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Cyclic Composition Operators on H^2

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Introduction. We say that the operator T on the Hilbert space H is *hypercyclic* if there is a vector v in H (called a *hypercyclic vector*) whose orbit, $\{T^n v : n \geq 0\}$, is dense in H, and *cyclic* if there is a vector (called a *cyclic vector*) whose orbit has dense linear span. We have undertaken the systematic study of cyclic phenomena for composition operators on the Hardy space H^2. In this paper, we announce our results concerning the cyclic behavior of *linear fractional composition operators*, composition operators whose symbols are linear fractional maps of the unit disk U into itself. For these operators, we have answered the cyclicity question, completely determining which are cyclic and which are hypercyclic. Our results appear in the table below. The general tenor of these results may be summarized in the following three principles.

- *The cyclic behavior of a linear fractional composition operator depends critically on the fixed point properties of its symbol.*
- *Every kind of cyclic behavior occurs within the class of linear fractional composition operators.*
- *When a linear fractional composition operator is cyclic or hypercyclic, then it is very strongly so. When it fails to show one of these properties, then it does so dramatically.*

Although the study of cyclic operators and cyclic vectors has been a standard part of operator theory for some time, the study of hypercyclicity has only recently piqued the interest of operator theorists. Hypercyclicity is clearly a very strong form of cyclicity. Note that hypercyclicity has the same connection with invariant subsets that cyclicity has with invariant subspaces. Note also that if the operator T has a hypercyclic vector, then every element

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in the orbit of this vector is also hypercyclic for T. Thus a hypercyclic operator has a dense set of hypercyclic vectors (and hence a dense set of cyclic vectors).

This paper is organized as follows. In the next section, we dispose of some preliminary matters and then prove a simple necessary condition for a composition operator to be cyclic: If C_φ is cyclic, then φ must be univalent. In section 2, we restrict our attention to linear fractional composition operators—those composition operators with the simplest possible univalent symbols—and sketch arguments indicating how questions regarding cyclicity may be resolved for these operators (details will appear in **[1]**). In the final section of the paper, we make a few remarks about cyclicity for composition operators induced by more general holomorphic self-maps of U.

Summary of Main Theorem

Cyclic Behavior for C_φ: φ a Linear Fractional Self-map of U, Not an Automorphism[1]

Fixed points of φ in $\mathbb{C} \cup \{\infty\}$	C_φ Hypercyclic?	C_φ Cyclic ?[2]	Example(s)
Interior and boundary	No[3]	No[5]	$\varphi(z) = \dfrac{z}{2-z}$
Interior and exterior	No[3]	Yes	$\varphi(z) = z/2$ $\varphi(z) = -z/(2+z)$
Boundary and exterior	Yes	Yes[4]	$\varphi(z) = \dfrac{1+z}{2}$
Boundary only	No[6]	Yes	$\varphi(z) = \dfrac{1}{2-z}$

[1] Every nonelliptic automorphism induces a hypercyclic composition operator (Proposition 2.3).
[2] Every cyclic linear fractional composition operator has a dense set of cyclic vectors. For these operators, every vector in H^2 is the sum of two cyclic vectors (Sec. 1).
[3] Proposition 2.2.
[4] Hypercyclic implies cyclic.
[5] Every finitely generated C_φ-invariant subspace has infinite codimension (Theorem 2.7).
[6] Only the constant functions can be limit points of a C_φ-orbit (Theorem 2.4(b)).

1. Preliminaries. The Hardy space H^2, the natural functional representation of the sequence space ℓ^2, is the Hilbert space of functions holomorphic on the open unit disk U whose Taylor coefficients in the expansion about the origin form a square summable sequence. The inner product inducing the H^2-norm is defined by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} \qquad (f, g \text{ in } H^2),$$

where $\hat{f}(n)$ and $\hat{g}(n)$ represent the nth Taylor coefficients of f and g, respectively. For each $\alpha \in U$, we let K_α denote the H^2-reproducing kernel for α so that

$$K_\alpha(z) = \frac{1}{1-\overline{\alpha}z},$$

and

$$\langle f, K_\alpha\rangle = f(\alpha) \quad \text{for all } f \in H^2.$$

In what follows, φ will always denote a holomorphic function that maps the unit disk U into itself, and C_φ will denote the composition operator on H^2 with symbol φ, that is

$$(C_\varphi f)(z) = f(\varphi(z)) \qquad (f \in H^2, z \in U).$$

Littlewood's Subordination Principle ([**8**], [**9**]) guarantees that, without any additional assumptions about the behavior of the holomorphic function φ, the operator C_φ is bounded on H^2.

For each positive integer n, we define φ_n to be the nth iterate of φ ($\varphi_n = \varphi\circ\varphi\circ\ldots\circ\varphi$, n times); we define $\varphi_0(z) = z$. In this paper, we will be especially interested in linear fractional composition operators; that is, those operators C_φ whose symbols φ are of the form

$$\varphi(z) = \frac{az+b}{cz+d},$$

where a, b, c, and d are complex numbers satisfying $ad - bc \neq 0$. As we indicated in the introduction, the fixed point properties of the symbols of such operators play a crucial role in determining cyclic behavior. Every nonelliptic linear fractional self-map of U has an attractive fixed point in the closure of U, and if the map is not parabolic, then it will have a repulsive fixed point lying outside U. The preceding statement regarding the existence of an attractive fixed point in $\overline{U}$ for nonelliptic linear fractional self-maps of U has the following remarkable generalization ([**5**], [**10**], [**2**]).

THEOREM (DENJOY-WOLFF). *If a holomorphic self-map of U is not an elliptic automorphism of U, then there is a unique point p in the closed unit disk such that $\varphi_n(z) \to p$ for each $z \in U$.*

The Denjoy-Wolff theorem plays an important role in all aspects of the study of composition operators.

We now turn our attention to cyclicity and hypercyclicity, beginning with a few easy observations. A hypercyclic operator must have dense range (since if f is hypercyclic for $T\colon H \to H$, then $\{T^n f\colon n \geq 1\}$ is dense in H). A cyclic operator need not have dense range (consider, e.g., the operator of multiplication by z on H^2), but the codimension of the closure of the range of a cyclic operator must be at most 1. Hypercyclicity and cyclicity are preserved under similarity.

The collection of cyclic or hypercyclic vectors for an operator is always G_δ. A cyclic operator need not have a dense collection of cyclic vectors (the outer functions—the cyclic vectors for multiplication by z on H^2—are not dense in H^2), but as we mentioned in the introduction, a hypercyclic operator always has a dense collection of hypercyclic vectors. Hence we have the following amusing "zero-one" law:

- *An operator has either no hypercyclic vector, or a dense G_δ set of them.*

From this result, Baire's Theorem yields a couple of interesting observations:

- *Every countable collection of hypercyclic operators has a common hypercyclic vector.*
- *If T is a hypercyclic operator on H, then every vector in H is the sum of two hypercyclic vectors.*

The second result was pointed out to us by Gilles Godefroy. To see why it is true, let E denote the collection of hypercyclic vectors for T, and suppose that $v \in H$. Since both E and $v - E$ are dense G_δ subsets of H, they have nonvoid intersection. To say that v_1 is in the intersection means that v_1 is hypercyclic and that $v_1 = v - v_2$ for some hypercyclic vector v_2.

We will rely heavily on the following sufficient condition for hypercyclicity. Essentially the same result was presented in **[6]** and proved independently in [7].

THEOREM 1.1 (Sufficient condition for hypercyclicity). *Suppose T is a continuous linear operator on a separable Banach space K for which the sequence of nonnegative powers (T^n) tends pointwise to zero on a dense subset X of K. Suppose further that there is a (possibly different) dense subset Y of K, and a (possibly discontinuous) map $S\colon Y \to Y$ such that $TS =$ identity on Y, and (S^n) tends pointwise to zero on Y. Then T is hypercyclic.*

Our focus on the cyclic behavior of composition operators begins with the following necessary condition for cyclicity.

PROPOSITION 1.2. *If C_φ is cyclic, then φ must be univalent.*

Proposition 1.2 is a consequence of the following Lemma and the observation that the closure of the range of a cyclic operator must have codimension at most 1. We remark that it is possible to prove a stronger version of Proposition 1.2 (see **[1]**): If C_φ is cyclic, then not only must φ be univalent on U, but it also must be univalent *almost everywhere on ∂U* (meaning that there is some set $E \subset \partial U$ having zero Lebesgue measure such that φ is univalent on $\partial U \setminus E$).

LEMMA 1.3. *If φ is not univalent, then the closure of the range of C_φ has infinite codimension in H^2.*

PROOF. If φ is nonconstant and not univalent on U, then there are points a and b in U with $a \neq b$ such that $\varphi(a) = \varphi(b)$. Choose $\varepsilon > 0$ small enough so that the disks $B(a, \varepsilon)$ and $B(b, \varepsilon)$ are contained in U and have empty intersection. Since φ is holomorphic, the set

$$G = \varphi(B(a, \varepsilon)) \cap \varphi(B(b, \varepsilon))$$

is open; of course, G is nonempty, since it contains the point $\varphi(a) = \varphi(b)$. Let $\{w_n\}_{n=1}^{\infty}$ be a set containing countably many (distinct) points from G. Let the sets $\{a_n\}$ and $\{b_n\}$ consist of points satisfying

$$a_n \in \varphi^{-1}(w_n) \cap B(a, \varepsilon); b_n \in \varphi^{-1}(w_n) \cap B(b, \varepsilon).$$

Now observe that for any positive integer n,

$$g_n(z) = K_{a_n}(z) - K_{b_n}(z)$$

belongs to the orthogonal complement of the closure of the range of C_φ. Since the set $\{g_n\}_{n=1}^{\infty}$ is linearly independent, the proof is complete. $\square$

In the next section, we restrict our attention to composition operators having the simplest univalent symbols, the linear fractional composition operators.

2. Results. Our results concerning the cyclic behavior of linear fractional composition operators are summarized in the following theorem.

THEOREM 2.1 (Main Theorem). *Let φ be a nonelliptic linear fractional self-map of U.*

(a) *Suppose φ has no fixed point in U. Then C_φ is hypercyclic unless φ is a parabolic nonautomorphism, in which case only the constant functions can adhere to C_φ orbits.*

(b) *C_φ is cyclic unless φ has both a fixed point in U and one on ∂U, in which case every finitely generated C_φ-invariant subspace has infinite codimension.*

For example, every nonelliptic linear fractional self-map of U without an interior fixed point induces a cyclic composition operator. In particular, all the parabolic self-maps of U induce composition operators that are cyclic, but among the parabolics, only the automorphisms induce hypercyclic operators. Composition operators with symbols that are nonelliptic disk automorphisms have been considered by Nina Zorboska, who in her dissertation **[11]** shows that every such operator is cyclic. The theorem above shows that such operators are actually hypercyclic.

Composition operators induced by elliptic linear fractional self-maps of U are an annoyance in that they don't fit into the classification scheme presented in Theorem 2.1. Fortunately, they can be disposed of quickly. An elliptic self-map of U must be a conformal automorphism of U that is conjugate by automorphisms to a rotation about the origin. A composition operator induced by such a map must therefore be similar to one induced by a rotation. A composition operator induced by a rotation is never hypercyclic (see the next proposition) and is cyclic if and only if the angle of rotation is an irrational multiple of π (see the comments following the proof of Theorem 2.6).

Thus our study focuses quickly on nonelliptic maps. For these, the following result shows that the program of investigation must branch into two separate lines, depending on whether the map in question does or does not have a fixed point in U (henceforth an *interior fixed point*).

PROPOSITION 2.2. *If φ has an interior fixed point, then C_φ is not hypercyclic.*

PROOF. Suppose that $\alpha \in U$ is a fixed point of φ, and let $f \in H^2$ be arbitrary. Since evaluation at α is a continuous linear functional on H^2, any function g in the closure of the C_φ-orbit of f must satisfy $g(\alpha) = f(\alpha)$. It follows that no C_φ-orbit may be dense in H^2; hence, C_φ is not hypercyclic. □

It therefore remains to determine which nonelliptic linear fractional self-maps of U with no interior fixed point induce hypercyclic composition operators, and which ones with an interior fixed point induce cyclic operators. We consider hypercyclicity first (part (a) of the Main Theorem).

PROPOSITION 2.3. *If φ is a nonelliptic conformal automorphism of U, then C_φ is hypercyclic.*

SKETCH OF THE PROOF. If φ is a nonelliptic conformal automorphism of U, then φ must have an attractive fixed point α on ∂U. If φ is a parabolic, then α is also the attractive fixed point for φ^{-1}, the inverse of φ under composition. If φ is not parabolic, it has another fixed point β on ∂U, and this point β is the attractive fixed point for φ^{-1}. In order to treat both cases simultaneously, we write $\alpha = \beta$ if φ is parabolic.

Let A_α be the set of functions holomorphic in a neighborhood of the closed unit disk that vanish at α, and define A_β similarly. It's not difficult to verify that these sets are dense in H^2 (e.g., one could apply Beurling's Theorem) and that the hypotheses of Theorem 1.1 are satisfied with $X = A_\alpha$, $Y = A_\beta$, $T = C_\varphi$, and $S = C_{\varphi^{-1}}$. □

All remaining questions regarding hypercyclicity for linear fractional composition operators are answered by the following theorem.

THEOREM 2.4. *Suppose φ is a linear fractional self-map of U that is not an automorphism, and that has no interior fixed point.*

(a) *If φ is not parabolic, then C_φ is hypercyclic.*
(b) *If φ is parabolic, then C_φ is not hypercyclic; in fact, only the constant functions can occur as limit points of C_φ-orbits.*

SKETCH OF THE PROOF. The function φ has its attractive fixed point α on ∂U.

(a) Suppose that φ is not parabolic. The operator C_φ is similar to an operator C_ψ, where ψ has its attractive fixed point at α and repulsive fixed point β ($|\beta| > 1$) on the ray originating at α and passing through the origin. Let Δ be the disk whose boundary passes through α and β and is tangent to the unit circle. Note that $U \subset \Delta$. It is not difficult to verify that ψ is a conformal automorphism of Δ; hence, the operator $C_\psi\colon H^2(\Delta) \to H^2(\Delta)$ is hypercyclic by Theorem 2.3. Since $H^2(\Delta)$ is contained in $H^2(U)$ as a dense subspace (the holomorphic polynomials are dense in both spaces) and has a stronger norm, the operator $C_\psi\colon H^2(U) \to H^2(U)$ must also be hypercyclic. Because hypercyclicity is preserved under similarity, C_φ must be hypercyclic.

(b) We now assume that φ has a unique fixed point, necessarily on the unit circle. Without loss of generality, we may assume that this fixed point is 1. We compute φ explicitly by employing the change of variable $w = (1+z)/(1-z)$, which sends U to the right half-plane Π, the fixed point 1 to ∞, and φ to the translation map

$$\Phi(w) = w + a \qquad (w \in \Pi),$$

where $\operatorname{Re} a > 0$, the strict inequality reflecting the fact that φ is not an automorphism of U. Pulling back to the unit disk, we obtain

$$\varphi(z) = \frac{(2-a)z + a}{-az + (2+a)} \qquad (z \in U),$$

and more generally, for $n = 0, 1, 2, \ldots$; the nth iterate φ_n of φ is obtained by replacing a by na in (1).

By referring to the half-plane realization of φ_n as translation by na, we see that the φ-orbit of any point in U converges nontangentially to 1. These orbits, however, approach 1 rather slowly as $n \to \infty$:

$$(*) \qquad \lim_{n\to\infty} n(1 - \varphi_n(z)) = \frac{2}{a} \qquad (z \in U).$$

We show that this slow approach to 1 prevents C_φ from being hypercyclic. (This situation should be compared to that involving the composition operator induced by, say, $\tau(z) = (1+z)/2$: τ has attractive fixed point 1 on ∂U, $\tau(z)$ tends to 1 at an *exponential* rate ($z \in U$), and C_τ *is* hypercyclic by part (a).)

Now fix $f \in H^2$ and $z \in U$. Write $s_n = \varphi_n(0)$ and $t_n = \varphi_n(z)$. The Cauchy-Schwarz inequality shows that

$$|f'(z)| = \frac{\|f\|}{(1-|z|)^{3/2}},$$

from which follows

$$|f(z) - f(w)| \le \|f\| \frac{|w - z|}{(1-|w|)^{3/2}} \qquad (z, w \in U, |z| < |w|).$$

Using this last estimate along with (*) above, one may show that

$$|f(t_n) - f(s_n)| \le Cn^{-1/2}$$

for some constant C. Thus,

$$\lim_{n\to\infty} [f(t_n) - f(s_n)] = 0.$$

Now, suppose $g \in H^2$ is a cluster point of the sequence $(f \circ \varphi_n)$. Then some subsequence $f \circ \varphi_{n_k}$ converges to g in H^2, hence pointwise on U. Thus

$$\begin{aligned} g(z) - g(0) &= \lim_{n\to\infty} [f(\varphi_{n_k}(z)) - f(\varphi_{n_k}(0))] \\ &= \lim_{n\to\infty} [f(t_n) - f(s_n)] \\ &= 0, \end{aligned}$$

and thus $g(z) = g(0)$, regardless of our choice of $z \in U$. This completes the proof. □

The proof of part (a) of the Main Theorem is now complete, save for some details that we omitted in the sketches of proofs above. We now turn to the proof of part (b). We have just seen that a parabolic nonautomorphism mapping U into itself induces a nonhypercyclic composition operator. Such operators are nevertheless cyclic.

THEOREM 2.5. *Every parabolic linear fractional self-map of U induces a cyclic composition operator.*

SKETCH OF THE PROOF. We have seen that nonelliptic disk automorphisms induce hypercyclic composition operators. Suppose that φ is a parabolic self-map of U that is not an automorphism. Without loss of generality, we may assume that 1 is the fixed point of φ; then, as we pointed out in the proof of Theorem 2.4 above, there is a complex number a with $\operatorname{Re} a > 0$ for which

$$\varphi(z) = \frac{(2-a)z + a}{-az + (2+a)} \qquad (z \in U).$$

For our purposes, a more convenient expression for φ is

$$\varphi(z) = \overline{\gamma} + \overline{\alpha} K_\beta(z), \tag{1}$$

where $\overline{\gamma} = (a+2)/a$, $\overline{\alpha} = 4/(a^2+2a)$, $\overline{\beta} = a/(2+a)$, and $K_\beta(z) = (1-\overline{\beta}z)^{-1}$. The fact that $\operatorname{Re} a > 0$ insures that none of the denominators in the definitions of α, β, and γ are zero. In addition, it guarantees that $\beta \in U$ so that K_β is the H^2-reproducing kernel for β.

We show that φ itself is a cyclic vector for the operator C_φ. Suppose that $f \in H^2$ is orthogonal to the C_φ-orbit of φ so that

$$\langle f, \varphi_n \rangle = 0 \quad \text{for} \quad n = 1, 2, \ldots. \tag{2}$$

Since the sequence of iterates $\{\varphi_n\}$ is uniformly bounded and pointwise convergent to 1 on the unit circle, it converges to 1 in the H^2 norm. It follows from (2) that

$$0 = \langle f, 1 \rangle = f(0). \tag{3}$$

Using (3), $\langle f, \varphi \rangle = 0$, and the expression (1) for φ, we find $f(\beta) = 0$. More generally, $f(\beta_n) = 0$, where $\beta_n = na/(2 + na)$. However, $\{\beta_n\}$ is not a Blaschke sequence; hence, $f \equiv 0$, and φ must be cyclic for C_φ. □

REMARK. It is possible to show that the collection of cyclic vectors for a composition operator induced by a parabolic linear fractional self-map of U is dense in H^2 (see [**1**]).

Our work so far has shown that every linear fractional self-mapping of U with attractive fixed point on the boundary induces a cyclic composition operator on H^2. We now turn our attention to mappings with an interior fixed point. Necessarily this fixed point is attractive, and since such a mapping cannot be parabolic, there is a repulsive fixed point somewhere outside U, either on ∂U, or outside the closure of U. Each of these cases gives rise to different cyclic behavior for the induced composition operator.

THEOREM 2.6. *If a linear fractional self-map has an attractive fixed point in U and a repulsive fixed point outside the closure of U, then the induced composition operator is cyclic on H^2.*

SKETCH OF THE PROOF. Suppose φ is a linear fractional map with a fixed point in the interior, but none on the boundary. Without loss of generality we may assume the fixed point is the origin. In this case φ can be written out explicitly as

$$\varphi(z) = \frac{z}{az + b},$$

where, by the Schwarz Lemma, $|b| > 1$.

We claim that for any $\alpha \in U$, the reproducing kernel K_α is a cyclic vector for C_φ. The proof of this claim may be carried out using techniques similar to those used in the proof of the preceding theorem. It is possible to show that a function f orthogonal to the C_φ-orbit of K_α must vanish on a sequence of points with limit point in U. We omit the details.

Another proof of Theorem 2.6 may be based on the fact that the collection of eigenvectors of the operator C_φ is dense in H^2 (see [**1**]). This alternative proof reveals that C_φ must have a dense collection of cyclic vectors. □

The reproducing kernels are also cyclic vectors for composition operators induced by rotations through an irrational multiple of π. For if φ is such a rotation, and ψ is its inverse, then f is orthogonal to the C_φ-orbit of K_α if and only if $f(\psi_n(\alpha)) = 0$ for $n \geq 0$. Since ψ is an irrational rotation, the points $\{\psi_n(\alpha)\}$ are dense in the circle of radius $|\alpha|$ in U, and so f must vanish identically.

Regarding the proof of our Main Theorem 2.1, only one case remains: that of composition operators induced by linear fractional maps having both interior and boundary fixed points. Such operators are highly noncyclic.

THEOREM 2.7. *Suppose φ is a linear fractional self-map of U that fixes both an interior and boundary point of U. Then C_φ is not cyclic; in fact, the closed linear span of any orbit has infinite codimension in H^2.*

Our proof of Theorem 2.7 is based on the following proposition, which is a generalization of the fairly well known fact that the adjoint of a cyclic operator may have only simple eigenvalues.

PROPOSITION 2.8. *Let T be a bounded linear operator on a Hilbert space. If the adjoint of T has an eigenvalue of multiplicity $\geq m > 1$, then any invariant subspace for T generated by $k < m$ vectors has codimension $\geq m - k$.*

SKETCH OF THE PROOF OF THEOREM 2.7. By Proposition 2.8 we need only find an eigenvalue for C_φ^* that has infinite multiplicity. We may without loss of generality suppose that the fixed points of φ are located at 0 and 1. In this case the change of variable $w = (1+z)/(1-z)$ converts φ into a linear fractional map of the right half-plane that fixes 1 and ∞, and therefore has the form $w \to sw + 1 - s$ for some $0 < s \leq 1$. Pulling this mapping back to the unit disk, we obtain

$$\varphi(z) = \frac{sz}{1-(1-s)z} \qquad (z \in U).$$

If $s = 1$, then $\varphi(z) \equiv z$, and 1 is an eigenvalue of infinite multiplicity for C_φ^*. If $0 < s < 1$, it is possible to represent C_φ^* as a combination of Toeplitz operators and a composition operator (see [4]). Using this representation of C_φ^*, one finds that the functions $f_\lambda(z) = z(1-z)^\lambda$, which belong to H^2 for $\operatorname{Re}\lambda > -1/2$, are eigenvectors for C_φ^* with corresponding eigenvalues $s^{\lambda+1}$. Now suppose λ, with $\operatorname{Re}\lambda > -1/2$, is fixed. For each integer k, set

$$\lambda(k) = \lambda + \frac{2\pi i k}{\log s}.$$

Then one checks easily that the collection of H^2 functions $\{f_{\lambda(k)} : k \in \mathbf{Z}\}$ is linearly independent, and

$$C_\varphi^* f_{\lambda(k)} = s^{\lambda(k)+1} f_{\lambda(k)} = s^{\lambda+1} f_{\lambda(k)}.$$

Thus $s^{\lambda+1}$ is an eigenvalue for C_φ^* that has infinite multiplicity, so the desired result follows from Proposition 2.8. □

3. Concluding remarks. We have seen that the cyclic behavior of a linear fractional composition operator depends critically on the fixed point properties of its symbol. In particular, a linear fractional composition operator C_φ is not cyclic if its symbol has both an interior and a boundary fixed point.

Unfortunately, this result does not hold for more general symbols. We have constructed an example of a cyclic composition operator C_ψ whose symbol has both an interior and a boundary fixed point. Hence, any scheme classifying the cyclic behavior of general composition operators must be based on more than just fixed point behavior. Our construction of C_ψ recalls Cowen's remarkable model for holomorphic self-maps of U [3] (as does the proof of our Main Theorem). We have also used this model to show that if φ maps U univalently onto a domain whose boundary is a Jordan curve lying in U, then C_φ is cyclic. It appears that further progress toward a more general theory of the cyclic behavior of composition operators will require a detailed understanding of the workings of Cowen's model.

ADDED IN PROOF. Since submitting this paper, we have obtained generalizations of several of our results concerning hypercyclic behavior. Call a holomorphic map $\varphi\colon U \to U$ a Jordan map if it extends continuously to a univalent map $\varphi\colon \overline{U} \to \overline{U}$. Let φ be a C^4 Jordan map with Denjoy-Wolff point 1 such that $\varphi(\overline{U}\backslash\{1\}) \subset U$. We have shown that if $\varphi'(1) < 1$, then C_φ is hypercyclic. If $\varphi'(1) = 1$, an elementary argument shows that $Re\varphi''(1) \geq 0$; and in this case, C_φ is hypercyclic when $\varphi''(1)$ is pure imaginary but not hypercyclic when $Re\varphi''(1) > 0$. (The C^4 requirement on φ can be weakened at the expense of complicating the statements of these results; proofs will appear in [1].)

We have also shown that the hyperbolic composition operators considered in this paper are *chaotic*—they fulfill the three criteria for chaos given by Devaney in *Introduction to Chaotic Dynamical Systems*; namely, they have a vector with dense orbit, possess a dense set of periodic points, and display sensitive dependence on initial conditions.

REFERENCES

1. P. S. Bourdon and J. H. Shapiro, *Cyclic phenomena for composition operators*, in preparation.
2. R. B. Burckel, *Iterating analytic self-maps of disks*, Amer. Math. Monthly **88** (1981), 396–407.
3. C. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), 69–95.
4. ———, *Linear fractional composition operators on H^2*, Integral Equations and Operator Theory **11** (1988), 151–160.
5. A. Denjoy, *Sur l'itération des fonctions analytiques*, C. R. Acad. Sci. Paris **182** (1926), 255–257.
6. R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. **100** (1987), 281–288.
7. C. Kitai, *Invariant closed sets for linear operators*, Ph.D. Thesis, University of Toronto, 1982.
8. J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23** (1925), 481–519.
9. J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math. **125** (1987), 375–404.
10. J. Wolff, *Sur l'itération des fonctions bornées*, C. R. Acad. Sci. Paris **182** (1926), 200–201.
11. N. Zorboska, *Composition operators on weighted Hardy spaces*, Ph.D. Thesis, University of Toronto, 1988.

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