_k(r,\theta)$ is a sum of products of powers of r, $\sin\theta$, and $\cos\theta$, where each such product contains $\sin\theta$ raised to at least the power $(2k - n)^+$.

If $2k > n$ then the k^{th} term on the right side of (1) is dominated by a constant multiple of

$$(1 - r)|\sin\theta|^{2k-n}/|1 - re^{i\theta}|^{2k+2},$$

which is itself dominated by a constant multiple of

$$(1 - r)/|1 - re^{i\theta}|^{n+2},$$

as desired. On the other hand, if $2k \le n$, then the term in question is dominated by a constant multiple of

$$(1 - r)/|1 - re^{i\theta}|^{2k+2}$$

which has the desired growth since the exponent in the denominator is $\le n + 2$. This completes the proof.

The next result generalizes the fact that the Poisson kernel P belongs to h^1, and also to h_0^p for every $p < 1$ (Proposition 2.5).

4.4 Proposition. *For $n = 1, 2, \ldots$, the derivative $P^{(n)}$ belongs to $h^{1/(n+1)}$, and also to h_0^p for all $p < 1/(n+1)$.*

Proof. By Lemma 4.3,

$$\lim_{r \to 1-} P^{(n)}(re^{i\theta}) = 0 \quad \text{if } e^{i\theta} \ne 1,$$

so the uniform integrability argument of Proposition 2.5 guarantees that the second conclusion of Proposition 4.4 follows from the first. To show that $P^{(n)} \in h^{1/(n+1)}$, let $\alpha = (n+2)/(n+1)$, fix $1/2 \le r < 1$, and successively apply Lemma 4.3 and the integral estimate 4.2 (a):

$$\begin{aligned}\int |P^{(n)}(re^{i\theta})|^{1/(n+1)} &\le C(1-r)^{\alpha-1}\int_{-\pi}^{\pi}|1-re^{i\theta}|^{-\alpha}\,d\theta\\ &\le C(1-r)^{\alpha-1}/(1-r)^{\alpha-1}\\ &= C\end{aligned}$$

where C is a positive constant independent of r. This completes the proof.

The crucial step in the proof of our main result is the following sharp special case.

4.5 THEOREM. *Suppose $0 < p < 1$ and let m be the unique positive integer such that $1/(m+1) \le p < 1/m$. Then there is a positive constant $C = C(p)$ such that for every subarc I of the unit circle centered at the point* 1,

$$\|P[\chi_I] - Q_m[\chi_I]\|_{h_p} \le C\|\chi_I\|_p$$

Proof. Suppose $I = \{e^{i\theta}: -\delta \le \theta \le \delta\}$, and let $2I$ denote the arc centered at 1 having twice the length of I. By Theorem 2.2 (a) the quasinorm

$$\sup\{M_p(u;r): 1/2 \le r < I\} \quad (u \in h^p)$$

is equivalent to the original h^p quasinorm, so it is enough to show that if $1/2 \le r < 1$, then

$$M_p^p(P[\chi_I] - Q_m[\chi_I]; r) \le C\delta \tag{1}$$

where C depends only on p and not on δ or r.

To this end, fix $1/2 \le r < 1$ and write the p^{th} power of the left side of (1) as $J + K$ where

$$J = \int_{2I}|P[\chi_I](r\omega) - Q_m[\chi_I](r\omega)|^p\,dm(\omega)$$

and K is the corresponding integral over $T \setminus 2I$.

We estimate J. Since $0 < p < 1$ we have $J \le J_1 + J_2$ where

$$J_1 = \int_{2I}|P[\chi_I]|^p(r\omega)\,dm\,(\omega) \le 2\delta$$

since $0 \leq P[\chi_I] \leq 1$ on Δ, and

$$J_2 = \int_{2I} |Q_m[\chi_I](r\omega)|^p \, dm(\omega).$$

To estimate J_2 we use Lemma 4.3 to bound the integrand:

$$\begin{aligned}
|Q_m[\chi_I](re^{i\theta})| &\leq \sum_{n=0}^{m-1} \left| \frac{1}{n!} P^{(n)}(re^{i\theta}) \int_I t^n \, dt/2\pi \right| \\
&\leq \sum_{n=0}^{m-1} |P^{(n)}(re^{i\theta})| \delta^{n+1}/(n+1)! \\
&\leq C \sum_{n=0}^{m-1} \frac{(1-r)\delta^{n+1}}{|1 - re^{i\theta}|^{n+2}} \\
&\leq C \sum_{n=0}^{m-1} \left(\frac{\delta}{|1 - re^{i\theta}|} \right)^{n+1}
\end{aligned}$$

By Proposition 4.2 (b) and the fact that $(n+1)p < 1$, the integral over $2I$ of the p^{th} power of the n^{th} term of the last sum is bounded by a constant multiple of

$$\delta^{(n+1)p} \delta^{1-(n+1)p} = \delta,$$

so

$$\begin{aligned}
J_2 &\leq C \sum_{n=0}^{m-1} \int_{2I} \left(\frac{\delta}{|1 - r\omega|} \right)^{(n+1)p} dm(\omega) \\
&\leq C\delta.
\end{aligned}$$

Thus $J \leq C\delta$, as desired.

We now estimate the integral K, which we rewrite as

$$K = \int_{2\delta < |\theta| < \pi} \left| \int_{|t| < \delta} [P(r, \theta - t) - Q_m(re^{i\theta}, t)] \frac{dt}{2\pi} \right|^p d\theta/2\pi. \tag{2}$$

We need a pointwise bound on the integrand of the inner integral, so fix θ and t with $2\delta < |\theta| < \pi$ and $|t| < \delta$. By Taylor's theorem, with t as the variable and $t = 0$ the center,

$$\begin{aligned}
|P(r, \theta - t) - Q_m(re^{i\theta}, t)| &= \left| P(r, \theta - t) - \sum_{k=0}^{m-1} P^{(n)}(re^{i\theta}) t^n/n! \right| \\
&= |P^{(m)}(re^{i\tau}) t^m/m!|
\end{aligned} \tag{3}$$

where τ is a number in the interval between θ and $\theta - t$. Thus $|\tau| \geq |\theta|/2$, so

$$(4) \qquad |1 - re^{i\tau}| \geq C|1 - re^{i\theta}|$$

where C is independent of r, θ, and t. By (3), Lemma 4.3, and (4), in that order, we obtain

$$(5) \qquad |P(r,\theta - t) - Q_m(re^{i\theta}, t)| \leq C\frac{|t|^m(1 - r)}{|1 - re^{i\theta}|^{m+2}}.$$

Recall that $1 \leq p(m + 1)$. Suppose we have strict inequality. The right side of (5) is dominated by a constant multiple of

$$|t|^m/|1 - re^{i\theta}|^{m+1}.$$

Substituting this estimate for the integrand of (2) yields

$$\begin{aligned} K &\leq C\delta^{(m+1)p}\int_{-\pi}^{\pi}|1 - re^{i\theta}|^{-(m+1)p}\,d\theta \\ &\leq C\delta^{(m+1)p}\delta^{1-(m+1)p} \\ &= C\delta \end{aligned}$$

where the next-to-last line follows from Lemma 4.2 (a), because $(m + 1)p > 1$.

Suppose, on the other hand, that $(m + 1)p = 1$. Then we substitute (5) directly into (2) to get

$$\begin{aligned} K &\leq C\delta^{(m+1)p}(1 - r)^p\int_{-\pi}^{\pi}|1 - re^{i\theta}|^{-(m+2)p}\,d\theta \\ &= C\delta(1 - r)^p\int_{-\pi}^{\pi}|1 - re^{i\theta}|^{-(1+p)}\,d\theta \\ &\leq C, \end{aligned}$$

again by Lemma 4.2 (a). Thus in either case $K \leq C\delta$, so estimate (1) is established, and the proof is complete.

3.6 THEOREM. *Suppose* $0 < p < 1$. *Then there exists* $C = C(p) < \infty$ *such that for each* f *in* $L^1(T)$ *there is a function* u *in* h_0^p *with*

$$\|P[f] - u\|_{h_p} \leq C\|f\|_p.$$

In fact u *can be taken to be a finite linear combination of rotates of the derivatives* $P^{(n)}$ *of the Poisson kernel defined in* 4.1 (d), *where* $0 \leq n < (1/p) - 1$.

Proof. Let S denote the collection of finite sums

$$(1) \qquad f = \sum c_j\chi_{I_j}$$

where each c_j is a real number and $\{I_j\}$ is a finite pairwise disjoint collection of intervals. We first prove the result for S. Let Y denote the linear span of the rotates of the functions $P^{(n)}$ for $0 \leq n < (1/p) - 1$. By Proposition 4.4 we know $Y \subset h_0^p$. Thus for $m < 1/p$ and $f \in L^1(T)$, each rotate of $Q_m[f]$, as defined in 4.1 (d), belongs to Y.

Suppose f is given by (1). Then by Proposition 4.5 and the remarks above, there exists for each j a function $u_j \in Y$ such that

$$\|P[\chi_{I_j}] - u_j\|_{h_p}^p \leq C^p m(I_j)$$

where C depends only on p. Let $u = \sum u_j$. Then $u \in Y$, and

$$\begin{aligned}\|P[f] - u\|_{h_p}^p &\leq \sum |a_j|^p \|P[\chi_{I_j}] - u_j\|_{h_p}^p \\ &\leq C^p \sum |a_j|^p m(I_j) \\ &= C^p \|f\|_p^p\end{aligned}$$

which is the desired estimate.

If $f \in L^1$, then we can choose $g \in S$ such that $\|f - g\|_1 < \|f\|_p$. By the paragraph above there exists u in Y with

$$\|P[g] - u\|_{h_p} \leq C\|g\|_p.$$

Then

$$\begin{aligned}\|P[f] - u\|_{h_p}^p &\leq \|P[f] - P[g]\|_{h_p}^p + \|P[g] - u\|_{h_p}^p \\ &\leq \|P[f] - P[g]\|_{h_1}^p + C^p\|g\|_p^p \\ &\leq \|f - g\|_1^p + C^p\|g\|_p^p \\ &\leq \|f\|_p^p + C^p\|g\|_p^p.\end{aligned}$$

But

$$\begin{aligned}\|g\|_p^p &\leq \|f\|_p^p + \|f - g\|_p^p \\ &\leq \|f\|_p^p + \|f - g\|_1^p \\ &\leq 2\|f\|_p^p\end{aligned}$$

so

$$\|P[f] - u\|_{h_p}^p \leq (1 + 2C^p)\|f\|_p^p$$

and the proof is complete.

We can now prove the main result of this section.

4.7 THEOREM. *For each* $0 < p < 1$ *the map* $f \to P[f] + h_0^p$, *defined for* f *in* $L^1(T)$, *has a unique extension to an isomorphism of* $L^p(T)$ *onto the quotient space* $h^p(\mathscr{P})/h_0^p$.

Proof. L^1 is dense in L^p, and $P[f]$ belongs to $h^p(\mathscr{P})$ for every f in L^1. Thus Theorem 4.6 asserts that the map in question takes a dense subspace of L^p continuously into $h^p(\mathscr{P})/h_0^p$; hence it has a unique continuous linear extension

$$\mathbf{P}\colon L^p \to h^p(\mathscr{P})/h_0^p.$$

Since the range of $\mathbf{P}$ contains all cosets $u + h_0^p$ where u is a harmonic polynomial (these are just the cosets $P[f] + h_0^p$, f a real trigonometric polynomial), we see that $\mathbf{P}$ has dense range. So we will be done if we can show that $\mathbf{P}$ is bounded below.

In fact we claim that $|||\mathbf{P}[f]||| \geq \|f\|_p$ for every f in L^p. It is enough to check this on a dense subset of L^p, so suppose $f \in L^1$. Write $u = P[f]$ and consider the dilate u_r for $0 \leq r < 1$. As $r \to 1-$ we know that $u_r \to f$ in L^1, hence in L^p, so $M_p(u; r) \to \|f\|_p$. Thus for every $u_0 \in h_0^p$ the definition of the h^p quasinorm and the subadditivity of $M_p^p(\cdot\,; r)$ yield

$$\begin{aligned}
\|P[f] + u_0\|_{h_p}^p &\geq \lim_{r \to 1-} M_p^p(u + u_0; r) \\
&\geq \lim_{r \to 1-} \left[M_p^p(u; r) - M_p^p(u_0; r)\right] \\
&= \lim_{r \to 1-} M_p^p(u; r) \quad (\text{since } u_0 \in h_0^p) \\
&= \|f\|_p^p.
\end{aligned}$$

Since u_0 is any member of h_0^p, this implies that $|||\mathbf{P}[f]||| \geq \|f\|_p$, which completes the proof.

4.8 *Remark.* So far, all the results derived in this paper which do not involve the conjugate function have analogues for the h^p spaces defined on the unit ball of $\mathbf{R}^N$ $(N > 2)$. The proofs of these generalizations differ little from the ones given here, except that in Theorem 4.5 we take m so that

$$N/(m + N) \leq p < N/(m + N - 1),$$

and view I as a Euclidean spherical cap.

Theorem 4.7 has immediate linear topological consequences, the most obvious of which is Theorem 3.8, the weak density of h_0^p in $h^p(\mathscr{P})$. Here are two others.

4.9 COROLLARY. *If a compact linear operator from $h^p(\mathscr{P})$ into a Hausdorff topological vector space annihilates h_0^p, then the operator is identically zero.*

Proof. Kalton [12] has shown that there is no nontrivial compact operator from L^p into any Hausdorff linear topological space E. So if T: $h^p(\mathscr{P}) \to E$ is compact and identically zero on h_0^p, then the operator S defined by

$$S(u + h_0^p) = Tu \quad (u \in h^p(\mathscr{P}))$$

is a compact operator from $h^p(\mathscr{P})/h_0^p$ into E. By Theorem 4.7 and Kalton's result, $S \equiv 0$ on $h^p(\mathscr{P})/h_0^p$, hence $T \equiv 0$ on $h^p(\mathscr{P})$.

It should be noted that Corollary 4.9 does *not* follow from the fact that h_0^p is weakly dense in $h^p(\mathscr{P})$. In fact there exist F-spaces with trivial dual which nonetheless admit nontrivial compact operators [17], [2; Section 5].

4.10 COROLLARY. *Every continuous linear functional on h_0^p has a continuous linear extension to h^p.*

Proof. Results of Kalton and Peck ([16; Theorem 5.2, page 74] and [15; Theorem 4.3, page 268]) show that if X is an F-space and Y a closed subspace such that the quotient X/Y is isomorphic to L^p, then Y has the desired extension property in X. Thus Theorem 4.7 shows that h_0^p has the extension property in $h^p(\mathscr{P})$. We will be finished if we can show that $h^p(\mathscr{P})$ has the extension property in h^p.

Suppose λ is a continuous linear functional on $h^p(\mathscr{P})$. Let N be the seminorm defined on h^p by

$$N(u) = \sup_{0 \le r < 1} |\lambda(u_r)| \quad (u \in h^p)$$

where u_r, the dilate of u by r, clearly belongs to $h^p(\mathscr{P})$. Since

$$|\lambda(u_r)| \le \|\lambda\| \, \|u_r\|_{h_p} \le \|\lambda\| \, \|u\|_{h_p}$$

we see that

$$N(u) \le \|\lambda\| \, \|u\|_{h_p} \tag{1}$$

for each $u \in h^p$. Since $u_r \to u$ in h^p as $r \to 1-$ for each $u \in h^p(\mathscr{P})$, we know that

$$|\lambda(u)| \le N(u)$$

for each u in $h^p(\mathscr{P})$. The Hahn-Banach theorem provides a linear extension Λ of λ to h^p, with $|\Lambda| \le N$ on h^p, and the continuity of Λ on h^p then follows from (1). This completes the proof.

It might appear at first glance that the extendability of continuous linear functionals from h_0^p to $h^p(\mathscr{P})$ should be a simple consequence of the fact that h_0^p is weakly dense in $h^p(\mathscr{P})$. This is not the case: if Y is a proper, closed, weakly dense subspace of an F-space X, then it may still happen that Y supports continuous linear functionals not extendable to X. In other words, the weak topology of Y may be strictly stronger than that of X. In fact Duren, Romberg and Shields [6; Section 5, Corollary 2, page 53] discovered that certain weakly dense subspaces of H^p have exactly this property.

We close this section with an approximation theorem which follows from Theorem 4.5.

4.11 THEOREM. *h_0^p is the closure in h^p of the linear span of the rotates of the Poisson kernel.*

Proof. Fix $0 < p < 1$ and let m be the positive integer such that

$$1/(m+1) \le p < 1/m.$$

Let X denote the closure in h^p of the linear span of the rotates of the derivatives $P^{(n)}$ defined in Section 4.1 (d), with $0 \le n \le m-1$. Proposition 4.4 guarantees that $X \subset h_0^p$. We claim that $h_0^p = X$.

To see this, fix v in h_0^p and $\varepsilon > 0$. By the definition of h_0^p and the fact that $v_r \to v$ in h^p as $r \to 1-$ (since $h_0^p \subset h^p(\mathscr{P})$) we can choose $0 < r_0 < 1$ such that

$$M_p^p(v; r) < \varepsilon/2 \quad \text{and} \quad \|v - v_r\|_{h_p}^p < \varepsilon/2$$

whenever $r_0 < r < 1$. Let f denote the restriction to T of v_r:

$$f(\omega) = v(r\omega) \quad (\omega \in T).$$

By Theorem 4.6 there exists u in X so that $\|P[f] - u\|_{h_p} \le C\|f\|_p$; that is,

$$\|v_r - u\|_{h_p} \le CM_p(u; r).$$

Thus

$$\|v - u\|_{h_p}^p \le \|v - v_r\|_{h_p}^p + \|v_r - u\|_{h_p}^p \le \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so $v \in X$. This shows that $h_0^p \subset X$, so the two spaces are the same.

We complete the proof by showing that the rotates of the Poisson kernel span a dense subspace of X. To do this we need only show that if $n \le m-1$ then each derivative $P^{(n)}$ is an h^p-limit of linear combinations of rotates of $P^{(n-1)}$. This requires an appropriate formula for $P^{(n)}$. Let

$$F(z) = \frac{1+z}{1-z} \quad (z \in \Delta)$$

and, writing $z = re^{i\theta}$, let $F^{(n)}$ denote the n^{th} derivative of F with respect to θ. Then $P^{(n)} = \operatorname{Re} F^{(n)}$ for $n = 0, 1, 2, \ldots$.

Now if g is holomorphic in Δ and g' is the usual complex derivative, then

$$\frac{\partial g}{\partial \theta}(z) = izg'(z).$$

In particular if n is a positive integer and $g_n(z) = (1 - z)^{-n}$, then

$$\frac{\partial g_n}{\partial \theta} = in[g_n - g_{n+1}],$$

so by induction,

$$F^{(n)} = i^n \sum_{j=1}^{n+1} c_j g_j \tag{1}$$

where the coefficients c_j are all real.

For $-\pi < t < \pi$ and z in Δ, write

$$\varphi_t(z) = t^{-1}\left[F^{(n-1)}(ze^{it}) - F^{(n-1)}(z)\right].$$

Then $\varphi_t \in H^p$ for each t. We are going to show that $\varphi_t \to F^{(n)}$ in H^p as $t \to 0$. Upon taking real parts, this implies that $\operatorname{Re} \varphi_t \to P^{(n)}$ in h^p, and since $\operatorname{Re} \varphi_t$ is a linear combination of rotates of $P^{(n-1)}$, the proof will be complete.

Since the functions $F^{(n)}$ and φ_t belong to H^p, it is enough to show that

$$\lim_{t \to 0} \int_T |\varphi_t - F^{(n)}|^p \, dm = 0. \tag{2}$$

For the rest of the proof we view φ_t and $F^{(n)}$ as functions on T. Since $\varphi_t \to F^{(n)}$ as $t \to 0$ at every point of T except 1, it is enough to prove that the family $\{\varphi_t\colon -\pi < t < \pi\}$ forms a bounded subset of L^q for some $q > p$. Then (2) follows from Vitali's Convergence Theorem.

Fix $p < q < 1/m$. By formula (1), φ_t is a linear combination of functions:

$$\Psi_t(e^{i\theta}) = t^{-1}\left[g_j(e^{i(\theta+t)}) - g_j(e^{i\theta})\right]$$

where $1 \le j \le n$. It is enough to prove that each such family

$$\{\psi_t\colon -\pi \le t \le \pi\}$$

is bounded in L^q. We see this by computing

$$\psi_t(e^{i\theta}) = e^{i\theta} t^{-1}(e^{it} - 1) \sum_{k=1}^{j-1} g_{k+1}(e^{i(\theta+t)}) g_{j-k}(e^{i\theta})$$

Since $j \leq n \leq m - 1$ and $q < 1/m$, Holder's inequality shows that each term in the series on the right belongs to L^q, with L^q norm independent of t. The quantity multiplying the sum is uniformly bounded in t and θ, so $\{\psi_t\colon -\pi < t < \pi\}$ is L^q bounded, as desired. This completes the proof.

4.12. *Final remarks.* The referee has pointed out that one can give a short proof of Theorem 4.7 ($h^p(\mathscr{P})/h_0^p = L^p$) by using in place of Theorem 4.5 another idea of Aleksandrov which depends more heavily on function theory. The argument is so simple and elegant that we feel it is worth presenting in some detail. We note however that Theorem 4.5 is still needed to prove the approximation theorem 4.11, as well as the higher dimensional analogue of Theorem 4.7.

Let $\mathscr{H}^p$ denote the space of (a.e.-equivalence classes of) measurable functions on T which are radial limits of functions in H^p. Thus $f \in \mathscr{H}^p$ if and only if there exists $F \in H^p$ with

$$f(\omega) = \lim F(r\omega) \quad (r \to 1-)$$

for a.e. ω in T. It is well known that the map $F \to f$ is an isometric linear transformation taking H^p onto $\mathscr{H}^p$, endowed with the "norm" of $L^p(T)$. Let $\overline{\mathscr{H}^p}$ denote the collection of complex conjugates of members of $\mathscr{H}^p$. Then when $0 < p < 1$ (and only then) we have

$$\mathscr{H}^p \cap \overline{\mathscr{H}^p} \neq \{\text{constants}\}.$$

For example the function $1/(1 - z)$ lies in this intersection, as does more generally the Cauchy transform of any singular measure on T. The space $\mathscr{H}^p \cap \overline{\mathscr{H}^p}$ was first studied by K. de Leeuw [4] as an example of an infinite dimensional closed rotation-invariant subspace of $L^p(T)$ $(0 < p < 1)$ which contains no nonconstant character. The linear topological properties of this space have been studied by Aleksandrov [1], [2], and Kalton [14]. As we remarked earlier, Theorems 4.5–4.7 are analogies for the Poisson kernel of results proved by Aleksandrov [1] for the conjugate function. In this analogy h_0^p plays the role of $\mathscr{H}^p \cap \overline{\mathscr{H}^p}$. In [1] Aleksandrov used his precursor of Theorem 4.5 to prove:

THEOREM A. *If* $0 < p < 1$ *then* $\mathscr{H}^p + \overline{\mathscr{H}^p} = L^p(T)$.

We now show how this result gives an easy proof of Theorem 4.7, and then we will indicate Aleksandrov's beautiful alternate proof of Theorem A.

Theorem A *implies* $h^p(\mathscr{P})/h_0^p = L^p$. As usual, if $0 < r < 1$ and $u \in h^p$, let $u_r(z) = u(rz)$ for $z \in \Delta$. If $u \in h^p(\mathscr{P})$ then it is easy to see that as $r \to 1-$ we have $u_r \to u$ in h^p, hence the restriction of u_r to T converges in

$L^p(T)$ to a function we will denote by Su. The map S is easily seen to be a norm-decreasing linear transformation of $h^p(\mathscr{P})$ into $L^p(T)$ with null space h_0^p. The image of $h^p(\mathscr{P})$ under S clearly contains both $\mathscr{H}^p$ and $\overline{\mathscr{H}^p}$, so it contains $\mathscr{H}^p + \overline{\mathscr{H}^p}$ which by Theorem A is $L^p(T)$. Thus S is a bounded linear map taking $h^p(\mathscr{P})$ onto $L^p(T)$, with null space h_0^p, so $h^p(\mathscr{P})/h_0^p$ is linearly homeomorphic to $L^p(T)$, and the proof is complete.

Proof of Theorem A ([2; Theorem 2.4]). Fix $0 < p < 1$. By a standard approximation argument it is enough to prove that there exists a constant $A_p < \infty$ such that for each trigonometric polynomial of these exists g and h in $\mathscr{H}^p$ with L^p-norms $\leq A_p\|f\|_p$, such that $f = g + \bar{h}$.

Fix the trigonometric polynomial f and choose a positive integer N so large that if $\gamma(\omega) = \omega^N$, then both γf and $\gamma \bar{f}$ lie in $\mathscr{H}^p$. Let $\varphi = \gamma/(\gamma - 1)$, so $\varphi \in \mathscr{H}^p \cap \overline{\mathscr{H}^p}$ and $\varphi + \bar{\varphi} = 1$. For t real let $\varphi_t(\omega) = \varphi(\omega e^{it})$ $(\omega \in T)$, and set $g_t = f\varphi_t$ and $h_t = \bar{f}\varphi_t$. Then both g_t and h_t are in $\mathscr{H}^p$ and $f = g_t + \overline{h_t}$. A routine computation shows that

$$\int_0^{2\pi} \|g_t\|_p^p \, dt/2\pi = \|\varphi\|_p^p \|f\|_p^p$$

so there exists t such that $\|g_t\|_p \leq \|\varphi\|_p \|f\|_p$. For this value of t,

$$\|h_t\|_p \leq \left(\|f\|_p^p + \|g_t\|_p^p\right)^{1/p} \leq \left(1 + \|\varphi_t\|_p^p\right)^{1/p} \|f\|_p$$

which completes the proof, with $g = g_t$, $h = h_t$, and $A_p = (1 + \|\varphi\|_p^p)^{1/p}$.

References

1. A.B. Aleksandrov, *Approximation by rational functions and an analogue of the M. Riesz theorem on conjugate functions for L^p spaces with $p \in (0,1)$*, Math. USSR Sbornik, vol. 35 (1979), pp. 301–316 (English translation of Math. Sb., vol. 107 (149) (1978), pp. 3–19).
2. ______, "Essays on non-locally convex Hardy spaces" in *Complex analysis and spectral theory*, V.P. Havin and N.K. Nikolskii, editors. Springer Lecture Notes in Mathematics, vol. 864, Springer, New York, 1981.
3. S. Banach, *Théorie des Opérations Lineares*, Monografje Matematyczne, vol. 1, Warsaw, 1932.
4. K. De Leeuw, *The failure of spectral analysis in L^p for* $0 < p < 1$, Bull. Amer. Math. Soc., vol. 82 (1976), pp. 111–114.
5. P.L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1980.
6. P.L. Duren, B.W. Romberg and A.L. Shields, *Linear functionals on H^p spaces with* $0 < p < 1$, J. Reine Angew. Math., vol. 38 (1969), pp. 32–60.
7. P.L. Duren and A.L. Shields, *Properties of H^p* $(0 < p < 1)$ *and its containing Banach space*, Trans. Amer. Math. Soc., vol. 141 (1969), pp. 255–262.
8. C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math., vol. 129 (1972), pp. 137–193.
9. G.H. Hardy and J.E. Littlewood, *Some properties of conjugate functions*, J. Reine Angew. Math., vol. 167 (1932), pp. 405–423.
10. J.P. Kahane, *Some random series of functions*, D.C. Heath, Lexington, Mass. 1968.

11. N.J. KALTON, *Basic sequences in F-spaces and their applications*, Proc. Edinburgh Math. Soc. (2), vol. 19 (1974), pp. 151–167.
12. ———, *Compact and strictly singular operators on Orlicz spaces*, Israel J. Math., vol. 26 (1977), pp. 126–136.
13. ———, *Quotients of F-spaces*, Glasgow Math. J., vol. 19 (1978), pp. 103–108.
14. ———, *Locally complemented subspaces in L^p-spaces for $0 < p < 1$*, Preprint, Univ. of Missouri, Columbia, Missouri.
15. ———, *The three space problem for locally bounded F-spaces*, Compositio Math., vol. 37 (1978), pp. 243–276.
16. N.J. KALTON and N.T. PECK. *Quotients of $L^p(0,1)$ for $0 \le p < 1$*, Studia Math., vol. 64 (1979), pp. 65–75.
17. N.J. KALTON and J.H. SHAPIRO, *An F-space with trivial dual and nontrivial compact endomorphisms*, Israel J. Math., vol. 20 (1975), pp. 282–291.
18. Y. KATZNELSON, *An introduction to harmonic analysis*, Wiley, New York, 1968.
19. R.E.A.C. PALEY and A. ZYGMUND, *A note on analytic functions in the unit circle*, Proc. Cambridge Philos. Soc., vol. 28 (1932), pp. 266–272.
20. N.T. PECK, *On nonlocally convex spaces*, Math. Ann., vol. 161 (1965), pp. 102–115.
21. W. RUDIN, *Real and complex analysis*, 2nd ed., McGraw Hill, New York, 1974.
22. ———, *Functional analysis*. McGraw Hill, New York, 1973.
23. ———, *Lumer's Hardy spaces*. Michigan Math. J., vol. 24 (1977), pp. 1–5.
24. ———, *Function theory in the unit ball of $\mathbf{C}^n$*, Springer, New York, 1981.
25. J.H. SHAPIRO, *Examples of proper, closed, weakly dense subspaces in non-locally convex F-spaces*, Israel J. Math., vol. 7 (1969), pp. 369–380.
26. ———, *Extension of linear functionals in F-spaces with bases*, Duke Math. J., vol. 37 (1970), pp. 639–645.
27. ———, *Mackey topologies, reproducing kernels, and diagonal maps on the Hardy and Bergman spaces*, Duke Math. J., vol. 43 (1976), pp. 187–202.
28. ———, "Remarks on F-spaces of analytic functions" in *Banach spaces of analytic functions*, J. Baker, C. Cleaver, and J. Diestel, editors, Springer Lecture notes in Mathematics, no. 604, Springer, New York, 1977.
29. ———, *Cauchy transforms and Beurling-Carleson-Hayman thin sets*, Michigan Math. J., vol. 27 (1980), pp. 339–351.
30. K. YOSIDA, *Functional analysis*, Springer, New York, 1965.
31. A. ZYGMUND, *Trigonometric Series, volume I*, Cambridge Univ. Press, Cambridge, 1959.

MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN