CYCLIC PHENOMENA FOR COMPOSITION OPERATORS

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Contents

ABSTRACT

We undertake a systematic study of cyclic phenomena for composition operators. Our work shows that composition operators exhibit strikingly diverse types of cyclic behavior, and it connects this behavior with classical problems involving complex polynomial approximation and analytic functional equations.

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Introduction

In this monograph we explore some links between function theory and operator theory that are created by Littlewood's Subordination Principle. Everyone knows that if φ is a holomorphic function that takes the unit disc U of the complex plane into itself, then for any function f holomorphic on U, the composition $f \circ \varphi$ is also holomorphic on U. But not so well known, and not at all obvious, is the following consequence of Littlewood's Principle ([36], [48, Chap. 1]):

> *Whenever the Taylor coefficients of f in its expansion about the origin form a square-summable sequence, then so do those of $f \circ \varphi$.*

Thus the *composition operator* C_φ, defined by

$$C_\varphi f = f \circ \varphi \quad (f \text{ holomorphic on } U),$$

takes the Hardy space H^2 into itself. Littlewood's Principle also supplies a uniform estimate which shows that C_φ is a bounded operator on H^2 (see [48, Chapter 1] or [19, Corollary 2.24] for example).

In the 1960s, John Ryff [44], Eric Nordgren [40], and Howard Schwartz [46] took up the problem of relating the properties of composition operators with the function theory of their inducing maps, and their efforts have inspired a continuing program of research on composition operators which includes investigations of spectra ([11, 12, 33, 15]), compactness ([50, 37, 47]), semigroups ([3, 4, 16, 52]), and subnormality ([16, 17, 18, 20]). Each paper in this necessarily incomplete list illustrates the subject's rich potential for connecting complex analysis with operator theory. For example, the study of compactness leads to geometric function theory and value distribution theory ([50, 37, 47]), while that of subnormality uncovers surprising relationships between function theory, semigroups, and classical Hilbert space operators (e.g., the Cesàro operator; see [20, 16, 52]).

Here we introduce a new direction into the composition-operator program: the study of *cyclicity*.

Cyclicity

Recall that an operator T on a linear topological space is said to be *cyclic* if there is a vector x in the space (called a *cyclic vector* for T) whose *orbit*

$$\mathrm{Orb}\,(T, x) = \{T^n x : n = 0, 1, 2, \ldots\}$$

has dense linear span. It may happen that the orbit itself is dense, without additional help from the linear span; in this case T is called *hypercyclic,* and x is a *hypercyclic vector.*

Because the closed linear span of $\mathrm{Orb}\,(T, x)$ is the smallest closed T-invariant subspace that contains the vector x, the concept of cyclicity is intimately connected with the study of invariant subspaces. Hypercyclicity has the same connection with invariant subsets, and, because of its independence of the notion of linearity, it also makes contact with dynamics (when formulated for mappings of arbitrary topological spaces, the concept is usually called *topological transitivity*, see for example [22, page 50]). Here we regard hypercyclicity as an extremely strong form of cyclicity.

No linear operator on a finite-dimensional space is hypercyclic (see [34], for example), but the property shows up surprisingly often in spaces of infinite dimension. Infinite-dimensional hypercyclicity was first observed by G.D. Birkhoff, who showed in 1929 that the translation operator $f(z) \to f(z+1)$ is hypercyclic on the Fréchet space of all entire functions [7]. More recently, hypercyclicity has been discovered lurking within important classes of Hilbert space operators (see [42, 34, 25, 26, 31, 5, 13], and for more detailed accounts [6, 48]).

Here we study the notions of cyclicity and hypercyclicity for composition operators on H^2. We proceed in two stages, first characterizing the cyclic and hypercyclic composition operators induced by linear-fractional self-maps of U, and then transferring these results to a wider setting by using linear-fractional self-maps to represent more general ones. It is this transference procedure that confronts us with the issues of iteration, polynomial approximation, and geometric function theory that form the heart of our work.

Cyclicity and Iteration

The study of cyclicity is especially natural for composition operators because the n-th power of C_φ is the composition operator induced by the n-th iterate of φ. That is, $C_\varphi^n = C_{\varphi_n}$, where

$$\varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi \quad (n \text{ times}).$$

This suggests that the cyclic behavior of a composition operator should be strongly influenced by the dynamical properties of its inducing map. Here is a simple illustration of how such properties, as determined in this case by fixed-point location, can influence the cyclic behavior of the induced composition operator.

Proposition 0.1 *Suppose that φ is a self-map of U that fixes a point z_0 in U. Then C_φ is not hypercyclic. Moreover, if φ is not an elliptic automorphism, then for each $f \in H^2$, the only limit point of $\operatorname{Orb}(C_\varphi, f)$ is the constant function $f(z_0)$.*

Proof. Suppose that φ fixes a point $z_0 \in U$. If φ is not an elliptic automorphism of U, then a little work with the Schwarz Lemma shows that $\varphi_n \to \varphi(z_0)$ pointwise on U. Hence, if a function g is a limit point of the orbit of f, say $g = \lim_j f \circ \varphi_{n_j}$, then by the continuity of point evaluation functionals on H^2, we see that for each $z \in U$:

$$g(z) = \lim_j f(\varphi_{n_j}(z)) = f(z_0),$$

as desired. If φ is an elliptic automorphism, then its iterates no longer converge to a constant, but we still have $g(z_0) = f(z_0)$. Thus every function in orbit-closure of f has value $f(z_0)$ at z_0, hence $\operatorname{Orb}(C_\varphi, f)$ cannot be dense. □

In the other direction, we have a non-Euclidean version of Birkhoff's original hypercyclicity theorem on entire functions (see Theorem 2.3):

> *Every composition operator induced by a non-elliptic disk automorphism is hypercyclic.*

Remark. Elliptic linear-fractional transformations are not really an issue for us. Since each such transformation is conjugate to a rotation, the corresponding composition operator is similar to one induced by a rotation, and it is an elementary exercise to analyze the cyclic properties of rotation-induced operators. For the convenience of the reader we present the argument in section 2 (Proposition 2.1; cf. also [48, §7.4, Prob. 11]).

Our complete result on the cyclic behavior of linear-fractional composition operators is Theorem 2.2, whose "non-automorphic" content we summarize in Table I. The comments appearing in the right-hand column of this table indicate the workings of an intriguing "zero-one law" wherein a linear-fractional composition operator either possesses a particular cyclic property strongly, or fails that property spectacularly. For example, noncyclic linear-fractional composition operators all have "infinite multiplicity," while cyclic ones have a dense set of cyclic vectors. In every nonhypercyclic case, only constant functions can adhere to orbits.

Table I
Theorem 2.2: Cyclic behavior of C_φ,
φ linear fractional, not an automorphism

Fixed points of φ (relative to U)	Cyclicity of C_φ	Examples	Comments
Interior & boundary	Not Cyclic	$\varphi(z) = \dfrac{z}{2-z}$	Every finitely generated invariant subspace has infinite codimension.
Interior & exterior	Cyclic, Not Hypercyclic	$\varphi(z) = \dfrac{-z}{2+z}$	Only constant functions can adhere to an orbit. C_φ has a dense collection of cyclic vectors.
Exterior & boundary (hyperbolic)	Hypercyclic	$\varphi(z) = \dfrac{1+z}{2}$	Hypercyclic $\Rightarrow$ Cyclic
Boundary only (parabolic)	Cyclic, Not Hypercyclic	$\varphi(z) = \dfrac{1}{2-z}$	Here, only constant functions can adhere to a C_φ-orbit.

In the sequel we will show that for more general composition operators, fixed point behavior still plays a decisive role in determining hypercyclicity, but when the issue is ordinary cyclicity, this importance is somewhat diminished. We show, for example, that in contrast to what happens in the first row of Table I, there exists a holomorphic (but not linear-fractional) self-map φ of U with interior and boundary fixed points, such that C_φ *is* cyclic (see the example following Corollary 3.7).

"Linear-Fractional" Classification of Arbitrary Maps

The Denjoy-Wolff Theorem. In a certain sense, every holomorphic self-map of U has an attractive fixed point: if there is not one in U, then there is a unique boundary point that serves the purpose. This is the content of the famous *Denjoy-Wolff Theorem*, which figures importantly in almost every aspect of the study of composition operators. To simplify its statement we adopt some terminology.

- We call a point $p \in \partial U$ a *boundary fixed point* of φ if φ has non-tangential limit p at p.
- We use the notation $\stackrel{\kappa}{\to}$ to indicate uniform convergence on compact subsets of U,
- If the derivative of φ has a nontangential limit at a boundary point p of U, we say φ has an *angular derivative* at p, and denote the limit by $\varphi'(p)$.

In the definition of angular derivative it is sometimes required that the non-tangential limit of φ at p (whose existence follows easily from that of the derivative) have modulus one (cf. [48, Chapter 4]). Although not explicitly required above, this additional condition will occur in our applications of the angular derivative.

Theorem 0.2 (Denjoy-Wolff) *Suppose φ is an analytic self-map of U that is not an elliptic automorphism.*

(a) *If φ has a fixed point $p \in U$, then $\varphi_n \stackrel{\kappa}{\to} p$ and $|\varphi'(p)| < 1$.*

(b) *If φ has no fixed point in U, then there is a point $p \in \partial U$ such that $\varphi_n \stackrel{\kappa}{\to} p$. Furthermore:*

- *p is a boundary fixed point of φ; and*
- *the angular derivative of φ exists at p, with $0 < \varphi'(p) \leq 1$.*

(c) *Conversely, if φ has a boundary fixed point p at which $\varphi'(p) \leq 1$ then φ has no fixed points in U, and $\varphi_n \stackrel{\kappa}{\to} p$.*

The fixed point p to which the iterates of φ converge is called the *Denjoy-Wolff* point of φ. Part (a), which has already been employed in the proof of Proposition 0.1, is not really part of the original theorem; it is included here only for convenience. For a proof of Theorem 0.2, and for further connections with the theory of composition operators, see [48, Chapter 5] or [19, Section 2.4]

Classification of linear-fractional maps. The Denjoy-Wolff Theorem suggests a "linear-fractional-like" classification of arbitrary holomorphic self-maps of U. For motivation, note that the *linear-fractional* self-maps of U fall into distinct classes determined by their fixed-point properties (cf. [48]: Chapter 0). These are:

- *Maps with interior fixed point.* By the Schwarz Lemma the interior fixed point is either attractive, or the map is an elliptic automorphism. In both cases the map is conjugate to a dilation $z \to \lambda z$ for some complex number λ with $0 < |\lambda| \leq 1$.

- *Hyperbolic maps with attractive fixed-point on ∂U.* These are the self-maps having derivative < 1 at a boundary fixed point.

- *Parabolic maps.* These have exactly one fixed point on the Riemann Sphere, and in order for the map to take U into itself, this fixed point must lie on ∂U. These maps are characterized by the fact that they have derivative $= 1$ at the fixed point.

The parabolic self-maps of U fall into two subclasses:

- *The automorphisms.* These are distinguished by the property that each orbit is separated in the hyperbolic metric (meaning that, for each $z \in U$, the hyperbolic distance between successive points of the orbit $(\varphi_n(z))$ stays bounded away from zero).

- *The nonautomorphisms.* For these, the orbits are not hyperbolically separated, i.e., the distance between successive orbit points tends to zero.

An elementary argument establishes these last two statements. The first just reflects the fact that automorphisms are hyperbolic isometries. The second is best viewed in the context of the right half-plane Π. Suppose ψ is a parabolic self-map of U with fixed point at 1, and let

$$T(w) = \frac{w+1}{w-1}, \quad \text{and} \quad \Psi \stackrel{\text{def}}{=} T \circ \psi \circ T^{-1}.$$

Thus T is a linear-fractional mapping of U onto Π that takes 1 to ∞, and one easily checks that $\Psi(w) = w + \psi''(1)$. It follows that $\psi''(1)$ has non-negative real part (otherwise Ψ could not map Π into itself), and since ψ is not an automorphism of U, $\psi''(1)$ cannot be pure imaginary. Now hyperbolic discs in Π of fixed radius have this property: their Euclidean size is proportional to the real part of their hyperbolic center (see section 4, or [48, Chapter 4] for the details). Our hypothesis on the translation distance $\psi''(1)$ insures that for each $w \in \Pi$ the Ψ-orbit $(\Psi_n(w))$ has unbounded real part, but fixed Euclidean distance $|\psi''(1)|$ between successive points. Thus for all sufficiently large n, the hyperbolic disc of radius ϵ about $\Psi_n(w)$ contains $\Psi_{n+1}(w)$, hence the orbit of w is not separated.

Motivated by the classification of linear-fractional self-maps of U, and encouraged by the restrictions the Denjoy-Wolff Theorem places on the values the derivative of an arbitrary self-map can take at the Denjoy-Wolff point, we introduce the following general classification scheme.

Definition 0.3 (Classification of arbitrary self-maps) A holomorphic self-map φ of U is of:

- *dilation type* if it has a fixed point in U;
- *hyperbolic type* if it has no fixed point in U and has derivative < 1 at its Denjoy-Wolff point;
- *parabolic type* if it has no fixed point in U and has derivative $= 1$ at its Denjoy-Wolff point.

As in the linear-fractional case, the maps of parabolic type fall into two subclasses:

- *Automorphic type:* Those having an orbit that is separated in the hyperbolic metric of U.
- *Non-automorphic type:* Those for which *no* orbit is hyperbolically separated.

Remark. It can be shown that either all orbits are separated or none are separated (for the maps of interest to us, we will prove this in section 4).

A major objective of this monograph is to determine the extent to which composition operators in each of the classes above share the same cyclic behavior as their linear-fractional role models. The possibility of such a connection is suggested by the following remarkable theorem of classical function theory.

Theorem 0.4 (The Linear-Fractional Model Theorem) *Suppose φ is a univalent self-map of U. Then there exists a univalent map $\sigma : U \to \mathbf{C}$ on*

U, and a linear-fractional map ψ such that $\psi(U) \subset U$, $\psi(\sigma(U)) \subset \sigma(U)$), and

$$\sigma \circ \varphi = \psi \circ \sigma. \tag{1}$$

Furthermore:

(a) ψ, viewed as a self-map of U, has the same type as ϕ.

(b) If ϕ is of hyperbolic type then ψ may be taken to be a conformal automorphism of U.

(c) If ϕ is of either hyperbolic or parabolic-automorphic type, then σ may be taken to be a self-map of U.

The fact that ψ maps the simply-connected domain $G = \sigma(U)$ into itself follows immediately from the functional equation (1). This equation establishes a conjugacy between the original map ϕ acting on the unit disc and the linear-fractional map ψ acting on G. Since the action of ψ is known, the subtleties of ϕ lie encoded in the geometry of G. All our investigations will be driven by the need to understand this geometry.

Definition 0.5 We call the pair (ψG) (or, equivalently, ($\psi\sigma$)) a *linear-fractional model* for ϕ).

Remarks. (a) *Some history.* The Linear-Fractional Model Theorem is the work of a number of authors, whose efforts stretch over nearly a century. The dilation case is due to Koenigs ([35]: 1884). In this case equation (1) is *Schröder's equation:* $\sigma \circ \varphi = \lambda\sigma$, where (necessarily) $\lambda = \varphi'(0)$ (see [48, Chapter 6] for more details). The hyperbolic case is due to Valiron. If one replaces the unit disc by the right half-plane, sending the Denjoy-Wolff point to ∞, then the resulting functional equation is again Schröder's equation, but this time λ is the reciprocal of the angular derivative of the original disc map at the Denjoy-Wolff point ([53]: 1931). Finally the parabolic cases were established by Baker and Pommerenke ([41, 2]: 1979), and independently by Carl Cowen ([14]: 1981). Once again the situation is best viewed in the right half-plane, rather than the unit disc, with the Denjoy-Wolff point placed at ∞. Then equation (1) is just $\sigma \circ \varphi = \sigma + i$ in the automorphic case [41], and $\sigma \circ \varphi = \sigma + 1$ in the nonautomorphic case [2]. In [14] Cowen unified the proof of the Linear-Fractional Model Theorem by means of a Riemann-surface construction that disposes of all the cases in one stroke (see also [19, Theorem 2.53]). He later introduced linear-fractional models into the study of composition operators, using them to investigate spectra

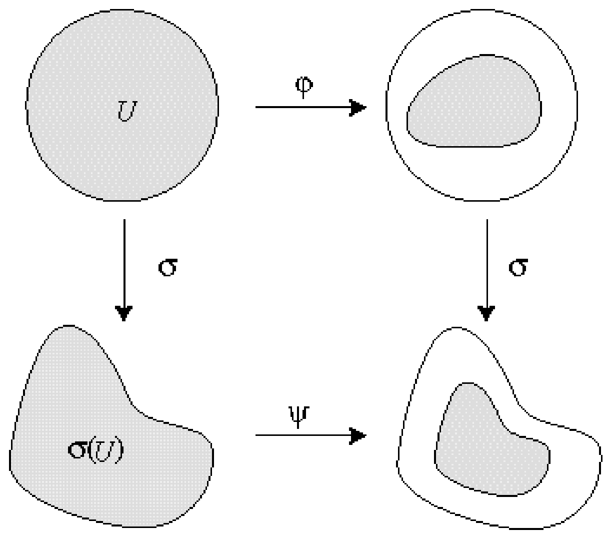

Figure 1: A Linear-Fractional Model

[15]. These models have also figured prominently in previously-mentioned work on subnormality [18] and compactness [49].

(b) *Role of the Theorem in this monograph.* The Linear-Fractional Model Theorem serves as a guide to, but not a prerequisite for, our efforts to generalize our results on cyclicity of linear-fractional composition operators. As the reader will see shortly, we require versions of the theorem that show how additional smoothness properties of φ influence the behavior of the intertwining map σ. For this we need to develop and analyze our own version of the model. This we do in section 4.

(c) *Distinguishing the parabolic models.* The problem of distinguishing the two parabolic cases of the Linear-Fractional Model Theorem is, in general, quite delicate. For our work, however, it poses no problem. We will establish in section 4 that if φ has enough differentiability at the Denjoy-Wolff point, then cases are distinguished by the second derivative of φ at that point. There is, however, some subtlety here; we will see in section 6 that, for example, C^2-differentiability at the Denjoy-Wolff point is not enough to allow the second derivative to distinguish the cases.

(d) *Necessity of Univalence.* We have stated the Linear-Fractional Model Theorem only for univalent maps φ. The result is true even if φ is not univalent, provided we are willing to give up the conclusion of univalence for the intertwining map σ. (In case φ is of dilation type, with fixed point

$p \in U$, we must also assume that $\varphi'(p) \neq 0$.) However only the univalent case will concern us here, because:

If C_φ is cyclic, then φ is univalent on U.

The necessity of univalence for *hypercyclicity* is easy to establish: if φ identifies two distinct points of U, then so does $f \circ \varphi_n$ for each $f \in H^2$ and each positive integer n, and therefore so does every limit point of the orbit $(f \circ \varphi_n)$ of f under C_φ. It follows that no orbit can be dense in H^2, so C_φ is not hypercyclic.

To prove the necessity of univalence for cyclicity requires more work (it is amusing to check why this is so). The full story is contained in Theorem 1.7, where it is shown that a certain "almost everywhere" boundary univalence is also necessary.

Transference

In order to transfer results on the cyclicity of linear-fractional composition operators to more general situations, we need to develop a version of the Linear-Fractional Model Theorem that predicts the boundary regularity properties of the intertwining map σ from those of φ. The sample result below makes clear the need for such analysis.

Theorem 0.6 (Sample Transference Theorem) *If φ is a univalent self-map of U of hyperbolic type, and the polynomials in σ are dense in H^2, then C_φ is hypercyclic.*

Sketch of Proof. Because φ has its Denjoy-Wolff point on ∂U and has angular derivative < 1 at that point, the Linear-Fractional Model Theorem provides a univalent self-map σ of U, and a hyperbolic disk automorphism ψ, so that the functional equation (1) is satisfied. Because ψ is a non-elliptic automorphism, our non-Euclidean analogue of Birkhoff's Theorem (Theorem 2.3) shows that C_ψ is hypercyclic. Let f be a hypercyclic vector for C_ψ, and assume that the polynomials in σ are dense in H^2. We claim that $f \circ \sigma$ is hypercyclic for C_φ. Applying (1), we obtain

$$C_\varphi^n (f \circ \sigma) = C_\sigma (f \circ \psi_n).$$

Hence, the orbit of $f \circ \sigma$ under C_φ is the image of $\mathrm{Orb}(C_\psi, f)$ under C_σ. Note C_σ has dense range (the image of the composition operator C_σ contains the set of polynomials in σ, which we're assuming to be dense). It follows that Orb $(C_\varphi, f\circ\sigma)$, being the image of the dense set Orb (C_ψ, f) under an operator with dense range, is dense in H^2. Thus C_φ is hypercyclic with hypercyclic vector $f\circ\sigma$. □

Our results on cyclicity have the same general flavor: If the polynomials in σ are dense in H^2, then C_φ inherits the same brand of cyclicity possessed by the linear-fractional composition operator C_ψ.

We will see in Section 1 that for C_φ to be cyclic, it is necessary that the polynomials in φ be dense in H^2. On the other hand, the transference technique introduced above requires the density of the polynomials, not in φ, but in σ. The crux of the cyclicity problem is that the desired polynomial approximation property need *not* be passed down from φ to σ (see, e.g., [48], §8.4, Problem 2, and §8.5). We devote much of the latter part of this monograph to overcoming this problem. The idea is to show that for maps φ that are sufficiently regular, the solutions σ of the functional equation (1) behave well enough to have the desired approximation property.

Our results here have interest that goes beyond the immediate study of cyclicity. We show, for example, that if the closure of $\varphi(U)$ is contained in U, then density of the polynomials in φ is equivalent to density of the polynomials in σ. As a consequence, we prove that, in this case, the connection between cyclicity and polynomial approximation is complete (Theorem 3.4):

> *Suppose* $\|\varphi\|_\infty < 1$*. Then* C_φ *is cyclic if and only if the polynomials in* φ *are dense in* H^2*.*

(This result fails if the assumption $\|\varphi\|_\infty < 1$ is dropped—see our remark following Theorem 3.4.)

In this monograph we draw our conclusions about polynomial approximation from the following fundamental result.

Theorem 0.7 (Walsh's Theorem) *Suppose G is a simply connected domain whose boundary is a Jordan curve. Let the holomorphic function F map U univalently onto G. Then the polynomials in F are dense in H^2.*

The result usually called Walsh's Theorem actually asserts that the polynomials in z are uniformly dense in $A(G)$, the subalgebra of $C(\overline{G})$ consisting of functions holomorphic on G (see, for example, [38, Theorem 3.9, page 98]). A theorem of Carathéodory asserts that F extends continuously and univalently to $\overline{G}$, so Walsh's original result asserts, in our situation, that the polynomials in F are dense in $A(U)$. Clearly $A(U)$ is dense in H^2, and this yields Theorem 0.7 (see [48, §8.1] for more details).

In particular (Corollary 3.5):

If φ maps the unit disk onto the interior of a Jordan curve lying in U, then C_φ is cyclic.

The Intertwining Map σ

To obtain the desired properties of σ we find an asymptotic representation of this map when φ obeys certain hypotheses of smoothness and "regularity." More precisely:

Definition 0.8 We call a map φ *regular* provided it is univalent and continuous on the closure of U, has Denjoy-Wolff point p on ∂U, and maps the closed disk into $U \cup \{p\}$.

We will require φ to have additional smoothness at its Denjoy-Wolff point; although this will vary from case to case, C^4-smoothness will always suffice. Our asymptotic representations of σ will yield the results on cyclicity listed in Table II below. The Denjoy-Wolff Theorem insures that $0 < \varphi'(p) \leq 1$, and we will see in Section 4 (Theorem 4.4, part (a)) that whenever $\varphi'(p) = 1$, then $\operatorname{Re}\varphi''(p) \geq 0$ (this also follows from the fact that the boundary of $\varphi(U)$ has curvature ≥ 1 at p). Thus the various cases represented by the rows of Table II exhaust all possibilities. Sections 3 through 5 are devoted to proving the assertions made in this table.

Comparison with Table I shows that if φ obeys our regularity and smoothness assumptions, then C_φ inherits the cyclic properties of its linear-fractional counterpart. When hypercyclicity is the issue, this will follow quickly from the asymptotic representations we obtain for σ in Section 4. However the cyclicity result described in the last row of Table II, while still making critical use of the regularity of σ, requires more effort; its proof occupies section 5.

Here is a quick outline of the rest of the monograph. Section 1 briefly summarizes the preliminary material on Hardy spaces, linear-fractional maps, angular derivatives, and cyclicity that forms the basis for the rest of the work. Linear-fractional cyclicity occupies section 2, while section 3 contains a detailed discussion of the linear-fractional model, the transference procedure, and its connection with polynomial approximation. In section 4 we obtain asymptotic representations of the intertwining map σ when φ is of hyperbolic and parabolic type, and in section 5 we complete the proof of our cyclicity theorem for regular maps. The work concludes with section

Table II
Cyclic behavior of C_φ: Denjoy-Wolff point at 1,
$\varphi \in C^4(1)$, *regular, and* $\varphi''(1) \neq 0$.

Hypothesis on $\varphi'(p)$	Hypothesis on $\varphi''(p)$	Cyclicity of C_φ	Type of φ (Model for φ)
< 1	None	Hypercyclic (Theorem 4.7)	Hyperbolic (Theorem 4.9)
=1	Pure imag. $\neq 0$	Hypercyclic (Theorem 4.16)	Parabolic automorphism (Theorem 4.12)
=1	Real part > 0	Cyclic, Not Hypercyclic (Thms. 4.6 & 5.2)	Parabolic non-automorphism (Theorem 4.12)

6, where we set out some open problems suggested by our work, and discuss further the matter of determining which of the two parabolic models (automorphic or non-automorphic) applies to a given φ.

1 Preliminaries

In this section we set forth additional terminology, and prove some basic results about the cyclic behavior of composition operators. We refer the reader to Duren's book [23] for more details about Hardy spaces, and to [19] and [48] for more on the basics of composition operators.

The Space H^2

Although our results hold for all the spaces H^p with $1 \le p < \infty$, we prefer to concentrate exclusively on the Hilbert space H^2, for which we can develop most of the necessary background within the next few paragraphs.

The Hardy space H^2 is the natural functional representation of the sequence space ℓ^2. It is the space of functions holomorphic in U whose Taylor coefficients in the expansion about the origin form a square summable sequence. More precisely,

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2 \qquad \Longleftrightarrow \qquad \|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

The inner product inducing the H^2 norm is given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \qquad (f, g \in H^2).$$

Associated to each point $\alpha \in U$ there is a function of particular interest to us: the *reproducing kernel* for α, defined by

$$K_\alpha(z) = \frac{1}{1 - \bar{\alpha} z} = \sum_{n=0}^{\infty} (\bar{\alpha} z)^n,$$

which clearly belongs to H^2, and has norm $1/\sqrt{1 - |\alpha|^2}$.

The property that gives K_α its name follows immediately from the definitions of K_α and inner product :

(2) $$f(\alpha) = \langle f, K_\alpha \rangle \quad (f \in H^2).$$

This shows that the linear functional of evaluation at $\alpha \in U$ is continuous on H^2, and provides the following growth estimate for functions in H^2:

$$|f(\alpha)| \leq \|f\| \, \|K_\alpha\| = \frac{\|f\|}{\sqrt{1-|\alpha|^2}}.$$

This shows, in particular, that convergence in H^2 implies uniform convergence on compact subsets of U. We will also need the corresponding estimate for derivatives, which is most easily obtained by applying the Cauchy-Schwarz inequality to the power series representation of f':

$$|f'(\alpha)| \leq \frac{\|f\|}{(1-|\alpha|)^{3/2}} \qquad (\alpha \in U, f \in H^2). \tag{3}$$

Life on the boundary of the unit disc is an important feature of Hardy space theory. For each $f \in H^2$ the radial limit

$$f^*(\omega) = \lim_{r \to 1^-} f(r\omega)$$

exists at almost every point ω of the unit circle. The boundary function f^* is the limit in $L^2 = L^2(\partial U, d\theta/2\pi)$ of the dilated functions f_r $(0 < r < 1)$ defined on the unit circle by $f_r(\omega) = f(r\omega)$, so it belongs to L^2, and has Fourier series representation

$$f^* \sim \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}.$$

Thus the mapping that associates the "interior" function f with the boundary function f^* is an isometry of H^2 onto the subspace of L^2 spanned by the orthonormal set $\{e^{in\theta} : n \geq 0\}$. From now on we drop the notation f^* and simply write $f(\omega)$ for the radial limit of f at ω. Thus we have for f and g in H^2:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta.$$

For more details we refer the reader to [23, Chapter 2].

We will need the following very special case of Beurling's Theorem on invariant subspaces of the shift operator (see [23, Theorem 7.4]). Fortunately, an elementary proof is available.

Proposition 1.1 *For $\omega \in \partial U$, let Z_ω denote the collection of functions holomorphic in a neighborhood of the closed unit disc which vanish at ω. Then Z_ω is dense in H^2.*

Proof. Suppose $f \in H^2$ is orthogonal to Z_ω. Then for all non-negative integers n we have $\langle f, (\omega - z)z^n \rangle = 0$; hence

$$\hat{f}(n+1) = \langle f, z^{n+1} \rangle = \bar{\omega}\langle f, z^n \rangle = \bar{\omega}\hat{f}(n).$$

From this it follows that $\hat{f}(n) = \bar{\omega}^n \hat{f}(0)$ for all n. Since ω is on the unit circle, and $f \in H^2$, this forces $\hat{f}(0) = 0$, and therefore all the Taylor coefficients of f must vanish. Thus $f \equiv 0$, so Z_ω is dense in H^2, as desired. □

Although our main results are set exclusively in the Hardy space H^2 of the unit disk, it is sometimes convenient to interpret some of the intermediate steps in a more general setting. If G is a simply connected plane domain, and σ is a univalent (holomorphic) mapping of U onto G, then the Hardy space $H^2(G)$, is the set of functions f holomorphic on G for which $f \circ \sigma \in H^2$. The inner product of two elements f and g in $H^2(G)$ is defined to be $\langle f \circ \sigma, g \circ \sigma \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product of the usual Hardy space H^2. In other words, the Hilbert space structure of $H^2(G)$ is simply the one imposed by declaring the composition operator $C_\sigma : H^2(G) \to H^2$ to be an isometry. The collection of functions $H^2(G)$ is easily seen to be independent of the particular univalent mapping σ used above, and although different maps σ give different Hilbert space norms on $H^2(G)$, it is also easy to check that these norms are all equivalent (the proof is simply the observation that every automorphism of U induces a Hilbert-space isomorphism of the original Hardy space H^2). See [23, Chapter 10] for more information about these generalized Hardy spaces .

Angular Derivatives

We indicated in the introduction that a boundary Denjoy-Wolff point is characterized by a condition on angular limits and angular derivatives. Here we make these notions precise. Given $0 < \alpha < \pi$ we define the family s_α of angular approach regions with vertex 1 as follows: s_α is the region in U between two chords symmetric with respect to the real axis that meet at 1 with angle α (see Figure 2). For any ω on the boundary of U, ωs_α is a family of angular approach regions with vertex ω.

We say that a function f defined on U has *angular limit* L at $\omega \in \partial U$, and write

$$\angle \lim_{z \to \omega} f(z) = L,$$

provided that given any α with $0 < \alpha < \pi$, the limit of $f(z)$ equals L as z approaches ω through ωs_α. Recall from the Introduction that we say a

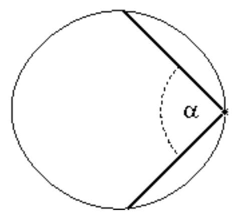

Figure 2: Angular Approach Region s_α.

self-map φ of U has an *angular derivative* at a point $\omega \in \partial U$ if the limit $\angle \lim_{z\to\omega} \varphi'(z)$ exists (finitely), and that when this happens, we denote the limit by $\varphi'(\omega)$: the *angular derivative of* φ *at* ω. The main classical result here the *Julia-Carathéodory theorem,* which provides an effective test of existence of the angular derivative (part (a)), and a geometric interpretation (part(b)).

Theorem 1.2 (Julia-Carathéodory) *Suppose that φ is an analytic self-map of U and that $\omega \in \partial U$. Then the following are equivalent.*

(a) $\liminf_{z\to\omega} \dfrac{1-|\varphi(z)|}{1-|z|} = \delta < \infty.$

(b) $\angle \lim_{z\to\omega} \dfrac{\eta - \varphi(z)}{\omega - z}$ *exists for some* $\eta \in \partial U$.

(c) φ *has an angular derivative at* ω, *and* $\angle \lim_{z\to\infty} \varphi(z) = \eta \in \partial U$.

Moreover, $\delta > 0$ *in* (a), *the boundary points* η *in* (b) *and* (c) *are the same, and the limit of the difference quotient in* (b) *coincides with that of the derivative in* (c), *with both equal to* $\omega\bar{\eta}\delta$.

For a proof of the Julia-Carathéodory theorem, see [10, §295–§303], [19, Theorem 2.44], or [48, Chapter 4]. Note that if φ fixes the boundary point ω, then the "derivative-like" limits in parts (a), (b), and (c) are are equal; in particular they agree when ω is the Denjoy-Wolff point of φ.

Recall from the Introduction that whenever the Denjoy-Wolff point p of φ lies on the boundary, the angular derivative $\varphi'(p)$ exists, and lies in the positive interval $(0, 1]$. This phenomenon arises from the inequality below, which provides additional insight concerning the behavior of φ near a boundary Denjoy-Wolff point (see [10, §297], [19, Lemma 2.41], or [48, Chapter 4]).

Theorem 1.3 (Julia-Carathéodory Inequality) *Suppose that φ has Denjoy-Wolff point $p \in \partial U$, and that $\varphi'(p) = \mu$. Then for any $z \in U$,*

$$\frac{|p-\varphi(z)|^2}{1-|\varphi(z)|^2} \leq \mu \frac{|p-z|^2}{1-|z|^2}.$$

Geometrically, the Julia-Carathéodory inequality says that if φ has its Denjoy-Wolff point on the boundary, then it takes each disk in U that is internally tangent to ∂U at that point into itself (see [48, §5.3] for example). This inequality will play an important role in our work with linear-fractional models in Section 4.

Cyclicity and Univalence

In the sequel, unless otherwise indicated, φ will denote an arbitrary self-map of U. We observed in the Introduction that univalence of φ is a necessary condition for C_φ to be hypercyclic, and we mentioned that univalence is also necessary for cyclicity. Here is the proof.

Suppose, for the sake of contradiction, that φ is *not* univalent. Then the open mapping theorem for holomorphic functions provides infinitely many pairs of distinct points a and b such that $\varphi(a) = \varphi(b)$. A simple computation shows that $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$ for each point $\alpha \in U$, where C_φ^* denotes the Hilbert-space adjoint of C_φ. Thus for each pair of points a and b as above,

$$C_\varphi^*(K_a - K_b) = K_{\varphi(a)} - K_{\varphi(b)} = 0.$$

This shows that if φ is not univalent, then the orthogonal complement of the range of C_φ is infinite dimensional. Now, the orthogonal complement of the range of a cyclic operator has dimension at most one (the orthogonal projection of any cyclic vector onto the complement spans). Thus C_ϕ is not cyclic, and we have established the necessity of univalence of φ on U for cyclicity.

In fact there is also a form of "boundary univalence" that is necessary for cyclicity. To prove it, we need an important special property of cyclic composition operators.

Theorem 1.4 *If C_φ is cyclic; then its range is dense in H^2.*

To appreciate this result, observe that not every cyclic Hilbert space operator has dense range. For example, the range of the "forward shift" on H^2 (i.e. the operator of "multiplication by z") has codimension 1, and so is not dense. But the operator is cyclic; in fact our proof of Proposition 1.1 shows that the function $\omega - z$ is a cyclic vector for each $\omega \in \partial U$.

Proof. Let M denote the closure of the range of C_φ and let n denote the codimension of M in H^2. Since C_φ is cyclic, $n = 0$ or $n = 1$. If $n = 0$, there is nothing to prove. We suppose that $n = 1$ and arrive at a contradiction.

Let h be a vector that spans the orthogonal complement of M. For any $g \in H^2$

$$\langle \bar{\varphi} h, g \circ \varphi \rangle = \langle h, \varphi g \circ \varphi \rangle = 0,$$

so $\bar{\varphi} h \in M^\perp$. Hence, if P denotes the orthogonal projection from L^2 onto H^2, then $P(\bar{\varphi} h) = \bar{\lambda} h$ for some complex number λ.

Now observe that

$$\langle h, (\varphi - \lambda) g \rangle = 0$$

for every $g \in H^2$. It follows that the closure of $(\varphi - \lambda) H^2$ must be contained in M. Recall now that, because C_φ is cyclic, φ must be univalent on U. Hence $\varphi - \lambda$ is univalent and factors as follows: $\varphi - \lambda = BF$, where B is a Blaschke product with at most one zero and F is outer ([23], Theorem 3.17). Because FH^2 is dense in H^2 (in fact the polynomial multiples of F are dense in H^2 by Beurling's Theorem), we have

$$BH^2 = \text{closure of } BFH^2 = \text{closure of } (\varphi - \lambda) H^2 \subset M.$$

Because B has at most one zero, the codimension of BH^2 in H^2 at most one. Since $BH^2 \subset M$ and M has codimension one in H^2, M must equal BH^2 and B must have exactly one zero in U. However, this contradicts the fact that 1 belongs M. □

The problem of determining which composition operators have dense range is non-trivial and, as the next result shows, is in fact, equivalent to the problem of characterizing those bounded simply connected domains G for which the polynomials are dense in $H^2(G)$.

Proposition 1.5 *Suppose that $G \subset U$ is simply connected, and that φ maps U univalently onto G. Then the following are equivalent:*

(a) *The polynomials are dense in $H^2(G)$;*

(b) *The polynomials in φ are dense in H^2;*

(c) *The composition operator $C_\varphi : H^2 \to H^2$ has dense range.*

We leave the simple proof of Proposition 1.5 to the reader. Combining Theorem 1.4 and Proposition 1.5 we obtain:

Corollary 1.6 *If C_φ is cyclic, then the set of polynomials in φ is dense in H^2. Equivalently, the set of polynomials in z is dense in $H^2(\varphi(U))$.*

We present deeper connections between polynomial approximation and cyclicity in sections 3 and 4.

We can now prove our final result on the necessity of univalence for cyclicity. Let us say that a function $f \in H^2$ is *univalent almost everywhere* on ∂U provided there is a set $E \subset U$ having zero Lebesgue measure such that f is univalent on $\partial U \backslash E$.

Theorem 1.7 *If C_φ is cyclic, then φ must be univalent on U, and univalent almost everywhere on ∂U.*

Proof. We have already noted that if C_φ is cyclic, then φ must be univalent. Theorem 1.4 shows that, in addition, C_φ must have dense range; in particular, there is a sequence (f_n) of functions in H^2 such that $f_n(\varphi(z)) \to z$ in H^2. Now choose a subsequence (f_{n_j}) such that $f_{n_j}(\varphi(z)) \to z$ almost everywhere on ∂U, and observe that off the set of measure zero on which $f_{n_j}(\varphi(z))$ may not go to z, φ must be univalent. □

To observe this Corollary in action, note that if φ maps U univalently onto the slit disk $U \backslash [0,1)$, then C_φ is not cyclic.

Motivated by the preceding results, we will focus primarily on univalent self-maps φ that extend continuously to ∂U and are *everywhere* univalent on ∂U. The image of the unit disk under such a map is a *Jordan domain* (the interior of a Jordan curve), and hence, Walsh's Theorem (Theorem 0.7) shows that C_φ satisfies the necessary conditions for cyclicity presented above.

We remark that the necessary conditions for cyclicity discussed in this section are not sufficient. We will show in the next section that, for example, if $\psi(z) = \frac{z}{2-z}$, then C_ψ is not cyclic (Theorem 2.8). Note, however, that C_ψ does have dense range. This follows, for example, from the fact that $\psi(U)$ is a Jordan domain—a disk, or from the elementary argument below:

Suppose that f is orthogonal to the range of C_ψ. Then because 1 is in the range, $0 = \langle f, 1 \rangle = f(0)$ so that $f = zg$ for some g in H^2. Because ψ^n belongs to the range of C_ψ, we have

$$\begin{aligned} 0 &= \langle f, \psi^n \rangle \\ &= \langle zg, \left(\frac{z}{2-z}\right)^n \rangle \end{aligned}$$

$$= \frac{1}{2^n}\langle g, \frac{z^{n-1}}{(1-(1/2)z)^n}\rangle$$

$$= \frac{1}{2^n(n-1)!}g^{(n-1)}(1/2).$$

Since g and all of its derivatives vanish at the point $1/2$ we see that $g \equiv 0$ and hence, $f \equiv 0$. It follows that C_ψ has dense range.

Hypercyclicity Basics

In what follows, X will always denote a separable Banach space, and T a continuous linear operator on X. Recall from the Introduction that a vector $f \in X$ is *hypercyclic* (for T) if its orbit $\text{Orb}\,(T, f)$ is dense in X, in which case T is a *hypercyclic operator.* Clearly only separable Banach spaces can support hypercyclic operators. Note also that the collection of hypercyclic operators is similarity invariant (as is the collection of cyclic operators).

Observe that

- The collection of hypercyclic vectors for T forms a G_δ set;

for this set can be written as:

$$\bigcap_{s,k}\{f \in X : \|T^n f - s\| < 1/k \text{ for some } n \geq 0\},$$

where k ranges through the positive integers, and s through a countable dense subset of X. This countable intersection is a G_δ set because, thanks to the continuity of T, the set in braces is open. Now if T has a hypercyclic vector, then everything in the orbit of this vector is also hypercyclic so that T has a *dense set* of hypercyclic vectors. Thus we have an amusing "zero-one law":

- An operator has either no hypercyclic vector, or a dense G_δ set of them.

From this result, Baire's Theorem yields a couple of interesting observations:

- Every countable collection of hypercyclic operators has a common hypercyclic vector.

- If T is a hypercyclic operator on X, then every vector in X is the sum of two hypercyclic vectors for T.

The second result was shown to us by Gilles Godefroy. To see why it is true, let E denote the collection of hypercyclic vectors for T, and suppose $x \in X$. Since both E and $x - E$ are dense G_δ subsets of X, they have nonvoid intersection. To say that h_1 is in the intersection means that it is hypercyclic, and that there exists hypercyclic h_2 for which $h_1 = x - h_2$.

We will rely heavily on the following sufficient condition for hypercyclicity. Essentially the same result was presented in [25], and proved independently by Kitai in [34] (see also [48, Chapter 7]).

Theorem 1.8 (Sufficient Condition for Hypercyclicity) *Suppose T is a continuous linear operator on a separable Banach space X, for which the sequence of non-negative powers (T^n) tends pointwise to zero on a dense subset of X. Suppose further that there is a (possibly different) dense subset Y of X, and a (possibly discontinuous) map $S : Y \to Y$ such that $TS =$ identity on Y, and (S^n) tends pointwise to zero on Y. Then T is hypercyclic.*

To view Theorem 1.8 in action, note how it provides a quick proof of the following result of Rolewicz [42] concerning the backward shift B on H^2, defined by $Bf(z) = z^{-1}(f(z) - f(0))$.

> *For each complex number $\lambda > 1$ the operator λB is hypercyclic on H^2.*

For the proof, just take $X = Y = \{p : p \text{ is a polynomial}\}$ in Theorem 1.8, let $T = \lambda B$, and let S be multiplication by $\lambda^{-1}z$. One checks easily that all the hypotheses of Theorem 1.8 are satisfied.

In the next section we will see how Theorem 1.8 shows, with almost equal ease, that composition operators induced by non-elliptic disc automorphisms are hypercyclic.

2 Linear-Fractional Composition Operators

A *linear-fractional transformation* is a mapping of the form

$$\varphi(z) = \frac{az+b}{cz+d} \ ,$$

where a, b, c, and d are complex constants and where, in order to insure that φ is non-constant, we also assume that $ad - bc \neq 0$. If such a mapping φ takes U into itself, we call the induced operator C_φ a *linear-fractional* composition operator. In this section we completely characterize the cyclic behavior of linear-fractional composition operators.

Linear-Fractional Basics

We require only the most basic facts about linear-fractional transformations and their classification (see, e.g., [1, §3.3]), [48, Chapter 0]). In particular, with the obvious conventions about ∞, each such transformation maps the Riemann sphere one-to-one and holomorphically onto itself. Acting as a mapping of plane sets, each linear-fractional transformation permutes the collection of circles and lines, and hence the collection of disks, disk-complements, and half-planes. Under functional composition, the set of all linear-fractional transformations forms a group that acts triply transitively on the sphere.

If $\varphi = T \circ \psi \circ T^{-1}$, where T and ψ are linear-fractional transformations, we say φ is *conjugate* to ψ (by T). The quadratic formula shows that every linear-fractional transformation φ, except the identity, has one or two fixed points in the sphere, and is therefore conjugate to a "normal form" φ that has respectively a single fixed point at ∞, or fixed points at 0 and ∞. In the former case φ is a translation: $\varphi(z) = z+\alpha$ for some complex number α, and in the latter it is a complex dilation: $\varphi(z) = \kappa z$ for some complex κ. The precise constants α and κ depend on the conjugating transformation. The reader may verify that in the one fixed point case, α can be any non-zero complex number, while in the two fixed point case either κ or its reciprocal can appear.

Linear-fractional transformations with one fixed point are called *parabolic*. Those with two fixed points are classified according to their normal forms: *elliptic* if $|\kappa| = 1$, *hyperbolic* if $\kappa > 0$, and otherwise *loxodromic*. The normal forms show us that every non-elliptic linear-fractional transformation has exactly one fixed point p that is attractive. By this we mean that for each z in the sphere, not itself a fixed point, $\varphi_n(z) \to p$, where, as before, φ_n denotes the n-fold composition of φ with itself. If φ is neither parabolic nor elliptic, then the other fixed point is *repulsive*, i.e., it is attractive for φ^{-1}. Observe that conjugation preserves derivatives at fixed points so that the derivative of a parabolic map at its fixed point is 1; the derivative of a hyperbolic map at its attractive fixed point is less than 1 (and at its repulsive fixed point greater than 1).

If a non-elliptic linear-fractional transformation φ happens also to be a self-map of U, then its attractive fixed point must clearly lie in the closure of U. We refer to a fixed point in U as an *interior* fixed point, one on ∂U as a *boundary* fixed point, and one in the complement of the closed disk as an *exterior* fixed point. Finally, there are these simple consequences of the Schwarz lemma:

- Any automorphism of U with interior fixed point must be elliptic.
- No self-map of U may have more than one interior fixed point.
- No linear-fractional self-map of U with interior fixed point can be parabolic (a parabolic map must have derivative one at its fixed point).

Note also that a linear-fractional self-map of U with two boundary fixed points must be an automorphism.

Cyclicity: First Observations

We now turn our attention to the cyclicity problem for linear-fractional composition operators. Recall from the Introduction that maps with interior fixed point are *never* hypercyclic. Thus the possibility of hypercyclicity arises only for linear-fractional self-maps of U with *no* interior fixed point; for the rest, the issue is confined to cyclicity.

We dispense once and for all with the elliptic case.

Proposition 2.1 *If φ is an elliptic automorphism of U, then C_φ is cyclic if and only if φ is conjugate to a rotation through an irrational multiple of π.*

Proof. Note that any elliptic self-map of U has to be an automorphism of U with interior fixed point, and hence must be conjugate (by automorphisms) to a rotation about the origin. Specifically, if φ is an elliptic automorphism of U with fixed point a, then there is a unimodular constant λ such that

$$(\psi_a \circ \varphi \circ \psi_a)(z) = \lambda z,$$

where ψ_a is the automorphism of U defined by

$$\psi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

(note that ψ_a, and hence C_{ψ_a}, is self-inverse). Thus $C_\varphi = C_{\psi_a} C_{\lambda z} C_{\psi_a}$ is similar to $C_{\lambda z}$.

Now, $C_{\lambda z}$ fails to be hypercyclic because its symbol has an interior fixed point. If $z \mapsto \lambda z$ is a rotation through rational multiple of π, then $C_{\lambda z}$ also fails to be cyclic because in this case the orbit of any function in H^2 under $C_{\lambda z}$ is a finite set. If, however, λ induces a rotation through irrational multiple of π, then $C_{\lambda z}$ is cyclic. To see this, let $\alpha \neq 0$ be a point in U and note that any function orthogonal to

$$\text{Orb}\,(C_{\lambda z}, K_\alpha) = \{K_{\bar{\lambda}^n \alpha} : n = 0, 1, 2, \ldots\}$$

must vanish at infinitely many points on the circle $|z| = |\alpha|$, hence must vanish identically on U. Because cyclicity is similarity invariant, the proof is complete. □

The Main Theorem

Having forever disposed of the elliptic case, we state the main result of this section: a complete characterization of the cyclic behavior of non-elliptic linear-fractional composition operators.

Theorem 2.2 (Linear-Fractional Cyclicity) *Let φ be a non-elliptic linear-fractional self-map of U.*

(a) *C_φ is cyclic unless φ has both a fixed point in U and one on ∂U. In this latter case, C_φ is strongly non-cyclic, in the sense that every finitely generated C_φ- invariant subspace has infinite codimension.*

(b) *If φ has no fixed point in U, then C_φ is hypercyclic unless φ is a parabolic non-automorphism. In this latter case C_φ is strongly non-hypercyclic, in the sense that the only functions that can adhere to C_φ-orbits are constant functions.*

The theorem says, for example, that if a linear-fractional self-map of U has no interior fixed point, then it induces a cyclic composition operator. In particular, every parabolic self-map of U induces a cyclic composition operator; but in this regard the theorem asserts that among the parabolics, only the *automorphisms* induce *hypercyclic* operators.

As we have seen (Proposition 0.1), if φ is a non-elliptic self-map of U with interior fixed point, then only the constant functions can adhere to C_φ-orbits. Thus the second part of the theorem asserts that, relative to hypercyclicity, the fixed point of a parabolic non-automorphism acts "as if it were inside the disc."

Nina Zorboska, in her dissertation [55], studied composition operators induced by non-elliptic disc automorphisms, and proved that they are all cyclic. The theorem above shows that they are actually hypercyclic. It also shows that, except for the basic necessary condition 1.4, the geometry of $\varphi(U)$ does not play a major role in determining cyclic behavior of composition operators. For instance, if

$$\varphi(z) = \frac{z}{2-z} \quad \text{and} \quad \psi(z) = \frac{1+2z}{3},$$

then the image of U under both maps is the disc of radius $2/3$ centered at the point $1/3$, yet according to the theorem, C_φ is non-cyclic (φ fixes 0 and 1), whereas C_ψ is hypercyclic (ψ fixes 1 and ∞, so it has no fixed point in U, and is not parabolic).

The Linear-Fractional Cyclicity Theorem 2.2 is really a summary of several results, the statements and proofs of which will occupy the rest of this section.

Proof of Part (b). Here we consider linear-fractional self-maps φ of U that have no interior fixed point. Three cases exhaust the possibilities:

- φ is an automorphism. In this case we show that C_φ is hypercyclic (Theorem 2.3).
- φ is not an automorphism and not parabolic, so that it has two fixed points: the attractive one necessarily on ∂U, the other necessarily outside the closure of U. We prove that in this case C_φ is again hypercyclic (Theorem 2.4(i)).
- φ is parabolic, but not an automorphism. In this case φ has only one fixed point, which necessarily lies on ∂U. We prove that C_φ is strongly non-hypercyclic (Theorem 2.4(ii)).

Now down to business.

Theorem 2.3 *Every non-elliptic automorphism of U induces a hypercyclic composition operator on H^2.*

Proof. Let φ denote the automorphism in question. Because φ is not elliptic, it has an attractive fixed point a on ∂U. If φ is not parabolic, there is one other (repulsive) fixed point b, which (because φ is an automorphism) must also lie on the boundary. If φ is parabolic, a is the attractive fixed point for both φ and φ^{-1}. In order to treat both cases simultaneously, we set $a = b$ if φ is parabolic.

Let Z_a be the set of functions holomorphic in a neighborhood of the closed unit disc, which vanish at a; and define Z_b similarly. According to the special case of Beurling's theorem discussed in Section 1 (Proposition 1.1), these sets are dense in H^2. Since $\varphi_n \to a$ pointwise on ∂U, with one possible exception (the repulsive fixed point of φ, if there is one), we see that for every $f \in Z_a$, the sequence $(f \circ \varphi_n)$ converges a.e. on ∂U to $f(a) = 0$. Since the sequence is uniformly bounded on ∂U, the Lebesgue Bounded Convergence Theorem insures that it also converges to zero in the norm of H^2. In other words, the sequence of positive powers of C_φ tends pointwise to zero on a dense subset of H^2.

Now C_φ is invertible on H^2: its inverse is C_ψ, where $\psi = \varphi^{-1}$. Since b is now the attractive fixed point for ψ, the operator C_ψ maps Z_b into itself, and the argument above shows that the sequence of non-negative powers of C_ψ tends to zero pointwise on on the dense set Z_b. Thus the hypotheses of our sufficient condition for hypercyclicity 1.8 are satisfied with $T = C_\varphi, S = C_\psi, X = Z_a$, and $Y = Z_b$. □

The next result completes the proof of part (b) of the Linear-Fractional Cyclicity Theorem.

Theorem 2.4 *Suppose φ is a linear-fractional self-map of U that is* not *an automorphism and does* not *have an interior fixed point.*

(i) *If φ is not parabolic, then C_φ is hypercyclic.*

(ii) *If φ is parabolic, then C_φ is not hypercyclic; in fact, the only possible limit points of C_φ-orbits are constant functions.*

Proof. (i) We use an elementary transference argument that foreshadows the work of the next section. Suppose φ is not parabolic. We will show that C_φ is hypercyclic by intertwining φ with a hyperbolic automorphism ψ, and using an approximation argument.

By the observations made earlier in this section, φ has its attractive fixed point on ∂U and its repulsive fixed point outside the closure of U. Now, φ is conjugate by an appropriate disk automorphism to a self-map having 1 as its attractive fixed point and ∞ as its repulsive fixed point. Such a self-map must have the form $z \mapsto az + (1-a)$, where $0 < a < 1$. Hence C_φ is similar to $C_{az+(1-a)}$; and we may therefore assume without loss of generality that

$$\varphi(z) = az + (1-a).$$

Note that φ is an automorphism of the half-plane $G = \{z : \text{Re } z < 1\}$. Hence if σ is a linear-fractional transformation mapping G onto U, then

$$\psi = \sigma \circ \varphi \circ \sigma^{-1} \tag{1}$$

is a disk automorphism with no interior fixed point. By Theorem 2.3, C_ψ is hypercyclic; let f be a hypercyclic vector for C_ψ. We claim that $f \circ \sigma$ is a hypercyclic vector for C_φ; that is, we claim $\text{Orb}\,(C_\varphi, f \circ \sigma)$ is dense in H^2.

Because sigma maps G onto U, and $U \subset G$, the disk $\sigma(U)$ must also be a subset of U. Hence, $f \circ \sigma$ is an H^2 function; moreover, the composition operator C_σ must have dense range since, for example, $\sigma(U)$ is a Jordan domain. By (1),

$$\text{Orb}\,(C_\varphi, f \circ \sigma) = C_\sigma \text{Orb}\,(C_\psi, f);$$

hence $\text{Orb}\,(C_\varphi, f \circ \sigma)$, being the image of the dense set $\text{Orb}\,(C_\psi, f)$, under the operator C_σ that has dense range, is itself dense in H^2. Thus C_φ is hypercyclic.

(ii) Suppose φ is parabolic, so it has a unique fixed point, necessarily on the unit circle. Without loss of generality we may assume this fixed point is 1. We compute φ explicitly by employing the change of variable $w = (1+z)/(1-z)$, which sends U to the right half-plane Π, the fixed point 1 to ∞, and φ to the translation map

$$\varphi(w) = w + a \qquad (w \in \Pi),$$

where $\text{Re } a > 0$ (the strict inequality reflecting the fact that φ is not an automorphism of U). Pulling back to the unit disc we obtain:

$$\varphi(z) = \frac{(2-a)z + a}{-az + (2+a)} \qquad (z \in U), \tag{2}$$

and more generally, for $n = 0, 1, 2, ...$; the n-th iterate φ_n of φ is obtained by replacing a by na in formula (2). Some calculations show that for $z \in U$:

$$1 - \varphi(z) = \frac{2(1-z)}{a(1-z)+2}, \quad \text{and} \quad \varphi(z) - \varphi(0) = \frac{4z}{(2+a)(2+a-az)}.$$

Upon replacing a by na in these expressions and letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} n(1-\varphi_n(z)) = \frac{2}{a}, \tag{3}$$

and

$$\lim_{n\to\infty} n^2[\varphi_n(z)-\varphi_n(0)] = \frac{4z}{a^2(1-z)} \ . \tag{4}$$

Now suppose $f, g \in H^2$, with g a cluster point of $\mathrm{Orb}\,(C_\varphi, f) = (f\circ\varphi_n)$. Our goal is to show that g must be constant on U. Recall formula (3) from Section 1:

$$|f'(z)| \le \frac{\|f\|}{(1-|z|)^{3/2}}.$$

Thus for any z and $w \in U$, with $|z| \le |w|$, it follows upon integrating f' over the line segment from z to w, and using the inequality above, that:

$$|f(z)-f(w)| \le \|f\|\frac{|w-z|}{(1-|w|)^{3/2}}. \tag{5}$$

By referring to the half-plane realization of φ_n as translation by na, where we recall that Re $a > 0$, we see that the φ - orbit of any point in U converges nontangentially to 1 (this is the only place where make essential use of the fact that φ is not an automorphism!). Now fix $z \in U$, and write $s_n = \varphi_n(0)$ and $t_n = \varphi_n(z)$. Then there exists a constant C such that:

$$|1-s_n| \le C(1-|s_n|), \text{and } |1-t_n| \le C(1-|t_n|) \quad \text{for all } n. \tag{6}$$

For convenience, let u_n denote either s_n or t_n: the one with larger magnitude. Then, letting "const.″ denote a constant independent of n which may, nonetheless vary from line to line, the previous estimates show:

$$\begin{aligned}
|f(t_n)-f(s_n)| &\le \|f\|\frac{|t_n-s_n|}{(1-|u_n|)^{3/2}} && \text{[by (5)]}\\
&\le \text{const.}\,\|f\|\frac{|t_n-s_n|}{|1-u_n|^{3/2}} && \text{[by (6)]}\\
&\le \text{const.}\,\|f\|\frac{n^{-2}}{n^{-3/2}} && \text{[by (3) and (4)]}\\
&= \text{const.}\,n^{-1/2}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}[f(t_n) - f(s_n)] = 0.$$

Now suppose $g \in H^2$ is a cluster point of the sequence $(f \circ \varphi_n)$. Then some subsequence $f \circ \varphi_{n_k}$ converges to g in H^2, hence pointwise on U. Thus

$$\begin{aligned} g(z) - g(0) &= \lim_{k\to\infty}[f(\varphi_{n_k}(z)) - f(\varphi_{n_k}(0))] \\ &= \lim_{k\to\infty}[f(t_{n_k}) - f(s_{n_k})] \\ &= 0. \end{aligned}$$

Thus $g(z) = g(0)$, regardless of our choice of $z \in U$, so g is constant, as desired. □

This completes the proof of part (b) of the Linear-Fractional Cyclicity Theorem.

Proof of Part (a). We begin this proof by showing that if φ is a parabolic non-automorphism, then—although by the last result C_φ is not hypercyclic—C_φ is cyclic (Theorem 2.5). We then consider non-elliptic self-maps of U with interior (attractive) fixed point. Because such maps cannot be parabolic, exactly one the following holds:

(a-1) φ has interior and exterior fixed points;

(a-2) φ has interior and boundary fixed points.

We show that C_φ is cyclic in case (a-1) and noncyclic in case (a-2) (Theorems 2.6 and 2.8 respectively), which will complete the proof of the Linear-Fractional Cyclicity Theorem.

Theorem 2.5 *Every parabolic linear-fractional self-map of U induces a cyclic composition operator on H^2.*

Proof. We have proved (Theorem 2.3) that parabolic automorphisms induce (hyper)cyclic composition operators, so we need only consider parabolic self-maps φ of U that are *not* automorphisms. For such a φ, we will show that the identity map u, defined on U by $u(z) = z$, is a cyclic vector for C_φ.

Without loss of generality we may assume that 1 is the fixed point of φ. Recall formula (2), which asserts that

$$\varphi(z) = \frac{(2-a)z + a}{-az + (2+a)}.$$

for some complex number a with $\operatorname{Re} a > 0$ (the strict positivity of $\operatorname{Re} a$ reflecting the fact that φ is not an automorphism). For our purposes a more convenient expression for φ is:

$$\varphi = \bar{\gamma} + \bar{\alpha} K_\beta,$$

where

$$\bar{\gamma} = \frac{a-2}{a}\ , \quad \bar{\alpha} = \frac{4}{a(a+2)}\ , \quad \bar{\beta} = \frac{a}{2+a}\ ,$$

and $K_\beta(z) = (1 - \bar{\beta} z)^{-1}$.

The requirement that $\operatorname{Re} a > 0$ insures that none of the denominators in the definitions of α, β, and γ is zero. In addition it guarantees that $\beta \in U$, so K_β is the H^2 reproducing kernel for the point β, as discussed in Section 1:

$$\langle f, K_\beta \rangle = f(\beta) \quad \text{for all } f \in H^2.$$

Now suppose $f \in H^2$ is orthogonal to the C_φ-orbit of u. That is, suppose

$$\langle f, \varphi_n \rangle = 0 \quad \text{for} \quad n = 0, 1, 2, ...,$$

where $\varphi_0 = u$. Since the sequence of iterates (φ_n) is uniformly bounded on ∂U, and pointwise convergent to 1 on the unit circle, it converges to 1 in the norm of H^2, so

$$0 = \lim \langle f, \varphi_n \rangle = \langle f, 1, \rangle = f(0).$$

Using this along with the orthogonality of φ and f, we have

$$0 = \langle f, \varphi \rangle = \langle f, \bar{\gamma} + \bar{\alpha} K_\beta \rangle = \gamma \langle f, 1 \rangle + \alpha \langle f, K_\beta \rangle = \alpha f(\beta),$$

so that $f(\beta) = 0$. But f is also orthogonal to φ_n for each n, and the formula for φ_n is obtained from that of φ by replacing a with na. Thus the last calculation actually shows that the function f vanishes identically on the sequence of points

$$\beta_n = \frac{na}{2+na} \quad (n = 1, 2, \ldots),$$

each of which belongs the unit disc. Because of the strict positivity of Re a we have:

$$1 - |\beta_n|^2 = \frac{4(1 + \text{Re } na)}{4 + 4\text{Re } na + |na|^2} \geq \text{const. } \frac{1}{n}\ ,$$

hence $\sum (1 - |\beta_n|) = \infty$, i.e., (β_n) is not a Blaschke sequence. Thus f, being in H^2, must vanish identically on U.

We have shown that only the zero vector can be orthogonal to the C_φ-orbit of u; therefore u is a cyclic vector for C_φ. □

To this point we have shown that every linear-fractional self-mapping of U with no interior fixed point (i.e., with its attractive fixed point on the boundary) induces a cyclic composition operator on H^2. We now turn our attention to non-elliptic mappings *with* an interior fixed point. Necessarily this fixed point is attractive, and since such a mapping cannot be parabolic, there is a repulsive fixed point somewhere outside U: either on the boundary or outside the closure of U (possibly at ∞). The results below show that each of these cases gives rise to different cyclic behavior for the induced composition operator.

Theorem 2.6 *If a linear-fractional self-map of U has attractive fixed point in U and repulsive fixed point outside the closure of U, then the induced composition operator is cyclic on H^2.*

PROOF. Suppose φ is a linear-fractional map with interior and exterior fixed points. Without loss of generality we may assume the interior fixed point is the origin. In this case φ can be written out explicitly as

$$\varphi(z) = \frac{z}{az+b},$$

where, by the Schwarz Lemma, $|b| > 1$ (since $\varphi'(0) = 1/b$). What is the restriction on a? The repulsive fixed point of φ is $(1-b)/a$ (to be interpreted as ∞ if $a = 0$), and the requirement that this point lie outside the closed unit disc is equivalent to the condition

$$\left|\frac{a}{1-b}\right| < 1. \tag{7}$$

We claim that for any non-zero $\alpha \in U$, the reproducing kernel $K_\alpha(z) = 1/(1-\bar{\alpha}z)$ is a cyclic vector for C_φ. A straightforward induction argument shows that for any non-negative integer n:

$$K_\alpha \circ \varphi_n(z) = \frac{as_n z + 1}{as_n z + 1 - \bar{\alpha}zb^{-n}} \ ,$$

where $s_0 = 0$, and for positive n:

$$s_n = \sum_{k=1}^{n} \frac{1}{b^k}.$$

Now fix a vector $g \in H^2$ that is orthogonal to the orbit $\{K_\alpha \circ \varphi_n : n = 0, 1, 2, \ldots\}$. We claim that g is the zero-function. To see this, note that the sequence $(K_\alpha \circ \varphi_n)$ converges to 1 in H^2 (since it converges to 1 uniformly on the closed unit disc); therefore

$$0 = \lim\langle g, K_\alpha \circ \varphi_n\rangle = g(0).$$

Recalling the proof of the last theorem, we write $K_\alpha \circ \varphi_n$ in terms of a reproducing kernel:

$$K_\alpha \circ \varphi_n = \lambda_n + \gamma_n K_{\beta_n},$$

where λ_n and γ_n non-zero complex constants, and

$$\beta_n = \frac{\alpha}{\bar{b}^n} - \bar{a}\bar{s}_n.$$

Thus our orthogonality hypothesis on g yields:

$$0 = \langle g, K_\alpha \circ \varphi_n\rangle = \lambda_n g(0) + \gamma_n g(\beta_n) = \gamma_n g(\beta_n).$$

Thus g vanishes identically on the sequence (β_n). Upon recalling that $|b| > 1$ we see from the definition that $s_n \to (b-1)^{-1}$, hence

$$\beta_n \to \frac{-\bar{a}}{\bar{b} - 1} \ ,$$

where, by inequality (7), this limit belongs to U. Thus g vanishes on a sequence with limit point in U, hence g is the zero-function. This shows that K_α is cyclic for C_φ. □

Our proof of the Linear-Fractional Cyclicity Theorem is almost finished; it only remains to consider composition operators induced by maps that have both an interior and a boundary fixed point. We will show that such operators are strongly non-cyclic in the sense that every finitely generated invariant subspace has infinite codimension. The key to this result is the following generalization of the fact that the orthogonal complement of the range of a cyclic operator has dimension at most one.

Proposition 2.7 *If the adjoint of a bounded linear operator T on a Hilbert space has a multiple eigenvalue, then T is not cyclic. If the adjoint of T has an eigenvalue of infinite multiplicity, then every finitely generated T-invariant subspace has infinite codimension.*

PROOF. Suppose T is cyclic on the Hilbert space $\mathcal{H}$, with cyclic vector f. Let λ denote an arbitrary complex number. The binomial theorem shows that the linear span of $\operatorname{Orb}(T, f)$ equals the linear span of $\operatorname{Orb}(T - \lambda, f)$, so

$T-\lambda$ is also cyclic. Hence, the kernel of $T^*-\bar{\lambda}$ (which equals the orthogonal complement of the range of $T-\lambda$) is at most one dimensional. Since λ was arbitrary, it follows that T^* has no multiple eigenvalues.

For the second part of the Proposition, we prove something a little more precise (which also contains the proof of the first part). For the moment, suppose only that T^* has an eigenvalue λ of multiplicity no less than some positive integer $m>1$. Suppose $g_1, g_2, \ldots, g_n$ are $n<m$ vectors in $\mathcal{H}$, and let $\mathcal{G}$ denote the closed, $T-$invariant subspace generated by these vectors; that is, $\mathcal{G}$ is the closed linear span of $\cup_{j=1}^{n}\mathrm{Orb}\,(T, g_j)$. We claim that $\mathcal{G}$ has codimension at least $m-n$. Thus in particular, if λ has *infinite* multiplicity, then given n, we can choose m as large as desired, so that $\mathcal{G}$ has infinite codimension, as desired.

We prove the claim by exhibiting $m-n$ linearly independent vectors in $\mathcal{H}$ that are orthogonal to $\mathcal{G}$. To this end, let $f_1, f_2, \ldots f_m$ be the linearly independent eigenvectors of T^* promised by our hypothesis, and let $\mathcal{F}$ denote their linear span. Let P denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{F}$, and observe that, because the subspace $\mathrm{span}\{Pg_1, Pg_2, \ldots, Pg_n\}$ of $\mathcal{F}$ has dimension $\leq n$, there exist $m-n$ linearly independent vectors $h_1, h_2, \ldots, h_{m-n} \in \mathcal{F}$ that are orthogonal to this subspace. In fact, these are the desired vectors orthogonal to $\mathcal{G}$.

To see this, we need only prove that $h_i \perp T^k g_j$ for each $1 \leq i \leq m-n$, each $1 \leq j \leq n$, and each non-negative integer k. Indeed:

$$\langle h_i, T^n g_j\rangle = \langle T^{*k} h_i, g_j\rangle = \lambda^k \langle h_i, g_j\rangle = \lambda^k \langle Ph_i, g_j\rangle = \lambda^k \langle h_i, Pg_j\rangle = 0,$$

which completes the proof. □

We now finish the proof of the Linear-Fractional Cyclicity Theorem.

Theorem 2.8 *Suppose φ is a linear-fractional self-map of the unit disc which fixes both an interior point and a boundary point of the disc. Then C_φ is not cyclic; in fact the closed linear span of any orbit has infinite codimension in H^2.*

PROOF. If φ is the identity function on U, then C_φ is the identity operator on H^2, so in this case the result holds trivially. Suppose now that φ not the identity function. We will show that C_φ^* has an eigenvalue of infinite multiplicity; the desired result will then follow from Proposition 2.7. We may without loss of generality assume that the fixed points of φ are located at 0 and 1. In this case the change of variable $w = (1+z)/(1-z)$ converts φ into a linear-fractional map of the right half-plane that fixes 1 and ∞, and therefore has the form $w \mapsto sw + (1-s)$ for some $0 < s < 1$. Pulling this mapping back to the unit disc, we obtain:

$$\varphi(z) = \frac{sz}{1-(1-s)z} \qquad (z \in U).$$

We need a representation for C_φ^*. Cowen [17] has computed the adjoint of any linear-fractional composition operator, and we could read the desired result off from his theorem. However, for reasons of variety and completeness, we employ the following direct argument. Let H_0^2 denote the closed subspace consisting of H^2 functions that vanish at the origin. Now the subspace of constant functions is invariant for any composition operator, and since φ fixes the origin, H_0^2 is invariant for C_φ. Since these two subspaces are complementary orthogonal subspaces in H^2, they are also invariant for C_φ^*.

We use a Cauchy integral argument to compute the action of C_φ^* on H_0^2. Fix $f \in H_0^2$, and n a positive integer. Write $\psi(z) = sz + 1 - s$, and set $g(z) = f(z)/z$. Then, resurrecting the notation $u(z) = z$:

$$\begin{aligned}
\langle C_\varphi^* f, u^n\rangle &= \langle f, C_\varphi(u^n)\rangle = \langle f, \varphi^n\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{\varphi(e^{i\theta})}^{\,n} d\theta \\
&= \frac{1}{2\pi}\int_{\partial U} f(\zeta)\left[\frac{s\bar{\zeta}}{1-(1-s)\bar{\zeta}}\right]^n \frac{d\zeta}{i\zeta} \\
&= \frac{s^n}{2\pi i}\int_{\partial U} \frac{g(\zeta)}{[\zeta-(1-s)]^n} d\zeta \\
&= \frac{s^n}{(n-1)!} g^{(n-1)}(1-s) \\
&= \frac{s}{(n-1)!}(g\circ\psi)^{(n-1)}(0).
\end{aligned}$$

Now the last quantity is the $(n-1)^{\text{st}}$ Taylor coefficient in the expansion of $sg \circ \psi$ about the origin, so upon recalling the definition of g in terms of f, and letting M_z and $M_{1/z}$ denote respectively the operator of multiplication by z on H^2, and multiplication by $1/z$ on H_0^2, we continue the calculation as follows:

$$\langle C_\varphi^* f, u^n\rangle = s\langle g\circ\psi, u^{n-1}\rangle = s\langle C_\psi M_{1/z} f, u^{n-1}\rangle = s\langle M_z C_\psi M_{1/z} f, u^n\rangle.$$

Thus we have shown that

$$C_\varphi^*\Big|_{H_0^2} = sM_zC_\psi M_{1/z}.$$

Now eigenfunctions for C_ψ are easy to find. Every function $(1-z)^\lambda$ for $\operatorname{Re}\lambda > -1/2$ lies in H^2 (because of the restriction on λ), and is an eigenfunction for C_ψ corresponding to the eigenvalue s^λ. For each such λ, define $f_\lambda(z) = z(1-z)^\lambda$. Then $f_\lambda \in H_0^2$, and by our formula for the adjoint of C_φ,

$$C_\varphi^* f_\lambda = s^{\lambda+1} f_\lambda.$$

Now suppose λ, with $\operatorname{Re}\lambda > -1/2$, is fixed. For each integer k set

$$\lambda(k) = \lambda + 2\pi i k/\log s.$$

Then one checks easily that the collection of H^2 functions $\{f_{\lambda(k)} : k \in Z\}$ is linearly independent, and

$$C_\varphi^* f_{\lambda(k)} = s^{\lambda(k)+1} f_{\lambda(k)} = s^{\lambda+1} f_{\lambda(k)}.$$

Thus $s^{\lambda+1}$ is an eigenvalue for C_φ^* that has infinite multiplicity, so the desired result follows from Proposition 2.7. □

Remarks on "Extreme Behavior"

A Banach space operator T is called *multicyclic* if there is exists a finite subset of the space for which the smallest T-invariant subspace is the whole space.

The preceding results show that if a linear-fractional composition operator is not cyclic, then it is not multicyclic either; in fact, since every finitely generated invariant subspace of such an operator has infinite codimension, it is very strongly non-multicyclic. Herrero introduced the corresponding hypercyclic idea: an operator T on a Banach space is called *multi-hypercyclic* if there is a finite subset of the space, the union of whose orbits is dense. Herrero [30] conjectures (in the context of Hilbert space) that every multihypercyclic operator is hypercyclic, and Salas [45] has verified this conjecture for bilateral weighted shifts. Our results show that it is also true for linear-fractional composition operators (recall that in every non-hypercyclic case, including that of an interior fixed point, the only functions that can adhere to an orbit are constant functions).

While non-cyclic composition operators are very strongly non-cyclic, the cyclic ones are, in the following sense, very strongly cyclic.

> *Every cyclic linear-fractional composition operator has a dense set of cyclic vectors.*

We have already noted that every hypercyclic operator has a dense G_δ set of (hyper)cyclic vectors. Each cyclic operator also has a G_δ set of cyclic vectors [51, Prop. 40], but this set need not be dense.

Indeed, consider the operator M_z of multiplication by z on H^2. It is cyclic with cyclic vector 1 (a fancy way of saying that the polynomials are dense in H^2); however, since convergence in H^2 implies uniform convergence on compact subsets of U, no cyclic vector can vanish at any point of U, and, by Hurwitz's Theorem, neither can any any norm limit of a sequence of cyclic vectors. Thus the collection of cyclic vectors for M_z is not dense in H^2.

We prove that the situation is different for linear-fractional composition operators, by showing in the next section (Corollary 3.3) that if φ is linear fractional with interior and exterior fixed points, then C_φ has a dense set of cyclic vectors. This leaves only the case where φ is a parabolic nonautomorphism unsettled. This can be handled by transferring the setting to the right half-plane, and showing that the collection of cyclic vectors for the resulting operator (translation by a vector with positive real part) is the dense set

$$\{F \in H^2(\Pi) : \hat{F} \text{ is nonzero a.e. on } (0, \infty)\}.$$

(Here $\hat{F}$ denotes the Fourier transform of the boundary function of F.) We leave the details to the reader.

The composition operators that we have shown to be hypercyclic actually have a very strong form of hypercyclicity. The reason is that (as was also noted in [25]) the proof of the Hypercyclicity Criterion actually gives a much stronger conclusion:

> *If the operator T satisfies the hypotheses of the Hypercyclicity Criterion 1.8 then for any subsequence (n_k) of positive integers, there exists $f \in X$ for which the set $\{T^{n_k} f\}$ is dense in X.*

Let us call operators for which the last conclusion is true *strongly hypercyclic*. Since we used the Hypercyclicity Criterion to establish hypercyclicity, and since the linear-fractional maps that do not satisfy its hypotheses are also not hypercyclic, our work actually shows:

> *Every hypercyclic linear-fractional composition operator is strongly hypercyclic.*

It turns out that our hypercyclic composition operators are actually "chaotic": In addition to having a vector with dense orbit, they have a dense set of periodic points and display "sensitive dependence on initial conditions" (see [26, section 6] for other instances of chaotic behavior in operator theory). On the other hand, not every hypercyclic Hilbert space operator is chaotic (see [31], [13]).

3 Linear-Fractional Models

In this section we describe in more detail our transference method for treating the cyclicity problem for more general composition operators C_φ. In the Introduction we observed that the linear-fractional self-maps of the unit disc fall naturally into four categories, determined by position of, and behavior at, the Denjoy-Wolff point; and that this determines a corresponding classification of arbitrary holomorphic self-maps of U into the following four types (Definition 0.3):

- *Dilation type*, if the Denjoy-Wolff point is in U,
- *Hyperbolic type*, if the Denjoy-Wolff point is on ∂U, and the map has derivative < 1 there.
- *Parabolic-automorphism type*, if the Denjoy-Wolff point is on ∂U, the derivative is $= 1$ there, and some orbit is (hence all orbits are) hyperbolically separated.
- *Parabolic-nonautomorphism type*, if the Denjoy-Wolff point is on ∂U, the derivative is $= 1$ there, and some orbit is (hence all orbits are) not hyperbolically separated.

The Linear-Fractional Model Theorem (Theorem 0.4) lends credence to this classification by guaranteeing that each univalent self-map of U is conformally similar to a linear-fractional self-map of U that has the same type, but is now viewed as acting on a more complicated domain. For maps φ of hyperbolic type we used this to derive a sample transference theorem (Theorem 0.6). Here is the general result that will guide the rest of our work.

Theorem 3.1 (Transference Principle) *Suppose that φ is a univalent self-map of U of either dilation, hyperbolic, or parabolic-automorphism type. Let σ be the intertwining map for φ promised by the Linear-Fractional Model Theorem. Suppose further that the set of polynomials in σ is dense in H^2. Then the cyclic behavior of the linear-fractional composition operator C_ψ transfers to C_φ. More precisely:*

- *If φ is of dilation type, then C_φ is cyclic, but not hypercyclic.*
- *If φ is of hyperbolic type, or parabolic-automorphism type, then C_φ is hypercyclic.*

Remarks. (a) The Transference Principle does not address the issue of whether the cyclic behavior of parabolic nonautomorphisms, established in section 2, transfers to more general self-maps of that type. In section 5 we show this is indeed the case for maps that are regular in the sense of Definition 0.8, and sufficiently smooth at the Denjoy-Wolff point.

(b) We have already proved the Transference Principle when φ is of hyperbolic type (Theorem 0.6). The same argument works for φ of parabolic-automorphism type; in fact it proves the following well-known general fact, which implies transference in both cases:

> *Suppose that A, S, and T are operators on a Hilbert space, with $TA = AS$. Suppose that A has dense range. Then if S is cyclic (resp. hypercyclic), so is T.*

The point is that if the vector f is cyclic (resp. hypercyclic) for S, then Af will be cyclic (resp. hypercyclic) for T provided A has dense range. This result arises in our work with $T = C_\varphi$, $A = C_\sigma$, and $S = C_\psi$, where $(\psi\sigma\ )$ is the appropriate linear-fractional model for φ.

Thus we need only prove the Transference Principle for maps of dilation type. A separate argument is required because in this case the intertwining map σ cannot always be taken to be a self-map of U. Koenigs [35] showed that in the dilation model $(\psi\sigma\ )$ for a map ϕ with interior fixed point, the intertwining map σ is unique up to a constant multiple. Hence, for instance, if $\phi(z) = \sigma^{-1} \circ (\frac{\sigma}{2})$ where σ maps U univalently onto an unbounded domain that is, say, star-like with respect to 0 (to ensure $\frac{\sigma(U)}{2} \subset \sigma(U)$), then no (nonzero) intertwining map in a dilation model for ϕ will even be bounded.

Theorem 3.2 *Suppose that σ maps U univalently onto a domain $G \subset \mathbf{C}$, and that there exists a complex number $\lambda \in U$ such that $\lambda G \subset G$. Suppose further that the polynomials in σ are dense in H^2. Let $\varphi = \sigma^{-1} \circ \lambda\sigma$. Then the composition operator C_φ is cyclic. Furthermore, the collection of cyclic vectors for this operator is dense in H^2.*

Proof. Let (a_n) be a sequence of nonzero complex numbers chosen so that

$$\sum_{n=0}^{\infty} |a_n| \, \|\sigma^n\| < \infty.$$

Then $v \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n \sigma^n$ belongs to H^2, and we claim that v is a cyclic vector for C_φ. To this end, observe that σ^n is an eigenvector for C_φ with corresponding eigenvalue λ^n. Let $f \in H^2$ be arbitrary and suppose that $\langle C_\varphi^k v, f\rangle = 0$ for $k = 0, 1, 2, \ldots$; it suffices to show that f is the zero-vector. We have for every non-negative integer k:

$$\begin{aligned} 0 &= \langle C_\varphi^k v, f\rangle \\ &= \langle \sum_{n=0}^{\infty} a_n \lambda^{nk} \sigma^n, f\rangle \\ &= \sum_{n=0}^{\infty} a_n \langle \sigma^n, f\rangle (\lambda^k)^n. \end{aligned}$$

Hence, if we define $h(z) = \sum_{n=0}^{\infty} a_n \langle \sigma^n, f\rangle z^n$, then $h(\lambda^k) = 0$ for $k = 0, 1, 2, \ldots$. Now h is analytic on U (in fact h is in the disk algebra because its coefficient sequence is absolutely summable), and the sequence (λ^k) has limit point 0; hence, $h \equiv 0$ on U. Because $a_n \neq 0$ for all n, we have $\langle \sigma^n, f\rangle = 0$ for all n. It follows that $f \equiv 0$ because, by hypothesis, the polynomials in σ are dense in H^2. This completes the proof that C_φ is cyclic, and with it, the proof of the Transference Theorem.

Finally, note that the collection of cyclic vectors for C_φ must be dense in H^2 because any polynomial in σ is the limit of cyclic vectors having the same form as v. □

First Applications of Transference

As a first application of Theorem 3.2, we present another proof that a linear-fractional composition operator is cyclic if its inducing map has both an interior and an exterior fixed point. This proof also reveals that the collection of cyclic vectors for such an operator is dense (cf. Theorem 2.6).

Corollary 3.3 *Suppose that φ is a linear-fractional self-map of U with interior and exterior fixed points. Then C_φ is cyclic, and has a dense collection of cyclic vectors.*

Proof. We may assume that φ has the form $\varphi(z) = \frac{z}{az+b}$ with $|b| > 1$ and $\left|\frac{a}{1-b}\right| < 1$. Set $\sigma(z) = \frac{az}{az+b-1}$. Observe that σ is holomorphic on a neighborhood of the closed unit disk, and that

$$\varphi = \sigma^{-1} \circ \frac{1}{b}\sigma.$$

Now, the polynomials in σ are dense in H^2 (this may proved directly, or is a consequence of the fact that $\sigma(U)$ is a Jordan domain—a disk); hence by Theorem 3.2, C_φ is cyclic and has a dense set of cyclic vectors. □

We now apply the Transference Principle to show that the cyclicity problem for C_φ is equivalent to a polynomial approximation problem *for* φ when $\|\varphi\|_\infty < 1$ (where $\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in U\}$). Note that the condition $\|\varphi\|_\infty < 1$ implies, by the Denjoy-Wolff theorem, that φ must have an interior fixed point.

Theorem 3.4 *Suppose that φ is analytic on U and that $\|\varphi\|_\infty < 1$. Then C_φ is cyclic if and only if the polynomials in φ are dense in H^2.*

Proof. We have already seen that density of the polynomials in φ is a necessary condition for cyclicity of C_φ. Our goal is to prove the converse.

Suppose that the set of polynomials in φ is dense in H^2 (or equivalently, that C_φ has dense range). Note that by the proof of Corollary 1.7, φ must be univalent on U (and univalent a.e. on ∂U). Because $\|\varphi\|_\infty < 1$, φ must have an interior fixed point; without loss of generality, we assume $\varphi(0) = 0$. Let $\lambda = \varphi'(0)$ and observe $0 < |\lambda| < 1$, where the first inequality follows from the univalence of φ and the second follows from the Schwarz lemma. The dilation model guarantees the existence of a univalent map $\sigma : U \to \mathbf{C}$ such that

$$\sigma \circ \varphi = \lambda\sigma. \tag{1}$$

Note that the intertwining relationship (1) may be rewritten $\sigma = \frac{1}{\lambda}\sigma \circ \varphi$, and it follows that $\sigma(U)$ is a bounded subset of $\mathbf{C}$ because $\|\varphi\|_\infty < 1$.

Because $\sigma(U)$ is bounded, we may choose a positive integer n large enough so that $\lambda^n\sigma$ maps U into itself. We claim that the range of the composition operator $C_{\lambda^n\sigma}$ is dense in H^2, or equivalently, that the set polynomials in $\lambda^n\sigma$ is dense. Since the set of polynomials in $\lambda^n\sigma$ equals the set of polynomials in σ, this will complete the proof of the theorem.

Because $\sigma(U)$ is an open set containing 0, there is a integer m such that the function υ defined by $\upsilon(z) = \sigma^{-1}(\lambda^m z)$ is a self-map of U. Since composition with υ preserves H^2, we have

$$\begin{aligned}
\text{Range } C_{\lambda^n\sigma} &\supset \{(f \circ \upsilon) \circ (\lambda^n\sigma) : f \in H^2\} \\
&= \{f \circ \sigma^{-1}\lambda^{m+n}\sigma : f \in H^2\} \\
&= \{f \circ \varphi_{m+n} : f \in H^2\} \\
&= \text{Range } C_{\varphi_{m+n}}.
\end{aligned}$$

Now $C_{\varphi_{m+n}}$ has dense range (because C_φ does); thus the range of $C_{\lambda^n\sigma}$ contains a dense set and is therefore dense. $\square$

Remark. Density of the polynomials in φ does not, in general, imply cyclicity of C_φ. To see this, let φ be any linear-fractional self-map of U that fixes both an interior and a boundary point. Then C_φ is not cyclic, by Theorem 2.8. However the polynomials in φ are dense in H^2, since $\varphi(U)$ is a disc (so either Walsh's Theorem or an elementary argument shows that the polynomials in z are dense in $H^2(\varphi(U))$).

By Walsh's Theorem, the polynomials in φ are dense in H^2 whenever $\varphi(U)$ is a Jordan domain, so we have the following:

Corollary 3.5 *If φ maps U univalently onto the interior of Jordan curve lying in U, the C_φ is cyclic.*

One can formulate corollaries similar to the one above with "Jordan domain" being replaced by "Carathéodory domain" or "image of a weak-star generator of H^∞".

Corollary 3.6 *Suppose that $\|\varphi\|_\infty < 1$. If C_φ is cyclic then C_φ has a dense collection of cyclic vectors.*

Proof. If C_φ is cyclic, then the set of polynomials in φ is dense in H^2 (Corollary 1.6). By the proof of Theorem 3.4, density of the set of polynomials in φ in this situation implies density of the set of polynomials in the corresponding σ. Hence, by Theorem 3.2, C_φ must have a dense set of cyclic vectors. $\square$

Corollary 3.7 *Suppose that φ is an analytic self-map of U and that $\|\varphi_n\|_\infty < 1$ for some $n \geq 1$. If the set of polynomials in φ is dense in H^2, then C_φ is cyclic and has a dense set of cyclic vectors.*

Proof. Density of the set of polynomials in φ implies density of the set of polynomials in φ_n. Furthermore, if f is cyclic for C_{φ_n}, then clearly f is cyclic for C_φ. Hence the result follows from Theorem 3.4 and Corollary 3.6. $\square$

Note that the preceding result yields another proof of Corollary 3.3, because if φ is a linear-fractional transformation with interior and exterior fixed points, then $\|\varphi_2\|_\infty < 1$.

Cyclicity and Fixed-Point Position

We conclude this section with the promised example showing how Theorem 2.8 can fail for general self-maps of φ.

Example: The mapping $\sigma(z) = \log(\frac{1+z}{1-z})$, takes U univalently onto the strip $S \stackrel{\text{def}}{=} \{z : -\pi/2 < \text{ Im } z < \pi/2\}$. Hence, the holomorphic function φ defined by

$$\varphi(z) = \sigma^{-1}\left(\frac{\sigma(z)}{2}\right) = \frac{1-\sqrt{1-z^2}}{z} .$$

maps U univalently onto the shaded region of Figure 3, and fixes 0, 1, and -1.

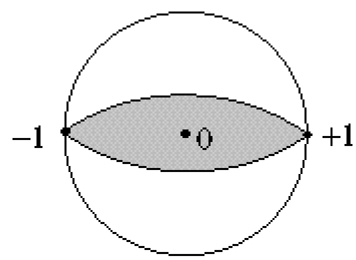

Figure 3: Image of φ.

We claim that C_φ is cyclic. In view of Theorem 3.2 (and Theorem 1.5), to verify this claim we need only show that the polynomials are dense in $H^2(S)$.

Theorem 3.8 *The polynomials are dense in* $H^2(S)$.

Proof. In the following argument, which was shown to us by Carl Sundberg, we work with the inner product on $H^2(S)$ induced by the point $z_0 = 0$. This inner product has integral representation

$$\langle f, g\rangle = \int_{\partial S} f\bar{g}\,d\omega \qquad (f, g \in H^2(S)),$$

where ω is harmonic measure on ∂S representing the origin. The reader may verify that

$$\text{(2)} \qquad \int_{\partial S} f\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(x+i\pi/2)+f(x-i\pi/2)\right]\frac{dx}{\cosh x} .$$

Let $\mathcal{P}$ denote the collection of polynomials in z, and suppose that $f \in H^2(S)$ is orthogonal to $\mathcal{P}$. Then

$$\int_{\partial S} \bar{z}^n f(z) d\omega(z) = 0 \qquad \text{for } n = 0, 1, 2, \ldots. \tag{3}$$

Moreover, by the definition of ω we have for each non-negative integer n,

$$\int_{\partial S} z^n f(z) d\omega(z) = 0. \tag{4}$$

Thus both f and $\bar{f}$ are orthogonal to $\mathcal{P}$; hence the same is true of the real and imaginary parts of f. We see from (2), that (3) and (4) hold if $\bar{z}$ replaces z; hence if $g(z) = f(\bar{z})$, then Re g and Im g are also orthogonal to $\mathcal{P}$. Define $H_1(z) = \operatorname{Re} f + \operatorname{Re} g$; and $H_2(z) = \operatorname{Re} f - \operatorname{Re} g$. Observe that $H_1(\bar{z}) = H_1(z)$ and $H_2(\bar{z}) = -H_2(z)$ a.e. on ∂S and that H_1 and H_2 are orthogonal to $\mathcal{P}$.

Claim: $H_1 = H_2 \equiv 0$. Observe that the claim implies Re $f \equiv 0$ and that Re $f \equiv 0$ implies $f \equiv 0$ (since $f(0) = \langle f, 1 \rangle = 0$). Hence, the proof will be complete once the claim is established. For this we define

$$I_j(\lambda) = \int_{\partial S} e^{\lambda \bar{z}} H_j(z) d\omega(z),$$

and observe that I_j is holomorphic for $|\text{Re } \lambda| < 1/2$. The decay of $d\omega$ justifies differentiation under the integral sign:

$$I_j^{(n)}(0) = \int_{\partial S} \bar{z}^n H_j(z) d\omega(z),$$

from which it follows, because H_j is orthogonal to $\mathcal{P}$, that $I_j \equiv 0$ for $|\text{Re } \lambda| < 1/2$. In particular,

$$0 = I_j(iy) = \int_{\partial S} e^{iy\bar{z}} H_j(z) d\omega(z)$$

for all real numbers y. It follows that the Fourier transforms of $H_j(t + i\pi/2)$ and $H_j(t - i\pi/2)$ vanish a.e. on the real line; thus, $H_j \equiv 0$ and the proof is complete. □

It is possible to use the transference method to construct examples of cyclic composition operators induced by self-maps φ that fix infinitely many points on ∂U. A result of Džrbašyan asserts that if G is a simply-connected domain that omits a sector, and has the further property that the linear measure of its intersection with the circle $\{|z| = r\}$ tends to zero rapidly enough as $r \to \infty$, then the polynomials in z are dense in the Bergman space of G (see [39, Theorem 10.1, page 153]). Bourdon has shown that when the polynomials are dense in the Bergman space of a simply connected domain, they are also dense in the Hardy space of that domain [8, Corollary 3.4]. Thus if we adjoin to the unit disc infinitely many disjoint spikes that go out to ∞ in such a way that the result is starshaped about the origin, and the spikes get thin fast enough to satisfy Džrbašyan's condition, then the polynomials will be dense in $H^2(G)$. Let σ be a Riemann map taking U onto G with $\sigma(0) = 0$ and set $\varphi = \sigma^{-1} \circ (\sigma/2)$. Then φ is a univalent self-map of U of dilation type that fixes infinitely many points on the boundary (one for each spike) and, by transference, C_φ is cyclic on H^2.

4 The Hyperbolic and Parabolic Models

We now turn our attention to the models that apply when a self-map of U has its Denjoy-Wolff point on ∂U. These are the hyperbolic and parabolic cases of the Linear-Fractional Model Theorem, and they have a simpler form if they are reformulated for the setting in which they were actually first proved: the right half-plane $\Pi = \{z : \text{Re } z > 0\}$. As in the Introduction, we accomplish this change of scene by conjugating with the linear-fractional map

$$T(z) = \frac{1+z}{1-z} \, ,$$

which maps U onto Π and takes the point 1 to ∞.

Let φ be a self-map of U that has Denjoy-Wolff point on ∂U so that φ has either a hyperbolic or a parabolic model. Without loss of generality (in terms of the cyclicity problem) we may assume that φ has Denjoy-Wolff point 1, so $\varphi(1) = 1$ and $0 < \varphi'(1) \leq 1$. We denote by Φ the self-map of the right half-plane that corresponds to φ via T:

$$\Phi = (T \circ \varphi \circ T^{-1}).$$

Clearly the sequence of Φ-iterates of any point in Π converges to ∞, so ∞ functions as the half-plane analogue of the Denjoy-Wolff point of Φ. We will also need to transfer to the right half-plane the alternative characterization of the Denjoy-Wolff point in terms of angular limits and derivatives.

For $0 < \alpha < \pi$, let $S_\alpha = \{w : |\arg w| < \alpha/2\}$, where as usual "arg" denotes the principal branch of the argument. Since the image of S_α under T^{-1} is contained in the angular approach region s_α of U and is tangent to s_α at 1 (see Figure 4), we see that angular limits in U correspond to limits in Π through the sectors S_α. Hence we say that a function F defined on Π has *angular limit* L at ∞, and write

$$\angle \lim_{w \to \infty} F(w) = L,$$

provided that given any α with $0 < \alpha < \pi$, $F(w)$ converges to L as w approaches ∞ through S_α. We say that a self-map Φ of Π has *angular*

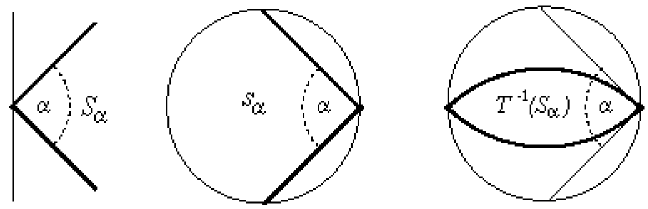

Figure 4: $T^{-1}(S_\alpha) \subset s_\alpha$

derivative Q at ∞ (and write $\Phi'(\infty) = Q$) provided that Φ' has an angular limit Q at ∞.

Transferring information from U to Π via T we have:

- If Φ is a self-map of Π with Denjoy-Wolff point ∞, then Φ has *angular* limit ∞ at ∞ and has angular derivative at ∞ equal to $1/\varphi'(1)$, where $\varphi'(1)$ is the angular derivative of $T^{-1} \circ \Phi \circ T$ at 1. In particular, $\Phi'(\infty) \geq 1$.
- The Julia-Carathéodory theorem shows that $\Phi'(\infty)$ may be calculated as the angular limit at ∞ of $\frac{\Phi(w)}{w}$.

For self-maps of Π with Denjoy-Wolff point ∞, the hyperbolic and parabolic parts of the Linear-Fractional Model Theorem have the following simple forms:

Theorem 4.1 (Right Half-Plane Models) *Suppose Φ maps the right half-plane into itself and has Denjoy-Wolff point at ∞. Let $c = \Phi'(\infty)$, so that $c \geq 1$.*

(a) The Hyperbolic Model: *If $c > 1$, then there exists a nonconstant analytic self-map ν of Π such that*

$$\nu \circ \Phi = c\,\nu. \tag{1}$$

In other words, (Ψ, ν) is a linear-fractional model for Φ, where Ψ is the hyperbolic automorphism of Π given by $\Psi(w) = c\,w$ ([53]).

(b) The Parabolic Models: *If $c = 1$, then there exists a nonzero complex number a with $\operatorname{Re} a \geq 0$, and a nonconstant analytic function ν defined on Π such that*

$$\nu \circ \Phi = \nu + a. \tag{2}$$

Moreover, in equation (2)*: If* $\operatorname{Re} a = 0$ *(the parabolic automorphism model), then* ν *may be taken to be a self-map of* Π [41], [2]*.*

Remarks. (a) When $\operatorname{Re} a \neq 0$ (the parabolic non-automorphism model), we cannot assert that the intertwining map ν may be taken to be a self-map of Π; in fact, we will show that for some maps of parabolic-nonautomorphic type, no "nice" intertwining map ν can be a self-map of Π (see Theorem 4.12, part (c), and the remarks following equation (28).

(b) In both models above, univalence of Φ implies univalence of ν.

Recall from Section 3 that in order to apply the Transference Principle (Theorem 3.1) when the self-map φ of U has linear-fractional model $(\psi\sigma\ )$, we must find conditions on φ that imply that the polynomials in σ are dense in H^2. Rather than work with φ and σ, we will work with their right half-plane equivalents—Φ and ν, then transfer the information obtained back to the disk setting. The Julia-Carathéodory inequality for self-maps of Π plays a crucial role in our analysis.

Theorem 4.2 (Julia-Carathéodory Inequality) *Suppose that* $\Phi : \Pi \to \Pi$ *has Denjoy-Wolff point* ∞ *and that* $\Phi'(\infty) = c$*. Then*

$$\operatorname{Re}\,\Phi(w) \geq c \operatorname{Re}\, w$$

for any $w \in \Pi$*; moreover, if for some* $w_0 \in \Pi$ *equality holds, then* Φ *must be an automorphism of* Π*.*

Proof. To obtain the inequality, translate the disk version (Theorem 1.3 with $p = 1$) to Π via T. If equality holds for some $w_0 \in \Pi$, then the analytic function $w \mapsto \Phi(w) - cw$ mapping Π to $\overline{\Pi}$ takes on a value in $\partial\Pi$ and hence must be a constant function with constant value, say, λ. Note that λ must be pure imaginary ($\lambda = \Phi(w_0) - w_0$); hence $\Phi(w) = cw + \lambda$ is an automorphism of Π. □

This inequality yields the following representation of self-maps of Π.

Theorem 4.3 *Suppose that* Φ *is a self-map of* Π *with Denjoy-Wolff point* ∞ *and that* $\Phi'(\infty) = c$*. Then*

$$\Phi(w) = cw + \rho(w)$$

where ρ *is a self-map of* Π*, and* ρ *has angular derivative* 0 *at infinity.*

Proof. The real part of $\rho(w) = \Phi(w) - cw$ is positive by Theorem 4.2, so ρ is a self-map of Π. Moreover,

$$\rho'(\infty) = \angle \lim_{w\to\infty} \frac{\rho(w)}{w} = \angle \lim_{w\to\infty} \frac{\Phi(w)}{w} - c = 0,$$

which completes the proof. □

To obtain further information about the nature of ρ, we assume that Φ has some smoothness near ∞ (i.e., that the original map φ has some smoothness near its Denjoy-Wolff point). This information will allow us to derive asymptotic representations of the intertwining maps ν in the right half-plane models of Theorem 4.1, and will lead to criteria that distinguish the parabolic automorphic type from the parabolic nonautomorphic type.

Expansions About the Denjoy-Wolff Point

We seek series representations for a holomorphic self-map φ of U about its Denjoy-Wolff point, when that point lies on the boundary. We assume (without loss of generality) that φ has Denjoy-Wolff point 1. By the Julia-Carathéodory theorem (1.2), φ' extends continuously to $\{1\} \cup s_\alpha$, for any $\alpha \in (0, \pi)$, where s_α is the angular approach region with angle α at 1. Hence,

$$\varphi(z) = 1 + \varphi'(1)(z-1) + \gamma(z), \tag{3}$$

where $\gamma(z) = o(|z-1|)$ as $z \to 1$ in s_α. If the expansion (3) holds with $\gamma(z) = o(|z-1|)$ as $z \to 1$ in the full disk U, we say that $\varphi \in C^1(1)$. More generally, if $0 \le \epsilon < 1$, we say that $\varphi \in C^{(n+\epsilon)}(1)$ provided that φ has the expansion

$$\varphi(z) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(1)}{k!}(z-1)^k + \gamma(z), \tag{4}$$

where $\gamma(z) = o(|z-1|^{n+\epsilon})$ as $z \to 1$ in U. It is not difficult to show that $\varphi \in C^{(n)}(1)$ if and only if $\varphi^{(n)}$ extends continuously to $U \cup \{1\}$.

Expansion (4) for φ yields a corresponding expansion about ∞ for its right half-plane analogue $\Phi = T \circ \varphi \circ T^{-1}$. For example, when $\varphi \in C^{1+\epsilon}(1)$ has Denjoy-Wolff point 1, we have

$$1 - \varphi(z) = \varphi'(1)(1-z) - \gamma(z), \tag{5}$$

where $\gamma(z) = o(|1-z|^{1+\epsilon})$. Upon phrasing the action of the map T in terms of the change of variable $w = (1+z)/(1-z)$, we obtain the equations

$$1 - z = \frac{2}{1+w}; \qquad \text{and} \qquad 1 - \varphi(z) = \frac{2}{\Phi(w)+1}$$

which, when substituted into equation (5) above yield

$$\Phi(w) + 1 = \frac{1}{\varphi'(1)} \frac{(w+1)}{1 - \frac{\gamma(z)}{\varphi'(1)} \frac{w+1}{2}}. \tag{6}$$

Because $\gamma(z) = o(\left|\frac{2}{w+1}\right|^{1+\epsilon})$, equation (6) shows that

$$\Phi(w) = \frac{1}{\varphi'(1)}\, w + \Gamma(w),$$

where $\Gamma(w) = o(|w+1|^{1-\epsilon})$ as $w \to \infty$ in Π. Similar manipulations yield the expansions for Φ appearing in Table III below (in the last row of this table we write the expansion of Φ in terms of $w+1$ instead of w in order to avoid needless worry about what is happening near the origin).

Table III

Expansions of Φ at ∞ (φ has Denjoy-Wolff point at 1)

Hypotheses on ϕ ($0 \le \epsilon < 1$)	$\Phi(w) =$	Growth of Γ as $w \to \infty$ in Π
$\phi \in C^{1+\epsilon}(1),\ \phi'(1) = 1/c$	$cw + \Gamma(w)$	$\Gamma(w) = o(\lvert w\rvert^{1-\epsilon})$
$\phi \in C^2(1),\ \phi'(1) = 1,$ $a = \phi''(1)$	$w + a + \Gamma(w)$	$\Gamma(w) = o(1)$
$\phi \in C^{3+\epsilon}(1), \phi'(1) = 1,$ $a = \phi''(1), b = \phi'''(1)$	$w + a + \frac{a^2 - \frac{2}{3}b}{w+1} + \Gamma(w)$	$\Gamma(w) = o\left(\frac{1}{\lvert w\rvert^{1+\epsilon}}\right)$

Remark. The constant $a^2 - \frac{2}{3}b$ that appears in row 3 of the table is a multiple of $(\mathcal{S}\varphi)(1)$, the Schwarzian derivative of φ at 1:

$$\begin{aligned} a^2 - \frac{2}{3}b &= -\frac{2}{3}(\varphi'''(1) - \frac{3}{2}\varphi''(1)^2) \\ &= -\frac{2}{3}\left(\left(\frac{\varphi''}{\varphi'}\right)'(1) - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2(1)\right) \\ &= -\frac{2}{3}(\mathcal{S}\varphi)(1). \end{aligned}$$

Consequences for Parabolic Type

As a first application of the expansions set out in the second and third rows of Table III we briefly discuss the role played by the second derivative in determining the properties of a map of parabolic type (recall Definition 0.3).

Theorem 4.4 *Suppose φ is a holomorphic self-map of U that is of parabolic type, has Denjoy-Wolff point at 1, and that $\varphi \in C^2(1)$. Then:*

(a) $\operatorname{Re}\varphi''(1) \geq 0$.

(b) *If either $\varphi''(1) = 0$ or $\operatorname{Re}\varphi''(1) > 0$, then φ is of nonautomorphic type.*

(c) *Conversely, if $\varphi''(1)$ is nonzero and pure imaginary,* and $\varphi \in C^{3+\epsilon}(1)$, *then φ is of automorphic type.*

Remarks. (a) The last two parts of Theorem 4.4 are not really needed for the sequel if the reader is willing to:

- Consider only maps φ with $C^{3+\epsilon}$-smoothness at the Denjoy-Wolff point, and
- interpret the terms "automorphic type" and "nonautomorphic type" to mean respectively: "$\operatorname{Re}\varphi''(1) = 0$ but $\varphi''(1) \neq 0$", and "$\varphi''(1) = 0$ or $\operatorname{Re}\varphi''(1) > 0$".

(b) The third statement of Theorem 4.4 is *false* for maps with less than C^3-smoothness at the Denjoy-Wolff point. We give examples of this phenomenon in section 6.

Proof. For part (a), suppose for the sake of contradiction that $a = \varphi''(1)$ has *strictly negative* real part. Then according to the information in the second row of Table III, we would have $\operatorname{Re}\Phi(-\frac{a}{2} + iy) < 0$ whenever the magnitude of the real number y is sufficiently large. This contradicts the fact that Φ maps the right half-plane into itself.

For parts (b) and (c) we begin with some elementary observations about the hyperbolic distance on Π (see [48, Chapter 4] for more on this point of view). Given $\epsilon > 0$ and $p \in \Pi$, let $\Delta(p,\epsilon)$ denote the hyperbolic disc of radius ϵ and center p, that is:

$$\Delta(p,\epsilon) = \{w \in \Pi : \varrho(w,p) < \epsilon\},$$

where ϱ is the hyperbolic distance in Π (we will never need to write down a formula for ϱ). The advantage of working in the right half-plane is that for $p, q \in \Pi$, the affine map

$$A_{p,q}(w) = \frac{\operatorname{Re} q}{\operatorname{Re} p}(w - i\operatorname{Im} p) + i\operatorname{Im} q$$

is a conformal automorphism of Π that takes p to q. Thus the disc $\Delta(q,\epsilon)$ is just $A_{p,q}(\Delta(p,\epsilon))$, so it has Euclidean dimensions $\frac{\operatorname{Re} p}{\operatorname{Re} q}$ times those of $\Delta(p,\epsilon)$.

To view this principle in action, fix $w_0 \in \Pi$ and consider the orbit

$$w_n = \Phi_n(w_0) \qquad (n = 1, 2, \ldots).$$

By the Julia-Carathéodory Inequality (Theorem 4.2), $\operatorname{Re} w_{n+1} \geq \operatorname{Re} w_n$, hence the Euclidean dimensions of $\Delta(w_n,\epsilon)$ increase with n. By row 2 of Table III, if $a = \varphi''(1) = 0$, then $|w_{n+1} - w_n| \to 0$, so by our comment on Euclidean dimensions, $w_{n+1} \in \Delta(w_n,\epsilon)$ for all sufficiently large n. Since ϵ is an arbitrary positive number, this implies that $\varrho(w_{n+1}, w_n) \to 0$, so that φ is of nonautomorphic type, as promised.

Continuing to focus on row 2, note that if $\operatorname{Re} a > 0$ then, as $n \to \infty$, we have $\operatorname{Re} w_n \to \infty$, whereas

$$|w_{n+1} - w_n| = |a + \Gamma(w_n)| \to |a|. \tag{7}$$

Thus as $n \to \infty$, the Euclidean distance between w_n and w_{n+1} stays bounded, while the Euclidean dimensions of $\Delta(w_n,\epsilon)$ do not. Thus $w_n + 1 \in \Delta(w_n,\epsilon)$ for all sufficiently large n, hence once again, $\varrho(w_{n+1}, w_n) \to 0$. This completes the proof of part (b).

For part (c), we will show a bit later in this section (Theorem 4.14) that the extra differentiability assumed there insures that $\operatorname{Re} w_n$ is bounded as

$n \to \infty$. Granting this, and writing

$$M = \sup_n \frac{\operatorname{Re} w_n}{\operatorname{Re} w_1} < \infty,$$

we see that the Euclidean dimensions of $\Delta(w_n, \epsilon)$ never get larger than M times those of $\Delta(w_0, \epsilon)$. Now we still have (7) above, which tells us that the orbit (w_n) is separated in the Euclidean metric (since $a \neq 0$). Thus for ϵ small enough, $\Delta(w_n, \epsilon)$ does not contain w_{n+1} for any n, which shows that the orbit (w_n) is hyperbolically separated. □

Recall that if a linear-fractional self-map of U is a parabolic nonautomorphism, then the induced composition operator is not hypercyclic on H^2 (Theorem 2.4). The following Lemma will allow us to adapt the argument that worked in the linear-fractional case to this general setting.

Lemma 4.5 *Suppose that $\varphi \in C^2(1)$ is of parabolic type, with Denjoy-Wolff point at* 1*, and that $\varphi''(1) \neq 0$. Let $z_0 \in U$ be arbitrary. Then the sequence of iterates $(\varphi_n(z_0))$ converges to* 1 *nontangentially if and only if* $\operatorname{Re} \varphi''(1) > 0$*. Moreover, if* $\operatorname{Re} \varphi''(1) > 0$*, then $(\varphi_n(z_0))$ is not a Blaschke sequence.*

Proof. Because φ has Denjoy-Wolff point 1, we know the sequence $(\varphi_n(z_0))$ converges to 1. What we must show is that this sequence is eventually contained in one of the angular approach regions s_α if and only if $\operatorname{Re} \varphi''(1) > 0$. Shifting the scene to the right half-plane, we must show that $\operatorname{Re} \varphi''(1) > 0$ if and only if the orbit of the point $w_0 = T(z_0)$ under $\Phi = T \circ \varphi \circ T^{-1}$ is eventually contained in one of the sectors S_α.

Now suppose for the moment that w is any point of Π. Upon substituting successively $\Phi(w), \Phi_2(w), \ldots$ for w in the expression for Φ in the second row of Table III we obtain

$$\Phi_n(w) = w + na + \sum_{j=0}^{n-1} \Gamma(\Phi_j(w)). \tag{8}$$

Let $w_n = \Phi_n(w_0)$ for $n > 0$. Since $w_n \to \infty$ in Π, we know that $\Gamma(w_n) \to 0$. Therefore,

$$\frac{1}{n} \sum_{j=0}^{n-1} \Gamma(w_j) \to 0 \qquad \text{as} \qquad n \to \infty.$$

This, along with equation (8) above shows that

$$\lim_{n\to\infty} \frac{w_n}{n} = \lim_{n\to\infty} \frac{w_0 + na + \sum_{j=0}^{n-1} \Gamma(w_j}{n} = a. \tag{9}$$

Because n is real and $a \neq 0$, it follows that

$$\arg(w_n) \to \arg a \quad \text{as} \quad n \to \infty,$$

where "arg" denotes the principal value of the argument. Hence, if $\operatorname{Re} a = 0$, then the terms of the sequence (w_n) will eventually lie outside of any of one the sectors S_α. On the other hand, if $\operatorname{Re} a > 0$ then $\{w_n\}$ will eventually lie in S_α for any α larger than $2 \arg a$.

To see that $(\varphi_n(z_0))$ is not a Blaschke sequence when Re $a > 0$; note that the Blaschke condition $\sum(1-|z_n|^2) < \infty$ for sequences in U transforms in the right half-plane to

$$\sum \frac{4 \operatorname{Re} w_n}{|w_n + 1|^2} < \infty, \tag{10}$$

where $w_n = T(z_n)$). Estimate (9) shows that when $\operatorname{Re} a > 0$, the growth of w_n and Re w_n are both comparable to n, hence this condition fails for Φ. □

Theorem 4.6 *Suppose φ is of parabolic type, and has C^2-smoothness at the Denjoy-Wolff point. If $\operatorname{Re} \varphi''$ does not vanish at the Denjoy-Wolff point (so that, by Theorem 4.4 φ is of non-automorphism type) then C_φ is not hypercyclic; in fact, only constant functions may adhere to C_φ-orbits.*

Remark. A proof similar to the one below shows that, even if φ'' vanishes at the Denjoy-Wolff point, C_φ will not be hypercyclic, provided φ has C^3 smoothness there. Our methods do not, however, appear to handle less smooth situations.

Proof. We model the proof on that of Theorem 2.4. We may without loss of generality assume that the Denjoy-Wolff point of φ is at 1. From the series representation (4) of φ, we have

$$\lim_{z \to 1} \frac{1 - \varphi(z)}{1 - z} = 1 \qquad \text{and} \qquad \lim_{z \to 1} \frac{\varphi(z) - z}{(z-1)^2} = \frac{a}{2}, \tag{11}$$

where $a = \varphi''(1)$. Now fix $z \in U$, and for notational convenience, set $z_n = \varphi_n(z)$. Since $z_n \to 1$ non-tangentially as $n \to \infty$, we have for each n,

$$|1 - z_n| \leq \text{const.}\,(1 - |z_n|) \tag{12}$$

where here (and throughout the rest of the argument) the constant is independent of n. Upon substituting z_n for z in the second part of (11) and using (12) we obtain

$$|z_{n+1} - z_n| \leq \text{const.}\,(1 - |z_n|)^2. \tag{13}$$

Similarly, the same substitution in the first part of (11) provides, in conjunction with (12),

$$1 - |z_{n+1}| \geq \text{const.}\,(1 - |z_n|). \tag{14}$$

Now fix $f \in H^2$ and use estimates (13) and (14), together with the functional difference estimate (5) of Section 2 to obtain

$$\begin{aligned} |f(z_{n+1}) - f(z_n)| &\leq \text{const.}\,\frac{|z_{n+1} - z_n|}{(\min\{1 - |z_n|, 1 - |z_{n+1}|\})^{3/2}} \\ &\leq \text{const.}\,\frac{(1 - |z_n|)^2}{(1 - |z_n|)^{3/2}} \\ &= \text{const.}\,(1 - |z_n|)^{1/2} \\ &\to 0 \qquad (n \to \infty). \end{aligned}$$

In other words,

$$\lim_{n\to\infty} [f(\varphi_{n+1}(z)) - f(\varphi_n(z))] = 0$$

for each $z \in U$. Hence, because convergence in H^2 implies pointwise convergence in U, any limit point g of Orb (C_φ, f) must satisfy

$$g(\varphi(z)) = g(z).$$

It follows that g must be constant, since it is constant on the sequence $(\varphi_n(z))$, which, by Theorem 4.5, does not obey the Blaschke condition. Hence, only constant functions may adhere to the C_φ-orbit of f. Since f was arbitrary, C_φ has no hypercyclic vectors. □

Remark. We will show in section 5 that if φ is a regular map (Definition 0.8) of parabolic-nonautomorphism type that has $C^{3+\epsilon}$ smoothness at the Denjoy-Wolff point, then although C_φ is not hypercyclic (as we just showed), it is, nevertheless, cyclic. This generalizes Theorem 2.5, which deals with linear-fractional maps, and it completes the proof of the statements made in the third row of Table II of the Introduction.

In the rest of this section we will generalize the positive hypercyclicity results of Section 2 from the linear-fractional case to the smooth, regular case. We will show that, when φ is regular and of either hyperbolic or parabolic-automorphism type, with some smoothness at the Denjoy-Wolff point (e.g., $C^{1+\epsilon}$ in the hyperbolic case; and $C^{3+\epsilon}$ in the parabolic case), then C_φ will be hypercyclic. We intend to use the Transference Principle; hence, we must obtain information about the intertwining maps σ in the hyperbolic and parabolic models for φ.

The Hyperbolic Case

Our goal is to prove the following result, which is summarized in the first row of Table II of the Introduction.

Theorem 4.7 (Hyperbolic Hypercyclicity) *If φ is a regular self-map of U that is of hyperbolic type and has $C^{1+\epsilon}$-smoothness at its Denjoy-Wolff point, then C_φ is hypercyclic.*

By a result of Kellogg, φ will have the required smoothness if, for example, it takes the unit disc onto a Jordan domain with C^2 boundary (see [27], Theorem 6, page 426). Thus:

Corollary 4.8 *Suppose φ has Denjoy-Wolff point 1 and $\varphi'(1) < 1$. If the boundary of $\varphi(U)$ is a C^2 Jordan curve which intersects ∂U at only the point 1, then C_φ is hypercyclic.*

We remark that since hypercyclicity of C_{φ_n} implies hypercyclicity of C_φ, Theorem 4.7 holds as long as some iterate of φ obeys the hypotheses, even if φ itself does not.

To prepare for the proof of Theorem 4.9, we assume that the Denjoy-Wolff point of φ is 1 and use the linear-fractional map T defined at the beginning of this section to translate the hypotheses on φ into corresponding conditions on $\Phi = T \circ \varphi \circ T^{-1}$. We denote the closed right half-plane $\{w \in \mathbf{C} : \operatorname{Re} w \geq 0\}$ by $\overline{\Pi}$, so $\overline{\Pi} \cup \{\infty\} = T(\overline{U})$. Our conditions on φ then transform into:

(H-1) Φ is continuous on $\overline{\Pi} \cup \{\infty\}$.

(H-2) $\Phi(\overline{\Pi}) \subset \Pi$.

(H-3) Φ has the following representation on $\overline{\Pi}$:

$$\Phi(w) = cw + \Gamma(w),$$

where $c = 1/\varphi'(1) > 1$, and Γ is a self-map of Π satisfying

$$|\Gamma(w)| \leq M(|w|^{1-\epsilon}) \qquad (w \in \overline{\Pi})$$

for some constant M, independent of w.

(H-4) Φ is univalent on $\overline{\Pi} \cup \{\infty\}$.

That the expansion (H-3) holds on Π was noted earlier (see the first row of Table III); that it holds on the closure on Π follows quickly from the continuity of Φ on $\overline{\Pi}$.

Theorem 4.9 (Hyperbolic Model) *Suppose that Φ is an analytic self-map of Π satisfying hypotheses* (H-1) *through* (H-3) *above. Then:*

(a) *Φ has a linear-fractional model (Ψ, ν) in which $\Psi(w) = cw$ (a hyperbolic automorphism of Π), and ν is an analytic self-map of Π that is continuous on $\overline{\Pi}$, and has the representation*

$$\nu(w) = w + \Lambda(w) \qquad w \in \overline{\Pi}, \tag{15}$$

where $\Lambda(w) = O(|w|^{1-\epsilon})$.

(b) *If, in addition, Φ is univalent on $\overline{\Pi}$ then $\nu(\overline{\Pi} \cup \{\infty\})$ is a Jordan subregion of the Riemann Sphere; in particular, $y \mapsto \nu(iy)$, y real, is a Jordan curve in Π such that $\operatorname{Im} \nu(iy) \to \infty$ as $y \to \infty$ and $\operatorname{Im} \nu(iy) \to -\infty$ as $y \to -\infty$.*

We claim that the preceding theorem will yield the Hyperbolic Hypercyclicity Theorem 4.7. For if φ satisfies the hypotheses of Theorem 4.7, then Theorem 4.9 guarantees the existence of an intertwining map ν for Φ such that the boundary of $\nu(\overline{\Pi} \cup \{\infty\})$ is a Jordan curve in $\overline{\Pi} \cup \{\infty\}$. It follows that $\sigma = T^{-1} \circ \nu \circ T$ maps ∂U onto a Jordan curve, hence the polynomials in σ are dense in H^2. Because σ intertwines φ and the hyperbolic automorphism $\psi = T^{-1}(cT)$, an application of Theorem 3.1 (the Transference Principle) establishes Theorem 4.7.

We are going to obtain the intertwining map ν in the Hyperbolic Model Theorem as a limit of a sequence of normalized iterates of Φ; specifically, we will show that the limit

$$\nu(w) = \lim_{n\to\infty} \frac{\Phi_n(w)}{c^n}. \tag{16}$$

exists uniformly on compact subsets of $\overline{\Pi}$. Having done this, it follows easily that ν is the desired intertwining map for Φ and Ψ. Indeed, for each $w \in \overline{\Pi}$,

$$\nu(\Phi(w)) = c \lim_{n\to\infty} \frac{\Phi_{n+1}(w)}{c^{n+1}} = c\,\nu(w).$$

Our realization of ν as a limit of normalized iterates is entirely analogous to the one used by Koenigs to obtain the intertwining map in the interior fixed point case. In the present case the result is due to Valiron [53], who assumed minimal hypotheses, and obtained ν as the limit of $\Phi_n(w)/|\Phi_n(w_0)|$, where w_0 denotes any fixed point of Π. Our stronger hypotheses (H-1) through (H-3) on Φ allow us to obtain ν by a less complicated argument, and more important, they allow us to obtain the expansion (15) for ν, from which we will deduce the extra regularity demanded by our methods. (We remark that for general self-maps, the sequence $\frac{\Phi_n}{c^n}$ need not converge; see the Lemma on p. 121 of [53].)

Our analysis of ν requires two preliminary lemmas, the first of which is obtained quickly by induction.

Lemma 4.10 *Suppose Φ has the expansion given in* (H-3)*. Then for any $w \in \overline{\Pi}$ and any positive integer n:*

$$\Phi_n(w) = c^n w + \sum_{j=0}^{n-1} c^{n-1-j}\Gamma(\Phi_j(w)), \tag{17}$$

and

$$|\Phi_n(w)| \leq |w| \prod_{k=1}^{n} \left(c + \frac{M}{|\Phi_k(w)|^{\epsilon}} \right). \tag{18}$$

Lemma 4.11 *If Φ satisfies hypotheses* (H-1) *through* (H-3)*, then there exist constants $t > 1$ and $\delta > 0$ such that for each $w \in \overline{\Pi}$ and each positive integer n,*

$$|\Phi_n(w)| \geq \delta\, t^n. \tag{19}$$

Proof. From expansion (H-3) and the triangle inequality we obtain

$$|\Phi(w)| \geq |w| \left(c - \frac{M}{|w|^{\epsilon}} \right). \tag{20}$$

Let t be any number satisfying $c > t > 1$. Equation (20) shows that there is an $R > 0$ such that if $|w| > R$ then $|\Phi(w)| \geq t|w|$. This inequality may be iterated to obtain the following inequality: For $|w| > R$

$$|\Phi_n(w)| \geq t^n |w| > t^n R.$$

We must obtain a similar inequality on the compact subset K of $\overline{\Pi}$ consisting of those points $w \in \overline{\Pi}$ satisfying $|w| \leq R$. Observe that the positivity and continuity of $\operatorname{Re}\Phi$ on K imply that $\alpha = \min\{\operatorname{Re}\Phi(w) : w \in K\} > 0$. Also observe that because $\operatorname{Re}\Gamma \geq 0$, it follows from equation (17) that

$$\operatorname{Re}\Phi_{n-1}(w) \geq c^{n-1}\operatorname{Re} w \tag{21}$$

for any w in $\overline{\Pi}$.

Combining these observations, we have for each $w \in K$

$$\begin{aligned} |\Phi_n(w)| &\geq \operatorname{Re}\Phi_n(w) \\ &\geq c^{n-1}\operatorname{Re}\Phi(w) \\ &\geq c^{n-1}\alpha, \end{aligned}$$

where the second inequality is obtained by replacing w with $\Phi(w)$ in inequality (21). We obtain the desired conclusion upon setting $\delta = \min\{R, \alpha/c\}$. □

Proof of Theorem 4.9. By equation (17),

$$c^{-n}\Phi_n(w) = w + \sum_{j=1}^{n} c^{-j}\Gamma(\Phi_{j-1}(w)).$$

Using first the growth estimate on Γ supplied by (H-3), and then inequality (18), we obtain:

$$\begin{aligned} \sum_{j=1}^{\infty} c^{-j}|\Gamma(\Phi_{j-1}(w))| &\leq M\sum_{j=1}^{\infty} c^{-j}|\Phi_{j-1}(w)|^{1-\epsilon} \\ &\leq M|w|^{1-\epsilon}\sum_{j=1}^{\infty} c^{-j}\prod_{k=1}^{j-1}\left(c + \frac{M}{|\Phi_k(w)|^{\epsilon}}\right)^{1-\epsilon} \\ &= \frac{M}{c^{1-\epsilon}}|w|^{1-\epsilon}\sum_{j=1}^{\infty} c^{-\epsilon j}\prod_{k=1}^{j-1}\left(1 + \frac{M}{c|\Phi_k(w)|^{\epsilon}}\right)^{1-\epsilon}. \end{aligned}$$

The sequence of products appearing in the numerator of the summands on the last line is uniformly bounded on $\overline{\Pi}$ because

$$\sum_{k=1}^{\infty} \frac{M}{c|\Phi_k(w)|^{\epsilon}} \leq \frac{M}{c\,\delta^{\epsilon}}\sum_{k=1}^{\infty} t^{-\epsilon k},$$

where $t > 1$ is the constant appearing in inequality (19). It follows that the series

$$\sum_{j=1}^{\infty} c^{-j}\Gamma(\Phi_{j-1}(w))$$

converges uniformly on compact subsets of $\overline{\Pi}$ to a function Λ, with

$$|\Lambda(w)| \leq \text{const.}\, |w|^{1-\epsilon} (w \in \overline{\Pi}) \tag{22}$$

Since Γ is continuous on $\overline{\Pi}$, so is Λ; in addition, since Γ is a self-map of Π and $c > 0$, it follows that Λ must also be a self-map of Π.

Thus the sequence of functions

$$c^{-n}\Phi_n(w) = w + \sum_{j=1}^{n} c^{-j}\Gamma(\Phi_{j-1}(w))$$

converges uniformly on compact subsets of $\overline{\Pi}$, and its limit function ν is a continuous self-map of $\overline{\Pi}$ having representation

$$\nu(w) = w + \Lambda(w). \tag{23}$$

We have already noted that ν, thus defined, satisfies the functional equation (1). Moreover we have just seen that Λ has the desired growth restriction, so part (a) of our theorem is proved. We now address the assertions of part (b).

Suppose that Φ is univalent on $\overline{\Pi}$. Since the sequence $(c^{-n}\Phi_n)$ converges to ν uniformly on compact subsets of Π, Hurwitz's theorem shows that ν must be univalent on Π. Of course this argument requires one to check that ν is not constant, but this is obvious from the fact that $\nu(w)/w \to 1$ as $w \to \infty$ (equation (23) and inequality (22)). Hypothesis (H-2) and the intertwining formula (1) now show that ν is univalent on $\overline{\Pi}$.

To see that $\nu(\overline{\Pi} \cup \{\infty\})$ is a Jordan subregion of the Riemann sphere note that the representation (23) of ν yields for each real number y,

$$\operatorname{Im}\nu(iy) = y\left(1 + \frac{\operatorname{Im}\Lambda(iy)}{y}\right).$$

This, along with estimate (22) shows that $\operatorname{Im}\nu(iy)$ converges to ∞ as $y \to \infty$, and to $-\infty$ as $y \to -\infty$. □

The Parabolic Case

Suppose that $\varphi \in C^{3+\epsilon}(1)$ is a regular self-map of U that is of parabolic type. We may assume without loss of generality that the Denjoy-Wolff point of φ is 1, so by definition, $\varphi'(1) = 1$. We know from the Linear-Fractional Model Theorem that φ has a linear-fractional model $(\psi\sigma\ \ )$, where ψ is a parabolic self-map of U. Our goal here is to re-derive this fact in such a way that we can make statements about the regularity of the intertwining map σ. As in the hyperbolic case, we will obtain this information by transferring the setting to the right half-plane, and deriving an asymptotic expansion for the corresponding intertwining map ν. However this time the expansion will be much more difficult to obtain than the one for the hyperbolic case.

Our hypotheses on φ translate into the following conditions on its right half-plane incarnation Φ.

(P-1) Φ is continuous on $\overline{\Pi} \cup \{\infty\}$.

(P-2) $\Phi(\overline{\Pi}) \subset \Pi$.

(P-3) Φ has the following representation on $\overline{\Pi}$:

$$\Phi(w) = w + a + \frac{b}{w+1} + \Gamma(w+1),$$

where $a = \varphi''(1)$ is non-zero and has non-negative real part, and $b = \varphi''(1)^2 - (2/3)\varphi'''(1)$, and for some positive constants C and ϵ,

$$|\Gamma(w+1)| \leq \frac{C}{|w+1|^{1+\epsilon}}$$

for all $w \in \overline{\Pi}$.

(P-4) Φ is univalent on $\overline{\Pi} \cup \{\infty\}$.

The expansion (P-3) was obtained from the third row of Table III (the continuity of Φ on $\overline{\Pi}$ allowing the extension of this expansion to Π). The following theorem shows that the constant $a = \varphi''(1)$ in that formula controls which parabolic model applies: φ has a parabolic automorphism model if $\operatorname{Re} a = 0$ (and $a \neq 0$), and a parabolic nonautomorphism model when $\operatorname{Re} a > 0$.

Theorem 4.12 (Parabolic Models) *Suppose that Φ is an analytic self-map of Π satisfying hypotheses (P-1) through (P-3) above. Then Φ has linear-fractional model (Ψ, ν) in which $\Psi(w) = w + a$ (a parabolic self-map of Π) and ν is a nonconstant analytic map on Π having the following properties.*

(a) *If* $\operatorname{Re} a > 0$ *in* (P-3), *then*

$$\nu(w) = w - \frac{b}{a}\log(1+w) + B(w) \qquad (w \in \overline{\Pi}), \tag{24}$$

where B *is a bounded holomorphic function on* Π *that is continuous on* $\overline{\Pi}$.

(b) *If* $\operatorname{Re} a = 0$ *then then representation (24) holds for all points in* H, *where* H *is the upper half of* $\overline{\Pi}$ *when Im* $a > 0$ *and the lower half when Im* $a < 0$, *and where* B *is bounded and continuous on* H, *holomorphic on its interior.*

(c) *The real part of* ν *is bounded below on* $\overline{\Pi}$ *if and only if* $\operatorname{Re} \overline{a}b \leq 0$.

(d) *If* Φ *is univalent on* $\overline{\Pi}$, *then* $\nu(\overline{\Pi} \cup \{\infty\})$ *is a Jordan subregion of the Riemann Sphere; in particular, the curve* $y \mapsto \nu(iy)$ (y real) is a Jordan arc in Π such that $\operatorname{Im} \nu(iy) \to \infty$ as $y \to \infty$ and $\operatorname{Im} \nu(iy) \to -\infty$ as $y \to -\infty$.

Remarks. (i) In case $\operatorname{Re} a = 0$, we will assume henceforth, for purposes of normalization, that $\operatorname{Im} a > 0$. Since our standing assumption is that $a \neq 0$, this entails no loss of generality.

(ii) Just as in the hyperbolic case, the intertwining map ν will emerge as a limit of a sequence of normalized iterates of Φ, and Hurwitz's theorem will show that if Φ is univalent then so is ν. Hence the real issue in part (d) is behavior of $\nu(iy)$ as $|y| \to \infty$. Note that if $\operatorname{Re} a > 0$ then upon substituting $w = iy$ ($y \in \mathbf{R}$) in (24) and taking imaginary parts, we obtain

$$\operatorname{Im} \nu(iy) = y + o(|y|) \qquad (|y| \to \infty),$$

from which follows (d). If $\operatorname{Re} a = 0$, then the same argument gives $\operatorname{Im} \nu(iy) \to \infty$ as $y \to \infty$. Showing $\operatorname{Im} \nu(iy) \to -\infty$ as $y \to -\infty$ is harder; this argument occupies the last part of the section.

(iii) Note that for $\operatorname{Re} a > 0$, Part (c) of Theorem 4.12, follows immediately from the representation (24). Indeed, upon taking real parts of both sides of (24) we obtain

$$\operatorname{Re} \nu(w) = \operatorname{Re} w - \frac{\operatorname{Re} \overline{a}b}{|a|^2} \log|1+w| + \text{a bounded function.}$$

Since $\log|1+w|$ is unbounded on $\overline{\Pi}$, the result follows immediately. Corollary 4.26 below establishes part (c) of the theorem for $\operatorname{Re} a = 0$.

(iv) After submitting this monograph for publication, we learned from reading [28, Chapter 1] that our work, in the case $\mathrm{Re}\, a > 0$, overlaps substantially with that of P. Fatou's [24, Chapter 2].

Consequences of The Parabolic Models Theorem

Before proving Theorem 4.12, we point out some of its consequences. Note that in part (c), $\mathrm{Re}\,\bar{a}b$ is just the real dot product of the plane vectors a and b. So part (c) can be restated:

> *The real part of ν is not bounded below on $\overline{\Pi}$ if and only if b makes an acute angle with a.*

The figure below illustrates the utility of this theorem: it is based on a computer generated image of the intertwining map ν for $\Phi(w) = w + 1 + 1/(w+1)$. Although it is not obvious from the numerical calculation, the fact that $\mathrm{Re}\,\bar{a}b = 1 > 0$ assures us that the real part of ν is not bounded below.

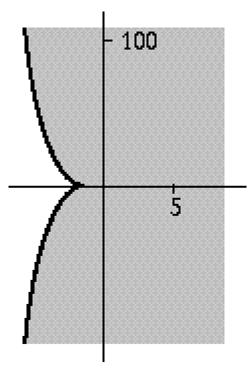

Figure 5: *The intertwining map ν for $\Phi(w) = w + 1 + 1/(w+1)$.*

The significance of (c) of Theorem 4.12 is that if $\mathrm{Re}\,\nu$ is bounded below by, say, the real number α, then $(\Psi, \nu - \alpha)$ is a parabolic model for Φ in which the intertwining map $\nu - \alpha$ is now a *self-map* of Π (a condition which, back in the unit disc, was vital to our transference technique).

The following proposition and (c) show that, under the assumptions of Theorem 4.12, the intertwining map ν *is* always bounded below in the automorphism case of the model, i.e., the case $\mathrm{Re}\, a = 0$.

Proposition 4.13 *If* $\mathrm{Re}\, a = 0$ *in the expansion* (P-3)*, then* $b \geq 0$*.*

Proof. Upon setting $w = iy$, with y real, in the expansion (P-3) and taking real parts on both sides, we obtain

$$y \operatorname{Re} \Phi(iy) = \frac{y \operatorname{Re} b + y^2 \operatorname{Im} b}{1 + y^2} + y\Gamma(iy + 1), \tag{25}$$

hence

$$\operatorname{Im} b = \lim_{|y| \to \infty} y \operatorname{Re} \Phi(iy).$$

But $\operatorname{Re} \Phi(iy) > 0$ by condition (P-2), so $y \operatorname{Re} \Phi(iy)$ has the sign of y. Letting y tend separately to $+\infty$ and $-\infty$ we see that $\operatorname{Im} b$ is respectively non-negative and non-positive, hence zero. Thus b is real.

To see that $b \geq 0$, use the information above in (25), multiply both sides of the resulting equation by y, and let $y \to \infty$. The result is:

$$b = \lim_{y \to \infty} y^2 \operatorname{Re} \Phi(iy),$$

from which it follows that $b \geq 0$. □

We can now interject a result that finishes the proof of Theorem 4.4.

Lemma 4.14 *Let $\varphi \in C^{3+\epsilon}(1)$ be a self-map of U of parabolic type, with Denjoy-Wolff point at 1. Suppose further that $\varphi''(1)$ is pure imaginary and $\neq 0$. Let Φ be the corresponding map of the right half-plane. Then the real part of every Φ-orbit is bounded.*

Proof. We are assuming that $\varphi''(1) = i\alpha$ for some real nonzero α. Then by Table III, Φ has the following expansion on $\overline{\Pi}$:

$$\Phi(w) = w + i\alpha + \frac{b}{w + 1} + \Gamma(w + 1)$$

where $b > 0$ (by Theorem 4.13) and

$$|\Gamma(w + 1)| \leq \frac{C}{|w + 1|^\tau} \ .$$

for some $\tau > 1$. Let $w_0 \in \Pi$ be arbitrary and for $n > 0$ set $w_n = \Phi_n(w_0)$. We show that $\sup_n \operatorname{Re} w_n < \infty$, and thus the sequence (w_n) is separated in pseudohyperbolic metric on Π. Note that

$$w_n = w + ni\alpha + \sum_{j=0}^{n-1} \frac{b}{w_j + 1} + \sum_{j=0}^{n-1} \Gamma(w_j + 1).$$

Hence,

$$\operatorname{Re} w_n \le \operatorname{Re} w + \sum_{j=0}^{n-1} \frac{b}{|w_j+1|} + \sum_{j=0}^{n-1} \frac{C}{|w_j+1|^{\tau}}. \tag{26}$$

Now because

$$\frac{w_n}{n} \to i\alpha,$$

we have

$$\begin{aligned} \operatorname{Re} w_n &\le \operatorname{Re} w + C_1 \sum_{j=0}^{n-1} \frac{b}{j+1} + C_2 \sum_{j=0}^{n-1} \frac{1}{(j+1)^{\tau}} \\ &\le C_1 + C_2 \log(n+1) \qquad (n \ge 1). \end{aligned}$$

Using this preliminary estimate and the fact that $|w_n| \approx n$, we can show that $\operatorname{Re} w_n$ is bounded:

$$\begin{aligned} \operatorname{Re} w_n &= \operatorname{Re} w + \operatorname{Re} \sum_{j=0}^{n-1} \frac{b}{w_j+1} + \sum_{j=0}^{n-1} \Gamma(w_j+1) \\ &\le \operatorname{Re} w + \sum_{j=0}^{n-1} \frac{b \operatorname{Re} w_j + b}{|w_j+1|^2} + \sum_{j=0}^{n-1} \frac{C}{|w_j+1|^{\tau}} \\ &\le \operatorname{Re} w + b \sum_{j=0}^{n-1} \frac{C_1 + C_2 \log(j+1)}{(j+1)^2} + \sum_{j=0}^{n-1} \frac{C}{(j+1)^{\tau}}. \end{aligned}$$

Since both sums on the right converge as $n \to \infty$, we see that $\operatorname{Re} w_n$ is bounded. □

Remark. The proof above shows that, under the hypotheses of Lemma 4.14, for each $w \in \overline{\Pi}$ there is a non-negative constant C such that

$$\operatorname{Re} \Phi_n(w) \le \operatorname{Re} w + C,$$

where C generally depends on w. This dependence of C on w will be described more explicitly in Propositions 4.25 and 4.29.

Lemma 4.14, in addition to finishing the proof of Theorem 4.4, provides additional information about the classification of parabolic-type maps that are very smooth at the Denjoy-Wolff point.

Theorem 4.15 *Suppose φ is of parabolic type, with Denjoy-Wolff point at 1. Suppose further that $\varphi \in C^{3+\epsilon}(1)$, and $\varphi''(1) \neq 0$. Then the following are equivalent:*

(a) *φ is of automorphism type (every φ-orbit is hyperbolically separated).*

(b) *Every φ-orbit converges to 1 tangentially.*

(c) *Every φ-orbit converges to 1 and lies outside some circle in U that is tangent to ∂U at 1.*

(d) *$\varphi''(1)$ is pure imaginary.*

Proof. Lemma 4.5 shows that (b) $\leftrightarrow$ (d), even under the weaker assumption $\varphi \in C^2(1)$. The second and third parts of Theorem 4.4 show that (a) $\leftrightarrow$ (d), while Lemma 4.14, with its conclusion interpreted for the unit disc, asserts that (d) $\rightarrow$ (c). Clearly (c) $\rightarrow$ (b), and this (somewhat redundantly) completes the circle. □

We return to our main theme. Recall that $b = -(2/3)(\mathcal{S}\varphi)(1)$, where $\mathcal{S}\varphi$ is the Schwarzian derivative of the original map φ, and 1 is the Denjoy-Wolff point of φ. Thus when translated back to the unit disk, Proposition 4.13 and the comments preceding it show that if $\varphi''(1)$ is pure imaginary (and nonzero), then

- $(\mathcal{S}\varphi)(1) \leq 0$, and
- φ has a parabolic model (ψ/σ) in which σ is a self-map of U.

Recall that when σ is self-map, we may apply the Transference Principle to obtain information about cyclicity:

Theorem 4.16 *Suppose that φ is a regular self-map of U that is of parabolic type, has Denjoy-Wolff point at 1, and has $C^{3+\epsilon}$-smoothness at 1. If $\varphi''(1)$ is pure imaginary (and nonzero), then C_φ is hypercyclic.*

Proof. We know that Φ, the right half-plane *alter ego* of φ, satisfies (P-1) through (P-4) so that Theorem 4.12 applies, and shows that Φ has a parabolic model (Ψ, ν) in which $\Psi(w) = w + a$, where $a = \varphi''(1)$. Moreover, by part (4) of the theorem, $\nu(\overline{\Pi} \cup \{\infty\})$ is a Jordan domain.

Suppose now that $\varphi''(1)$ is pure imaginary so that $\operatorname{Re} a = 0$. Then in the model for Φ, the map Ψ is a parabolic automorphism, and $\operatorname{Re} \nu$ is bounded below (by Proposition 4.13 and part (3) of Theorem 4.12). Choose a real number α such that $\nu+\alpha$ is a self-map of Π. Of course, $\nu+\alpha$ will still be a Jordan map and will still intertwine Φ and Ψ ; hence, $\sigma = T^{-1}\circ(\nu+\alpha)\circ T$ is a Jordan self-map of U that intertwines φ and the parabolic automorphism $T^{-1}\circ\Psi\circ T$. The Transference Principle now yields the result. □

If $\operatorname{Re}\varphi''(1) > 0$ then we have seen that C_φ is not hypercyclic (Theorem 4.6) and we have asserted that C_φ is nevertheless cyclic. In those situations where the intertwining map is a self-map of Π, this cyclicity follows from the Transference Principle by an argument just like the one above:

Theorem 4.17 *Suppose that φ is a regular self-map of U that is of parabolic type, has Denjoy-Wolff point at 1, and has $C^{3+\epsilon}$-smoothness at 1. Suppose further that $\operatorname{Re}\varphi''(1) > 0$ and $\operatorname{Re}\overline{\varphi''(1)}(\mathcal{S}\varphi)(1) \geq 0$. Then C_φ is cyclic.*

Proof. We know φ has a parabolic nonautomorphism model $(\psi\sigma\ )$ in which σ is a Jordan map. Moreover, the hypothesis $\operatorname{Re}\overline{\varphi''(1)}(\mathcal{S}\varphi)(1) \geq 0$ tells us that σ may be chosen to be a self-map of U. The cyclicity of C_ψ (Theorem 2.5) now easily transfers to C_φ: If f is a cyclic vector for C_ψ, the $f\circ\sigma$ is a cyclic vector for C_φ. □

In fact C_φ is cyclic when $\operatorname{Re}\varphi''(1) > 0$, even if $\operatorname{Re}\overline{\varphi''(1)}(\mathcal{S}\varphi)(1) < 0$. However this requires more work, and we postpone the proof to the following section (see Theorem 5.2). We now begin the task of proving the Parabolic Models Theorem.

Motivation for the Proof

We will obtain the intertwining map ν in Theorem 4.12 as a limit of a sequence of normalized iterates of Φ. Here is some motivation for our choice of normalization. Suppose that ν intertwines Φ and the translation $\Psi(w) = w+a$, and that ν is nice—say $\nu'(w)$ has nonzero limit c as $w\to\infty$ in Π (so in particular, c is the angular derivative of ν at ∞). We have for any $w\in\Pi$

$$\nu\circ\Phi_n(w) = \nu(w) + na.$$

Taking derivatives of both sides yields

$$\nu'(\Phi_n(w))\Phi_n'(w) = \nu'(w). \tag{27}$$

Now $\Phi_n(w)$ approaches ∞; hence letting $n\to\infty$ in 27 yields

$$\nu'(w) = c\lim_{n\to\infty}\Phi_n'(w).$$

Integrating, we see that $\nu(w)$ is the limit of $c\Phi_n(w) + t_n$ for some sequence of constants (t_n). Fix $\omega_0 \in \Pi$ and note that we may take t_n to be $\nu(w_0) - c\Phi_n(w_0)$. Hence,

$$\nu = c\left\{\lim_{n\to\infty}(\Phi_n - \Phi_n(w_0))\right\} + \nu(w_0). \tag{28}$$

The preceding equation also shows that ν is unique up to an additive constant—provided that we restrict our attention to maps whose derivatives extend continuously to ∞ with nonzero derivative at ∞. To see this, suppose υ is another such map intertwining Φ and $\Psi(w) = w + a$. Then

$$\begin{aligned}\upsilon(w) &= \upsilon'(\infty)\left\{\lim_{n\to\infty}(\Phi_n - \Phi_n(w_0))\right\} + \upsilon(w_0)\\ &= \frac{\upsilon'(\infty)}{c}\nu + \beta,\end{aligned}$$

where β is the constant $\upsilon(w_0) - \nu(w_0)\upsilon'(\infty)/c$. Hence, υ intertwines Φ and the translation mapping $w \mapsto w + \frac{\upsilon'(\infty)}{c}a$. But our assumption was that it intertwined Φ and $w \mapsto w + a$, thus $\frac{\upsilon'(\infty)}{c}$ must be 1, so that $\upsilon = \nu + \beta$.

Another consequence of (28) is that if we assume there is "nice" intertwining map, then the sequence

$$\Phi_n - \Phi_n(w_0) \tag{29}$$

must converge to a map that intertwines Φ and a translation. We will show that the sequence (29) *does* converge when Φ satisfies (P-1) through (P-3); moreover, it converges to the intertwining map ν advertised in the Parabolic Models Theorem.

Given this motivation, we now proceed more formally. Here, and in the remainder of this section, we will be assuming that Φ is a self-map of Π satisfying (P-1) through (P-3). Fix a "base point" point $w_0 \in \Pi$ and set, for each non-negative integer n,

$$\nu_n(w) = \Phi_n(w) - \Phi_n(w_0) \qquad (w \in \overline{\Pi}). \tag{30}$$

(It should be noted that on the left side of this definition, the symbol ν_n does *not* denote "the n^{th} iterate of ν") Each function ν_n is holomorphic on Π and continuous on $\overline{\Pi}$. We are going to:

- Show that the sequence (ν_n) converges to a function ν that is holomorphic on Π and continuous on $\overline{\Pi}$, and which obeys the functional equation
  $$\nu(\Phi(w)) = \nu(w) + a; \tag{31}$$

- Show that ν has the asymptotic representation (24), and
- Use this representation to study the boundary behavior of ν at ∞.

Two points deserve immediate comment here. First, the functional equation (31) will follow from the definition of ν_n and the hypotheses on Φ as soon as it is shown that $\nu_n \to \nu$ uniformly on compact subsets of $\overline{\Pi}$. Indeed, suppose the convergence has been established. Then

$$\nu_n(\Phi(w)) = \Phi_{n+1}(w) - \Phi_n(w_0) = \nu_{n+1}(w) + \Phi_{n+1}(w_0) - \Phi_n(w_0).$$

We see from representation (P-3) of Φ, and the fact that the Φ-orbit of each point tends to ∞, that $\Phi_{n+1}(w_0) - \Phi_n(w_0) \to a$, so the right hand side of the last displayed equation tends to $\nu(\Phi(w))$ while the left side tends to $\nu(w) + a$.

Observe that the choice of base point w_0 is irrelevant. For suppose we have shown that the sequence (ν_n) converges for a certain w_0. Let w_1 be any other point of $\overline{\Pi}$, and set

$$\tilde{\nu}_n(w) = \Phi_n(w) - \Phi_n(w_1) \qquad (w \in \overline{\Pi}).$$

Then for $w \in \overline{\Pi}$,

$$\nu_n(w) - \tilde{\nu}_n(w) = \Phi_n(w_1) - \Phi_n(w_0) = \nu_n(w_1) \to \nu(w_1). \tag{32}$$

Thus the sequence $(\tilde{\nu}_n)$ converges on $\overline{\Pi}$ to a function $\tilde{\nu}$ that differs from the original function ν by the constant $\nu(w_1)$.

Estimates on Orbit Magnitudes

We now derive the estimates on Φ-orbits that are required for the proof of the Parabolic Models Theorem. These are summarized in the following statement, in which $\overline{\Pi}_R$ denotes the part of the closed right half-plane that lies above the line $y = R$, and n is a non-negative integer. In what follows we use the symbol C (possibly lower case, possibly with subscripts), to denote a positive constant which may vary from one appearance to the next, but which *never* depends on the point w or the index of iteration n.

Proposition 4.18 (Fundamental Orbital Estimates) *If* $\operatorname{Re} a > 0$*, then*

$$c(|w| + n) \leq |\Phi_n(w)| \leq C(|w| + n) \tag{33}$$

for all $w \in \overline{\Pi}$ *and all* n.

If $\operatorname{Re} a = 0$*, then the upper estimate in (33) holds for all* $w \in \overline{\Pi}$*, and there exists* $R > 0$ *such that the lower one holds for all* $w \in \overline{\Pi}_R$.

Eventually we will see that the positive constant R can be taken to be 0.

Proof. The representation (P-3) insures that there is a positive constant R such that

$$|\Phi(w) - w - a| < \frac{|a|}{2} \tag{34}$$

whenever $w \in \overline{\Pi}$ and $|w| > R$. Thus $|\Phi(w) - w| < 3|a|/2$ whenever $|w| > R$. On the other hand, the continuity hypothesis (P-1) shows that $|\Phi(w) - w|$ is bounded on the compact half-disc $D = \{|w| \leq R\} \cap \overline{\Pi}$. Thus $|\Phi(w) - w|$ is bounded on all of $\overline{\Pi}$. Upon substituting $\Phi_j(w)$ for w in this uniform bound, summing, and using the triangle inequality, there results:

$$|\Phi_n(w) - w| < Cn \qquad (w \in \overline{\Pi}, n = 1, 2, \ldots), \tag{35}$$

from which it follows (upon taking $C > 1$ in the last equation) that for all $w \in \overline{\Pi}$,

$$|\Phi_n(w)| < |w| + Cn < C(|w| + n).$$

This proves the upper estimate in inequality (33) for all values of $a \neq 0$.

For the lower estimate we have to consider separate cases.

The case $\operatorname{Re} a > 0$**.** Suppose the constant a in (P-3) has strictly positive real part. The key to the lower estimate in inequality (33) is the fact that the slopes of the vectors $\Phi_n(w) - w$ are uniformly bounded both in n and w:

$$\sup\{\text{slope}[\Phi_n(w) - w] : w \in \overline{\Pi}, n = 1, 2, \ldots\} < \infty. \tag{36}$$

To prove this, take real parts on both sides of (P-3) to obtain a "real" version of inequality (34), from which follows

$$\operatorname{Re}[\Phi(w) - w] > \frac{1}{2} \operatorname{Re} a \qquad (w \in \overline{\Pi}, \quad |w| > R).$$

Also, the Julia-Carathéodory inequality (4.2) and hypothesis (P-2) insure that the real part of $\Phi(w) - w$ is strictly positive in $\overline{\Pi}$. These facts, along with the continuity of Φ, insure that there exists $c > 0$ such that $\operatorname{Re}[\Phi(w) - w] > c$ for all $w \in \overline{\Pi}$. (The constant c here is not the one that occurs in (4.2).) Upon replacing w by $\Phi_j(w)$ in this inequality and summing, we obtain

$$\operatorname{Re}[\Phi_n(w) - w] > cn \qquad (w \in \overline{\Pi}; n = 1, 2, \ldots). \tag{37}$$

This inequality, along with (35) shows that

$$\begin{aligned} \text{slope}[\Phi_n(w) - w] &= \frac{\operatorname{Im}[\Phi_n(w) - w]}{\operatorname{Re}[\Phi_n(w) - w]} \\ &\le \frac{|\Phi_n(w) - w|}{\operatorname{Re}[\Phi_n(w) - w]} \\ &\le \frac{Cn}{cn} = \frac{C}{c}, \end{aligned}$$

which proves (36).

Let μ denote the supremum in inequality (36). This inequality asserts that for each $w \in \overline{\Pi}$ and each positive integer n, the angular sector in $\overline{\Pi}$ with vertex at w and sides of slope $\pm\mu$ contains $\Phi_n(w)$. Thus the horizontal translate of this sector to one with a vertex on the imaginary axis also contains $\Phi_n(w)$, so a little trigonometry based on Figure 6 below, shows that

$$|\Phi_n(w)| \ge r = |\operatorname{Im} w| \sin\beta.$$

where β is the angle between the sides of the translated sector and the imaginary axis $(0 < \beta < \pi/2)$.

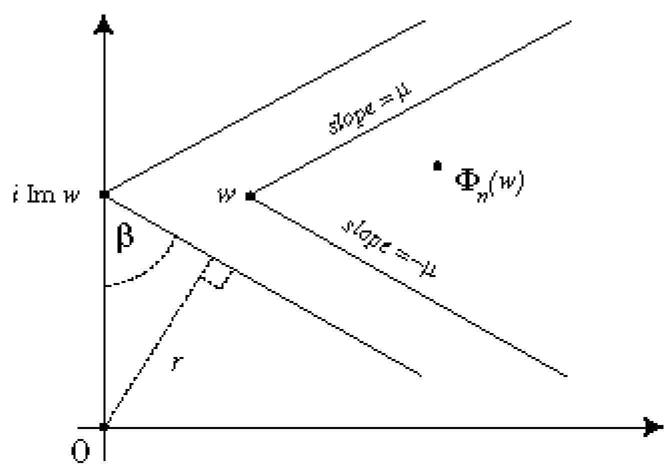

Figure 6: Lower estimate for $\Phi_n(w)$.

Furthermore, the lower estimate (37) yields

$$|\Phi_n(w)| \ge \operatorname{Re}\Phi_n(w) \ge cn + \operatorname{Re} w.$$

Upon averaging the last two inequalities, we obtain

$$|\Phi_n(w)| \geq \frac{1}{2}(cn + |\operatorname{Im} w| \sin\beta + \operatorname{Re} w),$$

which, because $\sin\beta > 0$, becomes the desired lower estimate:

$$|\Phi_n(w)| \geq c(n + |w|) \qquad (w \in \overline{\Pi}; n = 0, 1, 2, \ldots).$$

This completes the proof of the first part of Proposition 4.18.

The case $\operatorname{Re} a = 0$**.** Our standing assumption in this case is that $\operatorname{Im} a > 0$, so in condition (P-3) we have $a = \alpha i$ for some positive constant α. For definiteness, suppose $\alpha > 0$ (henceforth we make this a standing assumption whenever $\operatorname{Re} a = 0$).

Fix $w \in \overline{\Pi}$ with $|w| > R$, so inequality (34) holds, and this implies,

$$\frac{\alpha}{2} < \operatorname{Im}[\Phi(w) - w] < \frac{3\alpha}{2}. \tag{38}$$

Thus the image of each $w \in \overline{\Pi}_R$, has larger imaginary part than does w, so it too lies in $\overline{\Pi}_R$, as does—upon repetition of the argument—the entire orbit of w.

Upon substituting $\Phi_n(w)$ for w in (38) we obtain

$$\frac{\alpha}{2} < \operatorname{Im}\{\Phi_{n+1}(w) - \Phi_n(w)\} < \frac{3\alpha}{2}, \tag{39}$$

while summation of the lower inequality in (39) yields

$$\operatorname{Im}\Phi_n(w) \geq \operatorname{Im} w + n\frac{\alpha}{2}. \tag{40}$$

The Julia-Carathéodory inequality guarantees that

$$\operatorname{Re}\Phi_n(w) \geq \operatorname{Re} w,$$

and upon adding the last two inequalities we obtain the desired result:

$$\begin{aligned} |\Phi_n(w)| &\geq \frac{1}{\sqrt{2}}[\operatorname{Im}\Phi_n(w) + \operatorname{Re}\Phi_n(w)] \\ &\geq \frac{1}{\sqrt{2}}[\operatorname{Im} w + \operatorname{Re} w + \frac{n\alpha}{2}] \\ &\geq \frac{1}{\sqrt{2}}(|w| + \frac{n\alpha}{2}) \\ &\geq c(|w| + n). \end{aligned}$$

This completes the proof of Proposition 4.18. □

Proof of the Parabolic Models Theorem

Existence of ν, Part I. Our primary tools in this enterprise will be the series representation (P-3), and the orbit-magnitude estimate (33) of Proposition 4.18. In fact, these will allow us to prove most of the first two parts of the Parabolic Models Theorem.

We employ the following notation which emphasizes our focus on orbits while removing extraneous symbols. For each $w \in \overline{\Pi}$ write

$$w(n) \stackrel{\text{def}}{=} \Phi_n(w) + 1,$$

and

$$\Delta w(n) \stackrel{\text{def}}{=} w(n+1) - w(n) = \Phi_{n+1}(w) - \Phi_n(w).$$

In this notation (30) becomes

$$\nu_n(w) = w(n) - w_0(n). \tag{41}$$

Moreover, we can write the result of substituting $\Phi_j(w)$ for w in (P-3) as

$$\Delta w(j) = a + \frac{b}{w(j)} + \Gamma(w(j)), \tag{42}$$

from which follows

$$\Delta\nu_j(w) = b\left[\frac{1}{w(j)} - \frac{1}{w_0(j)}\right] + \Gamma(w(j)) - \Gamma(w_0(j)). \tag{43}$$

Now sum both sides of this equation, noting that $\nu_0(w) = w - w_0$, to get for each $w \in \overline{\Pi}$ and each non-negative integer n:

$$\nu_n(w) = w - w_0 + b\sum_{j=0}^{n-1}\left[\frac{1}{w(j)} - \frac{1}{w_0(j)}\right] + \sum_{j=0}^{n-1}\left[\Gamma(w(j)) - \Gamma(w_0(j))\right].$$

Now the general term of the first sum on the right is:

$$\begin{aligned}\frac{1}{w(j)} - \frac{1}{w_0(j)} &= -\frac{\nu_j(w)}{w(j)w_0(j)} \\ &= \frac{-w}{w(j)w_0(j)} + \frac{w_0 - (\nu_j(w) - \nu_0(w))}{w(j)w_0(j)}\ .\end{aligned}$$

Thus

$$\nu_n(w) = w - bwH_n(w) + B_n(w) + \beta_n(w), \tag{44}$$

where

$$H_n(w) = \sum_{j=0}^{n-1} \frac{1}{w(j)w_0(j)},$$

$$B_n(w) = -w_0 + b\sum_{j=0}^{n-1} \frac{w_0 - (\nu_j(w) - \nu_0(w))}{w(j)w_0(j)} ,$$

and

$$\beta_n(w) = \sum_{j=0}^{n-1} \left[\Gamma(w(j)) - \Gamma(w_0(j))\right].$$

To further explicate matters, it helps to use the notation $\Omega = \overline{\Pi}$ if the real part of the constant a in (P-3) is strictly positive, and $\Omega = \overline{\Pi}_R$ if $\operatorname{Re} a = 0$ where we recall that R is the radius of the half-disc outside of which inequality (34) holds, and $\overline{\Pi}_R$ is the quarter-plane that lies above the line $y = R$).

We are going to show that each of the three sequences of functions defined above converges uniformly in Ω to a function that is bounded on Ω and holomorphic on its interior, and by (44) this will establish the desired convergence of the sequence (ν_n).

From now on, we assume for technical reasons that the base point w_0 lies in Ω (a restriction only in the case $\operatorname{Re} a = 0$; in fact, recall our earlier observation that once convergence of the sequence (ν_n) is established for one choice of basepoint, then convergence follows for all basepoints, and the resulting limit functions differ from each other by constants).

From the fundamental estimate of Proposition 4.18 we know that if $w \in \Omega$ then $|w(n)| \geq c(|w|+n)$, while positivity of the real part of Φ implies that $|w(n)| \geq 1$. Upon averaging these inequalities there results

$$|w(n)| \geq \frac{c}{2}(|w| + n) + \frac{1}{2} \geq \frac{c}{2}n + \frac{1}{2} , \tag{45}$$

from which the desired convergence of the sequence (H_n) follows immediately. In addition, (45) yields

$$\sum_{j=0}^{n} |\Gamma(w(j))| \leq c$$

where, as always, the constant c is independent of n and w. This establishes the desired convergence of (β_n). For (B_n) we need only show that the series

$$\sum_{j=0}^{\infty} \frac{|\nu_j(w) - \nu_0(w)|}{|w(j)w_0(j)|}$$

has the desired convergence. For this we use equation (43) along with inequality (45) to obtain

$$|\Delta\nu_j(w)| \le b\left[\frac{1}{|w(j)|} + \frac{1}{|w_0(j)|}\right] + |\Gamma(w(j))| + |\Gamma(w_0(j))| \le c\frac{1}{j+1}\ ,$$

from which a summation yields

$$|\nu_j(w) - \nu_0(w)| \le \sum_{i=0}^{j-1} |\Delta\nu_i(w)| \le c\log(j+1)$$

for all non-negative integers j. Using this estimate along with (45), we see that

$$\frac{|\nu_j(w) - \nu_0(w)|}{|w(j)w_0(j)|} \le c\frac{\log(j+1)}{(j+1)^2} \qquad (j = 0, 1, 2, \ldots),$$

which establishes that the sequence (B_n) has the desired convergence on Ω. Here is what we have proved so far:

Proposition 4.19 (The existence and first representation of ν) *The sequence (ν_n) converges uniformly on compact subsets of Ω to a continuous function ν that is holomorphic on the interior of Ω. Moreover*

$$\nu(w) = w - bwH(w) + B(w),$$

where the function B is bounded and continuous on the region Ω and holomorphic on its interior; and

$$H(w) = \sum_{n=0}^{\infty} \frac{1}{w(n)w_0(n)},$$

where the series converges uniformly on Ω.

We remark that the work above actually shows both that as $|w| \to \infty$, the function $B(w)$ tends to a constant, and $H(w)$ tends to zero. We will not need this additional fact about B, and we are going to considerably refine the observation about H.

We devote the remainder of this section to (essentially) finishing the proof of the first two parts of Theorem 4.12. In view of Proposition 4.19, the desired result can largely be rephrased as:

$$wH(w) \overset{\mathrm{b}}{=} \frac{1}{a}\log(1+w),$$

where the "equation" $f(w) \overset{\mathrm{b}}{=} g(w)$ means that $f - g$ is a bounded function on Ω.

Our strategy is to show that for each $w \in \Omega$, the modified orbit element $w(n)$ is approximated closely enough by $w + an$ so that

$$wH(w) \overset{\mathrm{b}}{=} \sum_{n=0}^{\infty} \frac{w}{(w+an)(w_0+an)} \overset{\mathrm{b}}{=} \int_1^{\infty} \frac{w\ dt}{(w+at)(w_0+at)} \ ,$$

after which a direct calculation shows that the difference between the integral and $\frac{1}{a}\log(1+w)$ is bounded. The following inequalities make everything work.

Lemma 4.20 *For each $w \in \Omega$ and each positive integer n,*

(a) $|w + na| \geq C(|w| + n)$.

(b) $|w(n) - w - na| \leq C \log(n+1)$.

Proof. To obtain the first inequality, apply Proposition 4.18 to the map $\Phi(w) = w + a$.

For the second one, use equation (42) and inequality (33) to provide the estimate

$$|\Delta w(j) - a| \leq \frac{|b|}{|w(j)|} + \frac{C}{|w(j)|^{1+\epsilon}} \leq \frac{C}{|w| + j} \leq \frac{C}{j} \ .$$

The desired inequality follows upon summing both sides of this one, and using the triangle inequality. $\square$

Lemma 4.21 *For each $w \in \Omega$ and each positive integer n,*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{1}{w(n)} - \frac{1}{w+na} \right| \leq \frac{C}{|w|+1}.$$

Proof. Using respectively Lemma 4.20 and inequality (33), we obtain

$$\begin{aligned}
\left| \frac{1}{w(n)} - \frac{1}{w+na} \right| &= \frac{|w(n) - w - na|}{|w(n)||w+na|} \\
&\leq C \frac{\log(n+1)}{|w(n)|(|w|+n)} \\
&\leq C \frac{\log(n+1)}{(|w|+n)^2} \ ,
\end{aligned}$$

whereupon

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left|\frac{1}{w(n)}-\frac{1}{w+nb}\right| &\leq C\sum_{n=1}^{\infty}\frac{\log(n+1)}{n(|w|+n)^2}\\ &\leq \frac{C}{|w|+1}\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^2}\\ &\leq \frac{C}{|w|+1}\ ,\end{aligned}$$

as desired. □

Corollary 4.22 *For $w \in \Omega$,*

$$\left|H(w)-\sum_{n=1}^{\infty}\frac{1}{(w+na)(w_0+na)}\right| \leq \frac{C}{|w|+1}.$$

Proof. The quantity whose magnitude is to be estimated can be written as $\mathrm{I}(w)+\mathrm{II}(w)+\mathrm{III}(w)$, where

$$\mathrm{I}(w)=\frac{1}{w(0)w_0(0)}=\frac{1}{(w+1)(w_0+1)}$$

is the term corresponding to $n=0$ in the sum defining $H(w)$,

$$\mathrm{II}(w)=\sum_{n=1}^{\infty}\frac{1}{w_0(n)}\left[\frac{1}{w(n)}-\frac{1}{w+na}\right],$$

and

$$\mathrm{III}(w)=\sum_{n=1}^{\infty}\frac{1}{w+na}\left[\frac{1}{w_0(n)}-\frac{1}{w_0+na}\right].$$

Clearly $I(w)$ obeys the desired inequality. Upon using inequality (33) along with Lemmas 4.20 and 4.21 to estimate the magnitudes of terms in the sums defining $\mathrm{II}(w)$ and $\mathrm{III}(w)$, we obtain this inequality for those sums also. □

Let us summarize how far we have progressed toward proving the first two parts part of Theorem 4.12. We have shown that:

For every $w \in \Omega$,

$$\nu(w)=w-bw\sum_{n=1}^{\infty}\frac{1}{(w+na)(w_0+na)}+B(w) \tag{46}$$

where B is a bounded, continuous function on Ω.

The next step is to replace the sum in this representation by the corresponding integral.

Proposition 4.23 *For each $w \in \Omega$,*

$$\left|\sum_{n=1}^{\infty} \frac{1}{(w+na)(w_0+na)} - \int_1^{\infty} \frac{dt}{(w+ta)(w_0+ta)}\right| \leq \frac{C}{|w|+1} .$$

Proof. The difference in question has magnitude bounded by the infinite sum of the integrals

$$\int_n^{n+1} \left|\frac{1}{(w+na)(w_0+na)} - \frac{1}{(w+ta)(w_0+ta)}\right| dt.$$

A little calculation shows that the integrand above is

$$\left|\frac{a(t-n)(w+w_0)+(t^2-n^2)a^2}{(w+na)(w_0+na)(w+ta)(w_0+ta)}\right|,$$

which, since t lies between n and $n+1$, is dominated by a constant multiple of

$$\frac{|a||w+w_0|+2(n+1)|a|^2}{n^2(|w|+n)^2},$$

where we have also used Lemma 4.20 to get the lower bounds used for the terms of the denominator. The last quantity is clearly bounded by a constant multiple of $\frac{1}{n^2(|w|+1)}$, so the magnitude of the original difference is bounded by a constant multiple of

$$\sum_{n=1}^{\infty} \frac{1}{(|w|+1)n^2} \leq \frac{C}{|w|+1} ,$$

which yields the desired inequality. □

Upon substituting this result into (46), we obtain for $w \in \Omega$,

$$\nu(w) = w - bw \int_1^{\infty} \frac{1}{(w+ta)(w_0+ta)} dt + \text{ a bounded function.} \tag{47}$$

To complete the proof of the representation of ν promised in the Theorem 4.12, we need only evaluate the integral and estimate the answer. An easy calculation shows that

$$\int_1^{\infty} \frac{dt}{(w+ta)(w_0+ta)} = \frac{1}{a(w-w_0)} \log \frac{w+a}{w_0+a} ,$$

and the reader can easily check that the function

$$\log(w+1) - \log\frac{w+a}{w_0+a}$$

is bounded for $w \in \overline{\Pi}$. Thus $\frac{\log(w+1)}{aw}$ can replace the integral in the representation (46) above, causing only a bounded amount of damage. This completes the proof that ν has the representation (24) promised by Theorem 4.12, at least on the region Ω and for $w_0 \in \Omega$. □

Summarizing our progress to this point:

- The work above and the discussion immediately following the statement of Theorem 4.12 prove that Theorem *completely* for the case $\operatorname{Re} a > 0$.
- For the case $\operatorname{Re} a = 0$ we have proved that ν has the representation (24) on the part of $\overline{\Pi}$ that lies above the line $\{y = R\}$ (the Theorem promises this for the entire upper half of $\overline{\Pi}$), and have observed that this yields
$$\lim_{y\to+\infty} \operatorname{Im}\nu(iy) = +\infty,$$
which is half of what is needed to prove part (d) of Theorem 4.12.

Existence of ν, Part II. Our main objective now is to prove part (d) of Theorem 4.12 for the case where the parameter a is pure imaginary (along the way we complete the proof of parts (b) and (c)). Recall that we are assuming $a \neq 0$, and for definiteness are taking a to be *positive* imaginary: $a = \alpha i$ for some $\alpha > 0$.

In analytical terms, our goal is to prove that

$$\lim_{y\to-\infty} \operatorname{Im}\nu(iy) = -\infty. \tag{48}$$

In a curious twist, our proof of this will require a detailed analysis of the behavior of the *real part* of $\Phi_n(w)$. Proposition 4.13 and condition (P-3) combine to give the following asymptotic representation of $\operatorname{Re}\Phi$.

Lemma 4.24 (Representation of $\operatorname{Re}\Phi$) *For $w \in \overline{\Pi}$,*

$$\operatorname{Re}\Phi(w) = \operatorname{Re} w + b\frac{\operatorname{Re} w + 1}{|w+1|^2} + \Theta(w+1) \tag{49}$$

with

$$\Theta(w+1) \leq \frac{C}{|w+1|^{1+\epsilon}} \ ,$$

where C and ϵ are positive constants, independent of w.

Proof. Take real parts of both sides of the original representation (P-3). Use the fact that b is real (Lemma 4.13) and a pure imaginary. The result follows immediately. □

Before stating the first of our estimates on the real part of an orbit, recall our convention that the symbol "C," as well as any of its variants, denotes at every occurrence a constant that does not depend on either the index of iteration n or the point w.

Proposition 4.25 *For $w \in \overline{\Pi}$ with $\operatorname{Im} w > R$,*

$$\operatorname{Re} \Phi_n(w) \leq C_1 \operatorname{Re} w + C_2.$$

Proof. For the points w in question we know from Proposition 4.18 that

$$|\Phi_n(w)| \geq C(|w| + n).$$

Now because a is pure imaginary, it disappears from (P-3) upon taking real parts, so upon substituting $\Phi_j(w)$ for w in what results, and using the lower estimate above, we obtain

$$0 < \operatorname{Re} \Delta\Phi_j(w) \leq \frac{b}{C(|w| + j + 1)} + \frac{C}{(|w| + j + 1)^{1+\epsilon}} \leq \frac{C}{j+1}.$$

Upon summing both sides of this inequality, we obtain the preliminary estimate:

$$\operatorname{Re} \Phi_{n+1}(w) \leq \operatorname{Re} w + C(1 + \log(n+1)). \tag{50}$$

Now replace w by $\Phi_j(w)$ in the representation of $\operatorname{Re} \Phi$ provided by Lemma 4.24, and recycle the last inequality. The result is:

$$\begin{aligned} \operatorname{Re} \Delta\Phi_j(w) &= b\frac{\operatorname{Re} \Phi_j(w) + 1}{|\Phi_j(w) + 1|^2} + \Theta(\Phi_j(w) + 1) \\ &\leq b\frac{\operatorname{Re} w + C_1(1 + \log(j+1))}{(j+1)^2} + \frac{C_2}{(j+1)^{1+\epsilon}} \ , \end{aligned}$$

whereupon

$$\begin{aligned}\operatorname{Re}[\Phi_n(w) - w] \quad &\leq \quad (\operatorname{Re} w)\sum_{j=0}^{n-1}\frac{b}{(j+1)^2} + C_1\sum_{j=0}^{n-1}\frac{1+\log(j+1)}{(j+1)^2} \\ &+C_2\sum_{j=0}^{n-1}\frac{1}{(j+1)^{1+\epsilon}}\end{aligned}$$

which gives the desired result. □

We can now begin filling in the remaining pieces of the proof of Theorem 4.12.

Corollary 4.26 *The sequence (ν_n) converges uniformly on compact subsets of $\overline{\Pi}$ to a function ν that is holomorphic on Π and continuous on $\overline{\Pi}$. The resulting function ν obeys the functional equation (31) and has real part bounded below on Π. The base point w_0 in the definition of ν_n can be chosen anywhere in $\overline{\Pi}$, changing the definition of ν by only a constant.*

Proof. To get started we initially choose $w_0 \in \overline{\Pi}$ with $\operatorname{Im} w_0 > R$. By the last result, the sequence $(\operatorname{Re}\Phi_n(w_0))$ is bounded, say by the constant M, so for each $w \in \overline{\Pi}$ we have

$$\operatorname{Re}\nu_n(w) = \operatorname{Re}[\Phi_n(w) - \Phi_n(w_0)] \geq -M. \tag{51}$$

Thus $\{\nu_n\}$ is a normal family on Π; moreover we already know the sequence (ν_n) converges uniformly on compact subsets of an upper quadrant of $\overline{\Pi}$. Thus the sequence converges uniformly on compact subsets of Π, and so by (P-2) on all of $\overline{\Pi}$. That the limit ν has real part bounded below follows from (51). The other properties of ν, including the fact that the base point can now be taken to be anywhere in $\overline{\Pi}$, have already been discussed. □

Orbit structure. Our next task is to examine in some detail the structure of an arbitrary Φ-orbit. Recall that we are assuming that in the representation (P-3), the constant a is positive imaginary: $a = i\alpha$, $\alpha > 0$. Recall also the fundamental estimate (34) that holds for all points of $\overline{\Pi}$ that lie outside the disc $\{|w| \leq R\}$. In particular, this estimate holds for every point outside the closed rectangle $\mathcal{B}$ in $\overline{\Pi}$ defined by the inequalities $|\operatorname{Im} w| \leq R$, $0 \leq \operatorname{Re} w \leq R$. Henceforth we refer to this rectangle as the *bad rectangle*, and to its complement $\mathcal{G}$ as (of course) the *good region.*

Thus inequality (38) holds whenever w lies in the good region, and asserts that $\Phi(w)$ is at least $\alpha/2$ units higher than w. Of course whenever Φ_n is in the good region, we may substitute $\Phi_n(w)$ for w in this last statement to obtain:

> *Each point of a Φ-orbit rises at least $\alpha/2$ units above its predecessor, unless that predecessor lies in the bad rectangle $\mathcal{B}$.*

We define the *bad points* of an orbit to be the consecutive points from the first that has imaginary part between $-R$ and R, to the last that has imaginary part $< R$. The remaining points of the orbit are the *good points.* The reader should take note that these "bad points" include all the ones (if any) that lie in the bad rectangle $\mathcal{B}$, but they can include more. Figure 7 below illustrates the situation. Two orbits are shown; the one on the left intersects $\mathcal{B}$, while the one on the right does not. The solid dots indicate the bad points of each orbit.

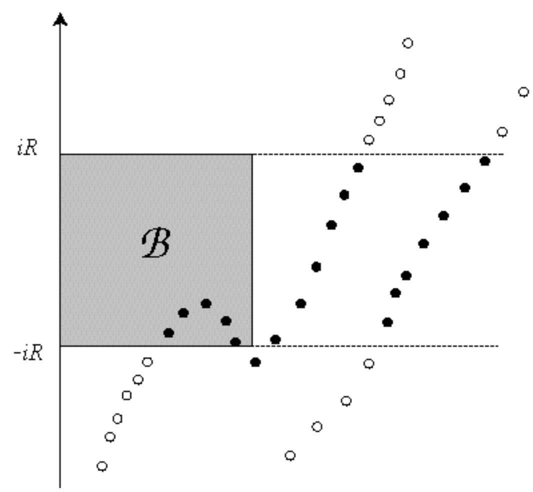

Figure 7: *Two orbits; the solid dots are the bad points.*

The observation just made about how orbits rise shows that if an orbit never encounters $\mathcal{B}$, then it has at most $4R/\alpha$ bad points, while the good points fall into a (possibly empty) finite initial segment, and an infinite final segment. Our next task is to show that something similar happens even if the orbit encounters the bad set.

Since each orbit tends to ∞, no orbit can have more than a finite number of points in $\mathcal{B}$. The key to the next result is that there is a bound on this number that is uniform over all starting points.

Proposition 4.27 *There exists $M < \infty$ such that no Φ-orbit has more than M bad points.*

Proof. First we prove that no more than a fixed number of points of any orbit can lie in the bad rectangle $\mathcal{B}$. Since ν is continuous on $\overline{\Pi}$ and $\mathcal{B}$ is compact, the image $\nu(\mathcal{B})$ is a compact subset of the plane. Let h denote the vertical height of this set.

Fix $w \in \overline{\Pi}$. The sequence $(\nu(w)+in\alpha)$ has no more than $(h/\alpha)+1$ points in $\nu(\mathcal{B})$; this implies that no more than $(h/\alpha)+1$ points of the orbit $(\Phi_n(w))$ can lie in $\mathcal{B}$.

Let μ denote the maximum of the continuous function $|\Phi(w) - w|$ over the compact set $\mathcal{B}$. Then the worst that can happen to an orbit which encounters $\mathcal{B}$ is that its last point in $\mathcal{B}$ has image with imaginary part equal to $-\mu - R$. Since the successor of every point not in $\mathcal{B}$ lies at least $\alpha/2$ units above the original point, it will take no more than $2(2R+\mu)/\alpha$ additional steps for the orbit to rise up to, or above, the line $y = R$. Thus an orbit which encounters $\mathcal{B}$ can have no more than $\frac{h+1+4R+2\mu}{\alpha} + 1$ bad points. We have previously noted that orbits that never encounter $\mathcal{B}$ fare better, so the proof is complete. □

Proposition 4.27 allows us to complete the proof of part (b) of the Parabolic Models Theorem.

Corollary 4.28 *If, in hypothesis (P-3), we have $a = \alpha i$ for some $\alpha > 0$, then*

$$\nu(w) = w - \frac{b}{a}\log(1+w) + B(w) \qquad (w \in H),$$

where $H = \{w \in \overline{\Pi} : \operatorname{Im} w \geq 0\}$, and B is a function that is bounded and continuous on H, and holomorphic on its interior.

Proof. We have already shown that ν has the desired form for any $w \in \overline{\Pi}$ with $\operatorname{Im} w \geq R$. Suppose that $0 \leq \operatorname{Im} w \leq R$ so that the first point in $\operatorname{Orb}(\Phi, w)$ is either in the bad rectangle $\mathcal{B}$ or to the right of $\mathcal{B}$. In either case, if M is the constant of Proposition 4.27, $\Phi_{M+1}(w)$ will be on or above the line $y = R$. Hence,

$$\nu(w) + i(M+1)\alpha = \Phi_{M+1}(w) - \frac{b}{a}\log(1+\Phi_{M+1}(w)) + B(\Phi_{M+1}(w))$$

Now (35) shows that $|w - \Phi_{M+1}(w)|$ is uniformly bounded on $\overline{\Pi}$, and it follows that ν has the desired representation on H. □

We now complete our analysis of the real part of an orbit.

Proposition 4.29 (Growth of real part of an orbit) *For each* $w \in \overline{\Pi}$ *and each positive integer* n*,*

$$\operatorname{Re}\Phi_n(w) \leq C_1 \operatorname{Re} w + C_2 \log(1 + |\operatorname{Im} w|) + C_3. \tag{52}$$

If $\operatorname{Im} w \geq 0$, *then the inequality holds with* $C_2 = 0$.

Proof. Recall that in the proof of Proposition 4.18 we saw that the series expansion (P-3) guarantees that the distances between consecutive points of orbits are uniformly bounded:

$$\delta = \sup\{|\Delta\Phi_n(w)| : n = 0, 1, 2, \ldots; w \in \overline{\Pi}\} < \infty. \tag{53}$$

Now we have already proved the last statement of the Proposition for starting points w with imaginary part $> R$ (Proposition 4.25). The extension of this statement to starting points in the upper half-plane, and indeed to starting points lying above the line $y = -R$, follows from this, the fact that orbits have no more than a bounded number of bad points, and the bound (53) above, which shows, among other things, that as the orbit runs through its no more than M bad points, it progresses to the right by no more than $M\delta$ units.

In particular, if an orbit intersects the bad rectangle $\mathcal{B}$, we may as well discard all the predecessors of the first point (call it w') in $\mathcal{B}$, for they lie to the left of this point. Applying the work of the last paragraph to the orbit originating at w', we see that all its members have real part $\leq C_1 R + C_3$. That is:

> *All the orbits that encounter* $\mathcal{B}$ *lie in a fixed vertical strip of finite width.*

Thus we need only prove inequality (52) for orbits that never hit the bad set, and have starting points w with imaginary part $< -R$. So fix such a point w, and recall that its orbit progresses to the right (as all orbits must, by the Julia-Carathéodory Inequality; Theorem 4.2), and upward, at each step increasing in height by at least $\alpha/2$. Let $\Phi_{N+1}(w)$ be the first point of the orbit whose imaginary part exceeds $-R$ (i.e., the first bad point).

We wish to estimate how far the initial good segment progresses to the right. For this we use the method of proof of Proposition 4.25, this time the idea is to count backward from $\Phi_N(w)$. Since each good point of an orbit lies at least $\alpha/2$ units above the previous point, we have for $0 \le j \le N$,

$$|\operatorname{Im}\Phi_j(w)| \ge R + (N-j)\frac{\alpha}{2}. \tag{54}$$

Upon substituting $\Phi_j(w)$ for w in the representation of $\operatorname{Re}\Phi$ found in Lemma 4.24, we obtain

$$\operatorname{Re}\Delta\Phi_j(w) = b\frac{\operatorname{Re}\Phi_j(w)+1}{|\Phi_j(w)+1|^2} + \Theta(\Phi_j(w)+1) \le \frac{C}{|\Phi_j(w)+1|}\ , \tag{55}$$

which, along with (54), yields the estimate

$$\operatorname{Re}\Delta\Phi_j(w) \le \frac{C}{R+\frac{\alpha}{2}(N-j)}\ .$$

Upon summation this yields:

$$\begin{aligned}
\operatorname{Re}[\Phi_N(w)-w] &= \sum_{j=0}^{N-1}\operatorname{Re}\Delta\Phi_j(w)\\
&\le C\sum_{j=0}^{N-1}\frac{1}{R+\frac{\alpha}{2}(N-j)}\\
&\le C\sum_{j=0}^{N-1}\frac{1}{N-j}\\
&\le C\sum_{j=1}^{N}\frac{1}{j}\\
&\le C\log(N+1),
\end{aligned}$$

hence

$$\operatorname{Re}\Phi_N(w) \le \operatorname{Re}w + C\log(N+1).$$

Now recall that $\Phi_N(w)$ is the last point of the orbit of w that lies below the line $y=-R$, and recall yet again that, since the orbit never encounters the bad set, each of its points lies at least $\alpha/2$ units higher than the previous one. Thus

$$\frac{N\alpha}{2} \le |\operatorname{Im}[\Phi_N(w)-w]| \le |\operatorname{Im}w| - R.$$

Upon substituting the resulting estimate for N into the previous inequality, we obtain

$$\operatorname{Re}\Phi_N(w) \le \operatorname{Re} w + C\log(|\operatorname{Im} w| + 1). \tag{56}$$

By Proposition 4.27, no orbit has more than M bad points, so the imaginary part of $\Phi_{N+M+1}(w)$ must exceed R. Since the orbit progresses no more than $(M+1)\delta$ units to the right during these additional $M+1$ steps, we can replace the index N by $N+M+1$ in inequality (56), at the expense of appropriately modifying the constant C (the reader should note that we have finally used the condition that $\Phi_N(w)$ is the *last* point of the orbit of w with imaginary part less than $-R$).

Finally, if $n > N+M+1$, then Proposition 4.25, with w replaced by $\Phi_{N+M+1}(w)$, shows that

$$\begin{aligned} \operatorname{Re}\Phi_n(w) &\le C_1 \operatorname{Re}\Phi_{N+M+1}(w) + C_2 \\ &\le C_1[\operatorname{Re} w + C\log(|\operatorname{Im} w| + 1)] + C_2, \end{aligned}$$

which, once the constants have been properly renamed, is the desired estimate. □

The dénoument. To finish the proof of Theorem 4.12, it remains only to show that

$$\lim_{y\to-\infty} \operatorname{Im}\nu(iy) = -\infty.$$

Proof. Consider the orbit that starts at iy, where $y < -R$. Let $N = N(y)$ be the first index n for which $\operatorname{Im}\Phi_n(iy) > R$. Thus

$$N \ge \frac{|y| + R}{\delta},$$

where δ, defined in (53), bounds the distance between consecutive points of any orbit. Let $w' = \Phi_N(iy)$, the first point of the orbit with imaginary part $> R$. By the functional equation for ν,

$$\nu(iy) = -iN\alpha + \nu(w'),$$

which, along with the previous estimate of N, yields

$$\operatorname{Im}\nu(iy) \le -\left(\frac{|y| + R}{\delta}\right) + \operatorname{Im}\nu(w'). \tag{57}$$

Now we have already established part (b) of the Theorem 4.12, and when applied to w' it yields

$$\nu(w') = w' + i\frac{b}{\alpha}\log(w'+1) + B(w').$$

Recalling that b is real, this implies

$$\operatorname{Im}\nu(w') = \operatorname{Im} w' + \frac{b}{\alpha}\operatorname{Re}\,\log(w'+1) + \operatorname{Re} B(w').$$

Recall that the function B is bounded on $\overline{\Pi}$. Moreover, w' is the *first* orbit point above the line $y = R$, so $\operatorname{Im} w' < R + \delta$. Thus the last equation yields the estimate

$$\operatorname{Im}\nu(w') \le R + \delta + \frac{b}{\alpha}\log|w'+1| + C.$$

Since the imaginary part of w' is bounded independent of y, its real part controls its magnitude. By Proposition 4.29 we have

$$\operatorname{Re} w' = \operatorname{Re}\Phi_N(iy) \le C_2\log(1+|y|) + C_3.$$

Putting the last two estimates together, we obtain (recalling that w' depends on y),

$$\operatorname{Im}\nu(w') \le C + o(y) \qquad (y \to \infty).$$

This estimate, along with (57) above yields

$$\operatorname{Im}\nu(iy) \le -C_1|y| + C_2 \qquad (y < -R),$$

from which the desired result follows immediately. This completes our proof of Theorem 4.12 $\square$

5 Cyclicity: Parabolic Nonautomorphism Case

We have proved the promised generalizations for all of our linear-fractional cyclicity results (Theorem 2.2) save one—that for parabolic nonautomorphisms. In terms of the summary presented in Table II of the Introduction, we have delivered on everything but the cyclicity assertion in the last row. In this section we complete this last bit of unfinished business.

We seek to generalize the part of Theorem 2.2 that deals with parabolic nonautomorphic self-maps of U, and for reference we recast this linear-fractional result as follows:

Theorem 5.1 *Suppose that φ is a linear-fractional self-map of U with Denjoy-Wolff point at 1. If φ is of parabolic type, and $\operatorname{Re}\varphi''(1) > 0$, then C_φ is cyclic, but it is not hypercyclic.*

We show here that this theorem generalizes to the class of composition operators with smooth, regular symbol. We are already part of the way there. We showed in Theorem 4.6 that if $\varphi \in C^2(1)$ is a self-map of U of parabolic type, with Denjoy-Wolff point 1, then the condition $\operatorname{Re}\varphi''(1) > 0$, implies C_φ is not hypercyclic. We have also shown (Theorem 4.17), that if φ is sufficiently smooth and regular, then C_φ is cyclic whenever $\operatorname{Re}\overline{\phi''(1)}(\mathcal{S}\phi)(1) \geq 0$. Thus, to complete our work we need to eliminate this last hypothesis on the Schwarzian derivative. The next few paragraphs explain the difficulties associated with this task.

Recall from the discussion following the statement of Theorem 4.12 that the hypothesis $\operatorname{Re}\overline{\phi''(1)}(\mathcal{S}\phi)(1) \geq 0$ makes "transference of cyclicity" easy to accomplish by allowing us to choose a linear-fractional model $(\psi\sigma\ )$ for φ in which σ is a *self-map* of U. Recall also the basic idea behind the transference technique in this situation:

> *If the polynomials in σ form a dense subset of H^2, and f is a hypercyclic (cyclic) vector for C_ψ, then $f\circ\sigma$ will be a hypercyclic (cyclic) vector for C_φ.*

Note that in order for this idea to work, σ must be a self-map of U, so that $f\circ\sigma$ will always be a Hardy-space function. Unfortunately, the intertwining map supplied by the Parabolic Models Theorem (Theorem 4.12) cannot always be taken to be a self-map of U. Indeed, part (c) of that theorem shows that the intertwining map that arises from our work *cannot* be taken to be a self-map of U when $\operatorname{Re}\overline{\phi''(1)}(\mathcal{S}\phi)(1) < 0$.

Despite this difficulty, the work of the last section has provided us with enough information about σ to prove the desired result:

Theorem 5.2 *Suppose that φ is a regular self-map of U of parabolic type, with Denjoy-Wolff point at 1. Suppose further that $\varphi \in C^{3+\epsilon}(1)$, with $\operatorname{Re}\varphi''(1) > 0$, and $\operatorname{Re}\overline{\phi''(1)}(\mathcal{S}\phi)(1) < 0$. Then C_φ is cyclic.*

We assume throughout the remainder of this section that φ is a self-map of U satisfying the hypotheses of Theorem 5.2. Our proof that C_φ is cyclic is organized as follows. First, we will apply the Parabolic Models Theorem to obtain a linear-fractional model $(\psi\sigma\ )$ for φ in which ψ is a parabolic nonautomorphism, and $G = \sigma(U)$ is a Jordan region containing U. Then we will prove that the linear span of $\{\psi_n : n = 0, 1, 2, \ldots\}$ is dense in $H^2(G)$ from which it will follow that the linear span of $\{\psi_n \circ \sigma : n = 0, 1, 2, \ldots\}$ is dense in $H^2(U)$. Then the intertwining relation $\psi_n \circ \sigma = \sigma \circ \varphi_n$, will show that σ is a cyclic vector for C_φ.

Applying the Parabolic Model

As in Section 4, our initial hypotheses on φ translate into hypotheses (P-1) to (P-4) (preceding the statement of Theorem 4.12) on its right half-plane analogue Φ, and the additional condition $\operatorname{Re}\overline{\phi''(1)}(\mathcal{S}\phi)(1) < 0$ becomes

(P-5) $\operatorname{Re}\overline{a}b > 0$.

So suppose Φ is a self-map of the right half-plane Π that satisfies (P-1)–(P-5). According to Theorem 4.12, there is a Jordan map ν that intertwines Φ and the parabolic nonautomorphism $\Psi(w) = w + a$, and has the following representation on $\overline{\Pi}$:

$$\nu(w) = w - \frac{b}{a}\log(1 + w) + B(w) \qquad (w \in \overline{\Pi}), \tag{1}$$

where B is a bounded holomorphic function on Π that is continuous on $\overline{\Pi}$.

Let $\mathcal{G}$ denote the Jordan domain $\nu(\Pi)$. The representation (1) gives rise to the following parameterization of $\partial\mathcal{G}$:

$$(2)\quad \operatorname{Re}\nu(iy) \stackrel{\text{b}}{=} -\frac{\operatorname{Re}\bar{a}b}{|a|^2}\log|1+iy|; \qquad \operatorname{Im}\nu(iy) \stackrel{\text{b}}{=} y - \frac{\operatorname{Im}\bar{a}b}{|a|^2}\log|1+iy|;$$

for y real. Here, as in Section 4, the symbol "$\stackrel{\text{b}}{=}$" is used to assert that the expression on the right differs from the one on the left by a bounded function (of the real variable y, this time).

It follows from (P-5) and (2) that $\operatorname{Re}\nu(iy)$ is bounded above. Thus upon replacing ν with $\nu - \alpha$ for a sufficiently large positive constant α, we may assume that $\partial\mathcal{G}$ lies between the imaginary axis and the curve $\Gamma : R \to \mathbf{C}$ defined by

$$(3)\qquad \Gamma(y) = iy - \frac{b}{a}\log(1+iy) - C$$

for C sufficiently large (see Figure 8).

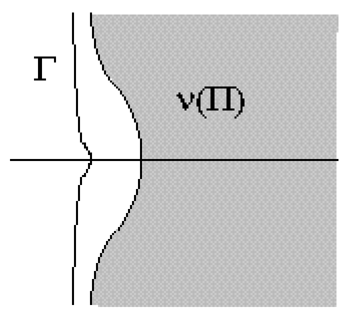

Figure 8: $\partial\mathcal{G}$ $(= \partial\nu(\Pi))$ lies between Γ and the imaginary axis.

A Cyclic Vector for C_φ

Let $(\psi\sigma$) be the the disk model for φ that corresponds to the right half-plane model (Ψ, ν) developed above. Hence, ψ is a parabolic nonautomorphism, and $G = \sigma(U)$ is Jordan domain containing U (because the image $\mathcal{G}$ of ν contains Π). Upon transferring Figure 8 to the unit disk, we see that Jordan curve ∂G must lie between ∂U and the curve $T^{-1}(\Gamma)$, as shown in Figure 9.

Recall that the inner product $\langle f, g\rangle$ of elements f and g in $H^2(G)$ may be computed by integrating the product $(f\circ\sigma)(\overline{g\circ\sigma})$ over the unit circle with respect to normalized Lebesgue measure m. Hence,

$$\langle f, g\rangle = \int_{\partial G} f\bar{g}d\mu,$$

where $\mu = m\sigma^{-1}$.

As we have already discussed, to show that σ is a cyclic vector for C_φ it suffices to prove the following:

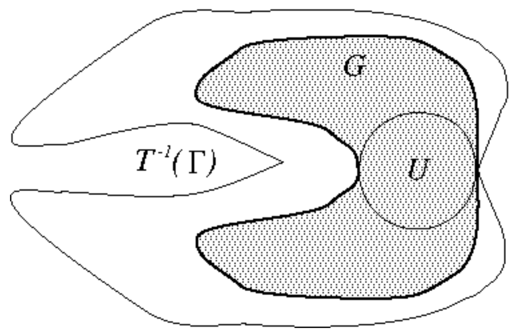

Figure 9: ∂G lies in the region between ∂U and the curve $T^{-1}(\Gamma)$.

Theorem 5.3 *The composition operator C_ψ is cyclic on $H^2(G)$, with cyclic vector $u(z) = z$.*

Proof. We proceed as in the proof of Theorem 2.5. In that proof we pulled the formula $\Psi(w) = w + a$ back to the unit disk to obtain the parabolic linear-fractional self-map of U:

$$\begin{aligned} \psi(z) &= \frac{(2-a)z + a}{-az + (2+a)} \\ &= \bar{\gamma} + \bar{\alpha}K_\beta, \end{aligned}$$

where $\bar{\gamma} = \frac{a-2}{a}$, $\bar{\alpha} = \frac{4}{a(a+2)}$, $\bar{\beta} = \frac{a}{2+a}$, and $K_\beta(z) = (1 - \bar{\beta}z)^{-1}$; and we obtained the formula for ψ_n by replacing a with na in these formulas.

Since Ψ maps $\partial\mathcal{G}$ into $\mathcal{G}$, it follows that ψ must map $\partial G\backslash\{1\}$ into G. Since $1 \in \partial G$ is the (attractive) fixed point of ψ, we have $\psi_n(z) \to 1$ for all $z \in \partial G$. Since the sequence (ψ_n) is uniformly bounded on ∂G we conclude from all these observations that the sequence $(\psi_n \circ \sigma)$ converges to 1 pointwise on ∂U and is uniformly bounded there. Thus $\psi_n \circ \sigma \to 1$ in $H^2(U)$; equivalently, $\psi_n \to 1$ in $H^2(G)$. Just as in the proof of Theorem 2.5 this shows that if $f \in H^2(G)$ is perpendicular to the C_ψ-orbit of u, then f must be orthogonal to the constant functions. Hence, just as before, we see that

$$0 = \langle f, K_{\beta_n} \rangle \qquad (n = 1, 2, 3, \ldots), \tag{4}$$

where $\beta_n = \frac{na}{na+2}$, and now (in contrast with the proof of Theorem 2.5) the inner product is that of $H^2(G)$.

Suppose that f is a function in $H^2(G)$ satisfying (4); we complete the proof of the theorem by showing f must be identically equal to zero.

Consider the "K-transform" of f:

$$F(\lambda) = \langle f, K_\lambda \rangle = \int_{\partial G} \frac{f(\zeta)}{1 - \lambda\bar{\zeta}} d\mu(\zeta). \tag{5}$$

The definition of F makes sense as long as $1/\bar{\lambda} \notin \partial G$; in particular this is true for all points λ in $M = (S^2 \backslash \bar{G})^*$ (where the "star" denotes reflection in the unit circle, and $\bar{G}$ is the closure of G relative to the Riemann sphere S^2; see Figure 10). In particular, $0 \in M$ because $\infty \in S^2 \backslash \bar{G}$. Differentiation under the integral sign shows that F is holomorphic on M.

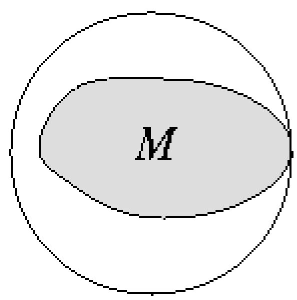

Figure 10: $M = (S^2 \backslash \bar{G})^*$.

We claim that to show $f \equiv 0$ it suffices to show $F \equiv 0$ on M. Indeed if $F \equiv 0$, then we can differentiate both sides of (5) with respect to λ to obtain for $n = 0, 1, 2, \ldots,$

$$0 = F^{(n)}(\lambda) = n! \int_{\partial G} \frac{\bar{\zeta}^n f(\xi)}{(1 - \lambda\bar{\zeta})^{n+1}} d\mu(\zeta) \qquad (\lambda \in M)$$

Setting $\lambda = 0$ in the preceding equation, we have

$$0 = \int_{\partial G} \bar{\zeta}^n f(\zeta) d\mu(\zeta) = \langle f, z^n \rangle \quad (n = 0, 1, 2, \ldots). \tag{6}$$

Since G is a Jordan domain, Walsh's theorem guarantees that the set of polynomials in z is dense in $H^2(G)$; hence (6) implies $f \equiv 0$.

Having established the claim, we now complete the proof of the Theorem by showing $F \equiv 0$. For this it suffices to show that $F \equiv 0$ on the open subset M_0 of M defined below. Note that the curve Γ pictured in Figure 8 lies in $S^2 \backslash \bar{\mathcal{G}}$. By (3), its reflection Γ^* in the imaginary axis satisfies

$$\begin{aligned} \Gamma^*(y) &= -\overline{\Gamma(y)} \\ &= iy + (\overline{b/a}) \log|1+iy| - (\overline{b/a}) \arg(1+iy) + C \\ &\stackrel{\text{b}}{=} iy + (\overline{b/a}) \log(1+iy), \end{aligned}$$

so for a sufficiently large constant C, all the points that lie on and to the right of the curve

$$\Gamma_0(y) \stackrel{\text{def}}{=} iy + \overline{(b/a)} \log(1+iy) + C$$

lie in $(S^2 \backslash \bar{\mathcal{G}})^*$. Hence, the function

$$\nu_0(w) \stackrel{\text{def}}{=} w + \overline{(b/a)} \log(1+w) + C \tag{7}$$

maps the right half-plane into a region $\mathcal{M}_0$ contained in $(S^2 \backslash \bar{\mathcal{G}})^*$, as shown in Figure 11. Let M_0 be the region in U that corresponds to $\mathcal{M}_0$ ($M_0 = T^{-1}(\mathcal{M}_0)$) and let σ_0 be the disk analogue of ν_0 so that $\sigma_0(U) = M_0$.

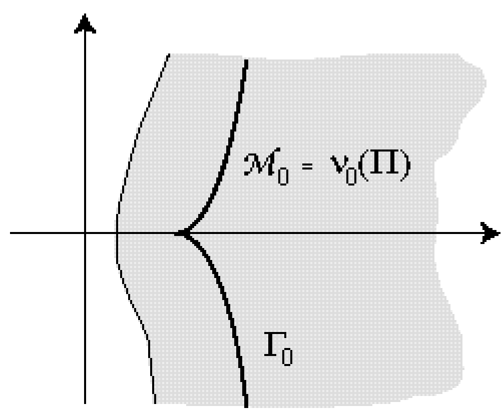

Figure 11: $(S^2 \backslash \bar{\mathcal{G}})^*$ *is shaded, and* $\mathcal{M}_0$ *lies to the right of* Γ_0.

Given the geometry of $\mathcal{M}_0$, any sequence converging to ∞ nontangentially in Π must eventually lie in $\mathcal{M}_0$; in particular this is true of the sequence

$T(\beta_n) = 1 + na$ (because $\operatorname{Re} a > 0$). Hence for some positive integer N we have $\beta_n \in M_0$ for all $n \geq N$.

We claim that the sequence $\left(\sigma_0^{-1}(\beta_n)\right)_{n=N}^{\infty}$ (in U) converges to 1 nontangentially and does not satisfy the Blaschke condition. This is easy to verify. Note that the sequence $(\beta_n)_{n=N}^{\infty}$ converges to 1 nontangentially and is not Blaschke. Note also that the form of ν_0 (7) shows that σ_0 is in $C^1(1)$ and $\sigma_0'(1) = 1$. Hence, $({\sigma_0}^{-1})'(1) = 1$ and it follows easily that image of (β_n) under ${\sigma_0}^{-1}$ must also converge nontangentially and fail to satisfy the Blaschke condition.

Observe that the function $F \circ \sigma_0$ defined on U vanishes at each point $\sigma_0^{-1}(\beta_n)$ for $n \geq N$ (by its definition (5) as the "K-transform" of f, and the fact that f is perpendicular to each function K_{β_n}). Thus the function $(F \circ \sigma_0)(\sigma_0')^2$ also vanishes at each of these points; we are going to show that this function lies in the Bergman space $L_a^1(U)$. From this we will conclude that $F \equiv 0$, since any Bergman space function vanishing on a non-Blaschke sequence *that converges to a boundary point nontangentially* must vanish identically (see [32]—the point is that a Bergman space function on U restricts to a Hardy space function on internally tangent subdisks of U).

For this Bergman space estimate, let A denote 2-dimensional Lebesgue measure, and compute:

$$\begin{aligned}
\int_U |(F \circ \sigma_0)(\sigma_0')^2| \, dA &= \int_{M_0} |F(\lambda)| \, dA(\lambda) \\
&= \int_{M_0} \left| \int_{\partial G} \frac{f(\zeta)\, d\mu(\zeta)}{1 - \lambda\bar{\zeta}} \right| \, dA(\lambda) \\
&\leq \int_{\partial G} |f(\zeta)| \int_{M_0} \frac{dA(\lambda)}{|1 - \lambda\bar{\zeta}|} \, d\mu(\zeta)\,.
\end{aligned}$$

Now fix $\zeta \in \partial G$ and consider the inner integral in the last line. Since $|\zeta| \geq 1$ and $M_0 \subset U$, we know that M_0 is contained in the disc B that has radius 2 and is centered at $1/\bar{\zeta}$. Then

$$\begin{aligned}
\int_{M_0} \frac{dA(\lambda)}{|1 - \lambda\bar{\zeta}|} &\leq \int_{M_0} \frac{dA(\lambda)}{|1/\bar{\zeta} - \lambda|} \\
&\leq \int_B \frac{dA(\lambda)}{|1/\bar{\zeta} - \lambda|}
\end{aligned}$$

$$= \int_{\{|\lambda|<2\}} |\lambda|^{-1}\, dA(\lambda)$$
$$= 4\pi.$$

Thus

$$\int_U |F \circ \sigma_0||\sigma_0'|^2\, dA \le 4\pi \|f\|_{H^2(G)} ,$$

which completes the proof that $(F \circ \sigma_0)(\sigma_0')^2 \in L^1_a(U)$. □

With this result, our characterization of cyclic phenomena for composition operators with smooth, regular symbol is complete.

6 Endnotes

In closing, we discuss some additional results, and speculate on further directions suggested by our work.

Orbit Separation and Parabolic Subtype

We observed in Theorem 4.15 that if φ is a map of parabolic type, and, at its Denjoy-Wolff point has $C^{3+\epsilon}$-smoothness and non-vanishing second derivative, then the following are equivalent:

(a) φ is of nonautomorphic type.

(b) Every φ-orbit converges to 1 non-tangentially.

In fact, the result below (which extends Theorem 3.5 of [14], and considerably simplifies its proof) shows that the implication (b)→(a) holds for arbitrary self-maps of U.

Theorem 6.1 *Suppose φ is a holomorphic self-map of U of parabolic type, and that some φ-orbit converges non-tangentially to the Denjoy-Wolff point. Then φ is of non-automorphic type.*

Proof. Again, we transfer the problem to the right half-plane, where by the Julia-Carathéodory inequality (4.2), $\Phi = T \circ \varphi \circ T^{-1}$ has the form

$$\Phi(w) = w + \Gamma(w), \tag{1}$$

with Γ a self-map of Π for which $\Gamma(w)/w \to 0$ as $w \to \infty$ nontangentially (see Theorem 4.3).

We are assuming that there exists a point $w_0 \in \Pi$ for which the orbit $w_n = \Phi_n(w_0)$ converges nontangentially to ∞. This means that

$$\inf_n \frac{\operatorname{Re} w_n}{|w_n|} = \delta > 0, \tag{2}$$

and from (1) above, that

$$\lim_n \frac{\Gamma(w_n)}{w_n} \to 0. \tag{3}$$

Substituting w_n for w in (1) and then using (2) and (3) we see that

$$|w_{n+1} - w_n| = o(\operatorname{Re} w_n) \quad \text{as } n \to \infty.$$

The rest of the argument proceeds in the same spirit as the proof of Theorem 4.15 (parts (b) and (c)). Fix $\epsilon > 0$ and recall that the Euclidean dimensions of $\Delta(w_n, \epsilon)$, the hyperbolic ϵ-disc about w_n are proportional to $\operatorname{Re} w_n$. But we just saw that the Euclidean distance between w_{n+1} and w_n is $o(\operatorname{Re} w_n)$, and these facts guarantee that $w_{n+1} \in \Delta(w_n, \epsilon)$ for all sufficiently large n. Since ϵ is arbitrary, it follows that $\varrho(w_n, w_{n+1}) \to 0$. □

Less Differentiability

We present a class a class of examples that shows how, under less than C^3-differentiability at the Denjoy-Wolff point:

- The converse of Theorem 6.1 fails (recall that we just observed that this converse holds under $C^{3+\epsilon}$-differentiability at the Denjoy-Wolff point).
- The determination of parabolic sub-type by the second derivative fails (note, however, that by Theorem 4.15 the second derivative *does* characterize parabolic sub-type for $C^{3+\epsilon}$ maps).

The Examples. For $0 < \alpha < 1$, let $\Phi_\alpha : \overline{\Pi} \to \Pi$ be defined by

$$\Phi_\alpha(w) = w + i + \frac{1/\alpha}{(w+1)^\alpha} \qquad (w \in \overline{\Pi}). \tag{4}$$

Let φ_α be the self-map of U corresponding to Φ_α ($\varphi_\alpha = T^{-1} \circ \Phi_\alpha \circ T$).

Theorem 6.2 *Suppose that $0 < \alpha < 1$. Then φ_α is univalent self-map of U satisfying:*

(a) *$\varphi_\alpha \in C^{2+\alpha}(1)$ with $\varphi_\alpha''(1) = i$.*

(b) *φ_α is of parabolic-nonautomorphic type (i.e., no φ_α orbit is separated).*

(c) *No φ_α orbit converges nontangentially.*

Proof. . To simplify notation we drop the subscript α. Since the derivative of Φ has positive real part, Φ is univalent on Π; hence, φ is univalent on U. That $\varphi \in C^{2+\alpha}(1)$ with $\varphi''(1) = i$ follows from our earlier work with series expansions about Denjoy-Wolff points (see the discussion preceding Table III). That no φ-orbit converges nontangentially follows from Lemma 4.5. To complete the proof, we need only show that no φ orbit is separated.

Returning to the right half-plane, let $w_0 \in \Pi$ be arbitrary, and for $n > 0$ let $w_n = \Phi_n(w_0)$. Observe that

$$w_n = w_0 + ni + \sum_{j=0}^{n-1} \frac{1/\alpha}{(w_j + 1)^\alpha}. \tag{5}$$

Hence,

$$\operatorname{Re} w_n \geq \operatorname{Re} w_0 + (1/\alpha) \sum_{j=0}^{n-1} \frac{\cos(\alpha\pi/2)}{|w_j + 1|^\alpha}. \tag{6}$$

Just as in the proof of Lemma 4.5, we have

$$\frac{w_n}{n} \to i \text{ as } n \to \infty. \tag{7}$$

Using (6) and (7), we find

$$\begin{aligned} \operatorname{Re} w_n &> C \sum_{j=0}^{n-1} \frac{1}{|w_j|^\alpha} \\ &> C \sum_{j=0}^{n-1} \frac{1}{j^\alpha} \\ &\to \infty \text{ as } n \to \infty. \end{aligned}$$

Thus the real part the the orbit of w_0 is unbounded, but as we have already observed this implies the orbit is not separated. □

We remark that it is possible to show that the map φ_1 is of automorphism type (all orbits nonseparated), which suggests that it may be possible to replace our $C^{3+\epsilon}$ hypothesis in Theorem 4.15 with just "C^3".

Further Directions

Let us agree that "map" means "univalent self-map of U". First, there is the question of cyclicity for composition operators induced by maps of dilation type (i.e., maps with a fixed point in U). Our results on this topic (except those concerning the trivial elliptic case) may be summarized as follows:

- If $\|\varphi_n\|_\infty < 1$ for some $n \geq 1$, then C_φ is cyclic if and only if the polynomials in φ are dense in H^2 (Corollary 1.6 and Corollary 3.7).
- An example: Each univalent map φ of the unit disc onto the lens-shaped region of Figure 3 (section 3) induces a cyclic composition operator (Theorem 3.8).
- Each linear-fractional map with a fixed point in U and one on ∂U induces a noncyclic composition operator (Theorem 2.2, part (b)).

There is an intriguing coincidence here: in the first two cases, the composition operators are all compact, while in the third, they are not compact (see [48, Chapters 2,3], for example). Is there a connection between cyclicity and compactness for composition operators induced by maps of dilation type?

The question that pervades all of our work on maps of hyperbolic and parabolic type (Denjoy-Wolff point on ∂U) is this: Does the cyclic behavior of C_φ match that of its "linear-fractional role model"? We summarize this question in Table IV, which may be viewed as the most general version of Table II of the Introduction.

Table IV

Conjectured cyclic behavior of C_φ (Denjoy-Wolff point of φ on ∂U)

Hypothesis on $\varphi'(p)$	Type of φ (Definition 0.3)	Cyclicity of C_φ
< 1	Hyperbolic	Hypercyclic ?
$=1$	Parabolic automorphism	Hypercyclic?
$=1$	Parabolic non-automorphism	Cyclic? Not Hypercyclic?

To settle this issue using our program requires a deeper understanding of the nature of the intertwining maps that occur in the Linear-Fractional

Model Theorem (Theorem 0.4). Much remains to be done, even if smoothness is assumed at the Denjoy-Wolff point. For example, we know that the third row of Table IV is true for regular maps φ with $C^{3+\epsilon}$-smoothness at the Denjoy-Wolff point, *and* nonzero second derivative there (Theorems 4.6 and 5.2). However we do *not* know if the cyclicity of C_φ persists when φ'' is allowed to vanish at the Denjoy-Wolff point.

In the direction of relaxing our smoothness hypotheses at the Denjoy-Wolff point, a first step might be to analyze the cyclic properties of specific examples such as the maps φ_α introduced above.

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