Compact, Nuclear, and Hilbert-Schmidt Composition Operators On H^2

J. H. SHAPIRO & P. D. TAYLOR

Communicated by the Editors

Introduction. Let φ be an analytic function taking the open unit disc U into itself. It is well known [15, Theorem 1], [5, sec. 2.6, page 29] that the *composition operator* C_φ defined by the equation

$$C_\varphi f(z) = f(\varphi(z)) \quad (z \text{ in } U)$$

is a bounded linear operator on each of the Hardy spaces H^p $(0 < p < \infty)$. In this paper we study the interplay between certain geometric properties of φ and functional analytic properties of C_φ ; emphasizing for simplicity the case where C_φ operates on the Hilbert space H^2. For univalent maps φ our results express quantitatively the fact that C_φ will be a compact, and perhaps even a Hilbert–Schmidt or trace class (*i.e.* nuclear) operator if the boundary of $\varphi(U)$ touches the unit circle infrequently and sharply; while it will fail to be compact if the boundary of $\varphi(U)$ touches the unit circle too smoothly, even at a single point. For example, it is easy to see that C_φ is compact (in fact, in the trace class) if φ takes U into a disc $|z| \leqq r < 1$. On the other hand, if $\varphi(z) = z$, then C_φ is the identity operator, which is not compact.

Much of our work generalizes to other H^p spaces, and we discuss these matters in the last section of the paper.

The first results of this type were obtained by H. J. Schwartz [17]. First Schwartz observed [17, Theorem 2.6, page 23] that the condition:

$$|\varphi(e^{it})| < 1 \text{ } a.e.$$

is necessary for the compactness of C_φ , but is not sufficient (here $\varphi(e^{it}) = \lim_{r\to 1-} \varphi(re^{it})$, where the limit exists *a.e.* by Fatou's radial limit theorem [5, Theorem 1.3]). In fact he showed that the function

$$\varphi(z) = (1 + z)/2, \tag{0.1}$$

which takes U onto a disc internally tangent to the unit circle, induces a non-compact composition operator, even though its boundary function has modulus

Indiana University Mathematics Journal, ©, Vol. 23, No. 6 (1973)

1 at only a single point of $|z| = 1$ [17, page 23]. Finally he showed that C_φ is compact whenever

$$\int_0^{2\pi} [1 - |\varphi(e^{it})|]^{-1}\, dt < \infty \tag{0.2}$$

[17, Theorem 2.7, page 26].

In this paper we continue Schwartz's work. Motivated by his example (0.1) we prove in section 2 (Theorem 2.1) that C_φ is not compact whenever φ has an *angular derivative* [1, sec. 299, page 32] at some point of the unit circle. In particular this happens whenever φ' is bounded and the boundary of $\varphi(U)$ touches the unit circle. We use this result to give examples of conformal maps which induce non-compact composition operators, yet take U onto domains whose boundaries touch the unit circle far more sharply than do internally tangent circles. We then give a sufficient condition for the compactness of C_φ (Theorem 2.3) which shows that there are conformal maps which induce compact composition operators on H^2, yet take U onto Jordan domains whose boundaries touch the unit circle smoothly (*i.e.* with continuously turning tangents). In fact, maps of this type can even induce Hilbert–Schmidt and trace class operators (Theorem 4.4, Proposition 5.2).

In section 3 we prove that Schwartz's condition (0.2) characterizes the Hilbert–Schmidt composition operators on H^2 (Theorem 3.1). As an application we show that C_φ is Hilbert–Schmidt whenever $\varphi(U)$ is contained in a polygon inscribed in the unit circle. The section concludes with a brief discussion of Hilbert–Schmidt composition operators on the space of analytic functions in U having finite Dirichlet integral.

In the fourth section we analyze in detail a family of univalent maps φ_α $(\alpha > 0)$ constructed from the functions

$$w = z(-\log z)^\alpha \quad (|z| < 1,\ \mathrm{Re}\, z > 0).$$

We show that each of the maps φ_α takes U onto a Jordan domain in U whose boundary contacts the unit circle at exactly one point, and does so smoothly. We show that the composition operator induced on H^2 by φ_α is compact for all $\alpha > 0$, and is Hilbert–Schmidt iff $\alpha > 2$. In particular this shows that there exist compact composition operators on H^2 which are not Hilbert–Schmidt.

In section 5 we briefly discuss trace class operators. Our main result is that C_φ is actually in the trace class whenever φ takes U into a polygon inscribed in the unit circle. We also give examples of trace class composition operators induced by conformal maps of U onto regions whose boundaries touch the unit circle smoothly.

In the final section we generalize the previous results to other H^p spaces $(p < \infty)$. In this context the *absolutely p-summing operators* [13, page 333] replace the Hilbert–Schmidt operators, and the *nuclear operators* [14, section 3.1.1, page 44] replace the trace class. We show that condition (0.2) characterizes the absolutely p-summing operators on H^p for $2 \leqq p < \infty$; and is sufficient for C_φ to be absolutely p-summing on H^p for $1 < p < 2$, and nuclear on H^1 (Theorem 6.2). In addition we show that C_φ is a nuclear operator on H^p ($1 \leqq p < \infty$) whenever φ takes U into a polygon inscribed in the unit circle (Theorem 6.3); and that the compactness of C_φ on H^p depends only on φ and not on p ($p < \infty$) (Theorem 6.1).

The paper begins with a preliminary section which sets out some definitions, notation, and background material.

We wish to thank Professors J. Caughran, S. Dragosh, D. Gregory, and T. McCoy for many helpful discussions of the material in this paper. In addition, we are most grateful to Professor J. Cima and Mr. James Thomsen for pointing our several errors in an earlier version of the manuscript.

1. Preliminaries. Here we set out some prerequisites from each of the three areas which occur in this paper: H^p theory, geometric function theory, and operator theory.

H^p spaces. The main reference for the material is [5, Chapters 1–3]. For $0 < p < \infty$ the *Hardy space* H^p is the space of functions f analytic in $U = (|z| < 1)$ such that

$$||f||_p^p = \sup_{0 \leqq r < 1} (1/2\pi) \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty .$$

For $1 \leqq p < \infty$ the functional $|| \ ||_p$ is a norm which makes H^p into a Banach space, while for $0 < p < 1$ the metric

$$d(f, g) = ||f - g||_p^p$$

makes H^p into a complete linear metric space [5, Chapter 2, page 37]. H^∞ is the Banach space of bounded analytic functions on U, taken in the supremum norm:

$$||f||_\infty = \sup |f(z)| \quad (z \text{ in } U).$$

Let σ denote normalized Lebesgue measure on the unit circle $|z| = 1$, and let $L^p = L^p(\sigma)$. We denote the n^{th} Fourier coefficient of a function g in L^1 by $\hat{g}(n)$ (n any integer). For each function $f(z) = \sum_0^\infty a_n z^n$ in H^p, the *radial limit*

$$f^*(e^{it}) = \lim f(re^{it}) \quad (r \to 1-)$$

exists *a.e.* $[\sigma]$ by Fatou's theorem [5, Theorem 2.3, page 17], f^* belongs to L^p; and if $p \geqq 1$ then $\hat{f}^*(n) = a_n$ for $n \geqq 0$, and $= 0$ for $n < 0$. In fact the map $f \to f^*$ is a linear isometry taking H^p onto a closed subspace of L^p which, if $p \geqq 1$, consists of all members of L^p whose negative Fourier coefficients vanish [5, Theorem 3.4, page 38]. From now on we will always write $f(e^{it})$ instead of $f^*(e^{it})$.

Throughout most of this paper we will be working in H^2 which, by the above remarks, is a Hilbert space in the L^2 inner product:

$$\langle f, g\rangle = \int f\bar{g}\, d\sigma = \sum f(n)\bar{g}(n) \qquad (f,\, g \text{ in } H^2).$$

In the last section, however, we will concern ourselves with general H^p spaces, and will require the following growth estimate.

Theorem 1.1. [5, Chapter 3, page 36]. *If f belongs to H^p $(0 < p < \infty)$, then*

$$|f(z)| \leqq 2^{1/p}\, \|f\|_p\, (1 - |z|)^{-1/p}$$

for all z in U.

This estimate has two important consequences: it shows that evaluation at a point of U is a bounded linear functional on H^p, and that the bounded subsets of H^p are normal families.

Geometric function theory. We will frequently, and without further explanation, use the *Riemann Mapping Theorem* [8, page 26, Theorem 1], which asserts that every simply connected plane domain having at least one boundary point is conformally equivalent to the open unit disc. A one-to-one conformal map of U onto such a domain will occasionally be referred to as a *Riemann map*. We will require the following results, which give information about the boundary behavior of Riemann maps.

By a *Jordan domain* we mean a simply connected plane domain whose boundary is a closed Jordan curve. A famous theorem of Caratheodory [8, page 44, Theorem 4] states that a Riemann map of U onto a Jordan domain R extends (uniquely) to a homeomorphism of the closed unit disc onto the closure of R. If φ is such a Riemann map, then we will denote its extension to the closed disc again by φ, and will refer to its values at particular points of the unit circle (*e.g.* "$\varphi(1) = 1$") without further explanation. The following result, which is needed in section 2, shows how the properties of the boundary of R influence the derivative of φ. Recall that a plane curve is called *smooth* if it has a tangent line which turns continuously as the point of tangency moves along the curve [8, Chapter X, sec. 1, page 423].

Theorem 1.2. *Let φ be a one-to-one conformal map of U onto a Jordan domain R.*

(*a*) [8, *page* 419, *Theorem* 1]. *If R has a rectifiable boundary, then φ' is in H^1, the boundary function $\varphi(e^{it})$ is absolutely continuous, and*

$$(d/dt)\varphi(e^{it}) = ie^{it}\varphi'(e^{it}) \text{ a.e.}$$

(*b*) [8, *page* 425, *Theorem* 5]. *If R has a smooth boundary, then φ' is in H^p for all $p < \infty$.*

We will often deal with curves in the closed unit disc which have points of contact with the unit circle and are smooth in a neighborhood of each such point. We say that such curves *touch the unit circle smoothly*.

Operator theory. A linear operator on a Banach space is called *compact* if it takes the unit ball into a set whose closure is compact. The same definition is used to define compact operators on spaces like H^p $(0 < p < 1)$. Clearly every compact operator is bounded, and the compact operators on a Banach space form a two-sided ideal in the ring of all bounded linear operators.

A linear operator T on an infinite dimensional Hilbert space is called a *Hilbert–Schmidt* operator if there exists an orthonormal basis (e_n) in H such that

$$\sum_n \|Te_n\|^2 < \infty \tag{1.1}$$

(cf. [4, Defn. XI.6.1, page 1010], [14, sec. 2.5, page 40], or [16, Chapter 2, page 29]). It is easy to see that every Hilbert–Schmidt operator is bounded, and that for an arbitrary linear operator T on H the (possibly infinite) sum on the right side of (1.1) does not depend on the particular choice of orthonormal basis (e_n) [14, sec. 2.5, page 41], [16, Chapter 2, page 29, Lemma 1]. A linear operator T on H is said to belong to the *trace class* [16, Chapter 3, Theorem 5, page 42] if there exist orthogonal sequences (e_n) and (f_n) in H with $\sum \|e_n\|\,\|f_n\| < \infty$, and

$$Tx = \sum \langle x, e_n\rangle f_n$$

for each x in H. Both the Hilbert–Schmidt and trace classes form two-sided idelas in the ring of bounded linear operators on H, both are proper subsets of the compact operators, and the trace class is a proper subset of the Hilbert–Schmidt class [16, Chapter 3, Theorem 2, page 41; and Theorem 5, page 42]. In section 5 we will require the following characterization of the trace class.

Theorem 1.3. [16, *Chapter* 3, *Theorem* 5, *page* 42]. *Suppose T is a linear operator on a Hilbert space H. Then the following three statements are equivalent*:

(*a*) *T is in the trace class.*
(*b*) *T is the composition of two Hilbert–Schmidt operators.*
(*c*) *There exists an orthonormal basis (e_n) for H such that* $\sum \|Te_n\| < \infty$.

In the last section of the paper we will discuss the following generalizations of Hilbert–Schmidt and trace class operators to arbitrary Banach spaces. Suppose F is a Banach space, T is a linear transformation on F, and $1 \leqq p < \infty$. Then T is called *absolutely p-summing* [13, page 333] if there is a constant $A > 0$ such that for any finite subset $f_1, f_2, \cdots, f_N$ of F;

$$\sum_{n=1}^{N} \|Tf_n\|^p \leqq A \sup \sum_{n=1}^{N} |\lambda(f_n)|^p,$$

where the supremum on the right extends over all λ in the unit ball of the dual of F. Clearly every absolutely p-summing operator is bounded, with norm $\leqq A^{1/p}$; moreover it is known that every absolutely p-summing operator on a reflexive Banach space is compact [13, Satz 14]. A striking result of Pełczynski [12] and Pietsch [13, Satz 11] asserts that an operator on Hilbert space is absolutely p-summing iff it is Hilbert–Schmidt. Thus the absolutely p-summing operators provide one way of generalizing Hilbert–Schmidt operators to arbitrary Banach spaces.

A linear transformation T on F is called *nuclear* if there exist sequences (f_n) and (λ_n) in F and the dual of F respectively such that

$$\sum \|\lambda_n\| \, \|f_n\| < \infty, \tag{1.2}$$

and for each f in F,

$$Tf = \sum \lambda_n(f) f_n \,. \tag{1.3}$$

Note that condition (1.2) insures that the series in (1.3) converges uniformly on the unit ball of F. It follows from this that every nuclear operator is compact [14, Satz 3.1.6, page 47].

Our definition of the trace class shows that every trace class operator on a Hilbert space is nuclear. In fact the converse is also true [7, sec. 2.3–2.5, pp. 37–56]: *an operator on Hilbert space is nuclear iff it is in the trace class.* We will need the following theorem, which relates these classes of operators.

Theorem 1.4. (*a*) [13, *Satz* 5, *page* 335]. *Every absolutely p-summing operator is absolutely q-summing for all* $1 \leqq p \leqq q$.

(*b*) [13, *Theorem* 5, *page* 347]. *The composition of two absolutely p-summing operators is absolutely $p/2$-summing.*

(*c*) [13, *Theorem* 6, *page* 348]. *The composition of two absolutely 2-summing operators is nuclear* (*cf. Theorem* 1.3).

(*d*) [14, *Satz* 3.2.5 *and Satz* 3.2.13]. *Every nuclear operator is absolutely 1-summing.*

In addition, both the absolutely p-summing and nuclear operators form two-sided ideals in the ring of bounded linear operators on F ([13, Satz 4, page 335], and [14, Satz 3.1.7, page 47]).

We close this section with some simple but extremely useful results about composition operators on H^p spaces.

Proposition 1.5. *Suppose I is a left ideal in the ring of bounded linear operators on H^p $(0 < p \leqq \infty)$, and suppose φ and ψ are analytic functions taking U into itself, where φ is one-to-one. If $C_\varphi \in I$ and $\psi(U) \subset \varphi(U)$, then $C_\psi \in I$.*

For example, suppose R is a simply connected domain contained in U. If the Riemann map of U onto R generates a compact (resp. trace class, Hilbert–Schmidt) composition operator on H^2, then so does any other analytic function taking U into R.

Proof of Proposition 1.5. We have $\psi = \varphi \circ \omega$, where $\omega = \varphi^{-1} \circ \psi$ is an analytic function taking U into itself. Thus $C_\psi = C_\omega C_\varphi$ where $C_\varphi \in I$ and C_ω is bounded. Since I is a left ideal, it follows that $C_\psi \in I$; and the proof is complete.

Finally we record two results due to Schwartz [17] concerning compact composition operators.

Proposition 1.6. (*a*) [17, *Theorem* 2.5, *page* 22]. *A composition operator C_φ is compact on H^p $(0 < p \leqq \infty)$ iff for every norm bounded sequence (f_n) in H^p which converges to zero uniformly on compact subsets of U, the image sequence $(C_\varphi f_n)$ converges to zero in norm.*

(*b*) [17, *Theorem* 2.6, *page* 23]. *If C_φ is compact on H^p, then $|\varphi(e^{it})| < 1$ a.e.*.

We omit the details of the straightforward proofs: part (a) follows quickly from the fact that bounded subsets of H^p are normal families, while part (b) is proved by applying C_φ to the functions z^n $(n = 0, 1, 2, \cdots)$, all of which lie in the unit ball of H^p.

2. Compact composition operators. From now on φ will always denote a non-constant analytic function taking U into itself. In this section we give sufficient conditions for both the compactness and the non-compactness of C_φ, and use these conditions to discuss some examples. We will see later (section 6) that the results of this section are valid for general H^p spaces $(p < \infty)$. For the present, however, we consider only the case $p = 2$.

Our sufficient condition for non-compactness is stated in terms of the *angular derivative* of φ. Suppose z^* is a point on the unit circle. We say φ has an *angular derivative* at z^* if there exists a point w^*, also on the unit circle, and a complex number c such that for any triangle $\triangle$ with interior contained in U and a vertex at z^* we have

$$\lim \frac{\varphi(z) - w^*}{z - z^*} = c \qquad (z \to z^*, z \text{ in } \triangle) \tag{2.1}$$

(cf. [1, sec. 299, page 32]). We emphasize that w^* is required to lie on the unit circle, so if φ has an angular derivative at z^*, then the boundary of $\varphi(U)$ touches the unit circle at the "image point" w^*. The following theorem illuminates the geometric significance of the angular derivative.

Theorem of Julia–Caratheodory [1, *sec.* 299, *page* 32], [11, *sec.* 5.3, *page* 57]. *There exists a number $0 < a \leqq \infty$ such that for every triangle $\triangle$ in U with a vertex at* 1,

$$\lim \left|\frac{\varphi(z) - 1}{z - 1}\right| = a \qquad (z \to 1, z \text{ in } \triangle).$$

If $a < \infty$ then (2.1) *holds with $z^* = w^* = 1$ and $c = a$.*

This result implies that a function φ which has an angular derivative at a point z^* on the unit circle is angle preserving there, in the sense that it maps a curve in U which terminates at z^* and makes an angle $0 < \alpha < \pi$ with the unit circle onto a curve which also terminates on the circle, and makes the same angle with it. In particular we would expect the boundary of $\varphi(U)$ to touch where the last integral converges because $\alpha > 1$. Thus R satisfies the hypotheses of the angular derivative theorem, so φ has an angular derivative at the point $z = 1$. Thus C_φ is not compact by Theorem 2.1.

We will see in section 4 that the conclusion of Corollary 2.3 can fail if we merely require $\alpha = 1$ in the hypothesis. In addition we will see that the condition "$\varphi' \in H^\infty$" in the hypothesis of Corollary 2.2 cannot be replaced by "$\varphi' \in H^p$ for all $p < \infty$." For some other results about the existence of the angular derivative we refer the reader to [8, page 426, Theorem 6] and [19].

We now give a sufficient condition for the composition operator induced by a univalent map to be compact.

Theorem 2.4. *Suppose Γ is a closed, rectifiable Jordan curve in the closed unit disc which touches the unit circle at exactly one point w_0, and is convex in a neighborhood of this point. Suppose φ maps U univalently onto the region interior to Γ with $\varphi(1) = w_0$. If there exists a set E of measure zero on the unit circle such that*

$$\lim |\varphi'(e^{it})| = \infty \quad (t \to 0,\ e^{it} \notin E), \tag{2.4}$$

then C_φ is a compact operator on H^2.

In section 4 we will use this result to give examples of univalent maps which induce compact composition operators on H^2, yet take U onto Jordan domains whose boundaries touch the unit circle smoothly. For the proof of the theorem we require the following generalization of the Fejer–Riesz inequality.

Gabriel's Theorem [6], [2]. *If Γ is a convex curve in the closed unit disc, then for every f in H^2,*

$$\int_\Gamma |f(w)|^2\, |dw| \leqq 4\pi\, \|f\|_2^2 .$$

Proof of Theorem 2.4. Suppose (f_n) is a sequence in the unit ball of H^2 which converges to zero uniformly on compact subsets of U. By Proposition 1.6 we will be done if we can show that $\|C_\varphi f_n\|_2 \to 0$. Let $\psi = \varphi^{-1}$ (composition inverse). By Theorem 1.2 we have $\varphi' \in H^1$, hence $|\varphi'(e^{it})| > 0$ *a.e.* [5, Theorem 2.2, page 17]. It follows readily from this that ψ' is an absolutely continuous function of arclength on Γ (cf. [8, pp. 419–421]), so the usual change of variable formula holds, and yields

$$2\pi\, \|C_\varphi f_n\|_2^2 = \int_{|z|=1} |f_n(\varphi(z))|^2\, |dz| = \int_\Gamma |f_n(w)|^2\, |\psi'(w)|\, |dw|. \tag{2.5}$$

Let $\epsilon > 0$ be given. Since φ is absolutely continuous on the unit circle, it takes sets of measure zero on the circle to sets of arclength measure zero on Γ. Thus we see from (2.4) that after correcting ψ on a set of arclength measure zero on Γ,

$$\lim \psi'(w) = 0 \quad (w \in \Gamma,\ w \to w_0),$$

hence $|\psi'| < \epsilon/4$ on some convex subarc Γ_ϵ of Γ which contains w_0 in its interior. From this we obtain for all n:

$$\int_{\Gamma_\epsilon} |f_n(w)|^2 \, |\psi'(w)| \, |dw| \leqq (\epsilon/4) \int_{\Gamma_\epsilon} |f_n(w)|^2 \, |dw| \leqq \epsilon\pi,$$

where the last inequality follows from Gabriel's Theorem, since $||f_n||_2 \leqq 1$. Since (f_n) converges to zero uniformly on compact subsets of U, and $\Gamma - \Gamma_\epsilon$ is compact, we have for all sufficiently large n:

$$\int_{\Gamma - \Gamma_\epsilon} |f_n(w)|^2 \, |\psi'(w)| \, |dw| \leqq (\epsilon/2) \int_{\Gamma - \Gamma_\epsilon} |\psi'(w)| \, |dw| \leqq \epsilon\pi,$$

where the last inequality follows from the fact that the integral on the right is, after a change of variable, just the integral of $|dz|$ over part of the unit circle. These estimates, together with (2.5) show that $||C_\varphi f_n||_2 < \epsilon$ for all sufficiently large n, which completes the proof.

We do not know if condition (2.4) in Theorem 2.4 can be replaced by the condition "φ does not have an angular derivative at any point of the unit circle." In fact, for arbitrary φ we do not know if the non-existence of the angular derivative of φ at each point of the unit circle is sufficient for C_φ to be compact. By Theorem 2.1 we know it is necessary.

3. Hilbert-Schmidt composition operators. In this section we show that condition (0.2) characterizes the Hilbert–Schmidt composition operators on H^2. We give some geometric applications of this result, and discuss Hilbert–Schmidt composition operators on the Dirichlet space.

Theorem 3.1. *C_φ is a Hilbert–Schmidt operator on H^2 iff φ satisfies condition* (0.2); *that is,*

$$\int_0^{2\pi} [1 - |\varphi(e^{it})|]^{-1} \, dt < \infty .$$

Proof. The functions

$$e_n(z) = z^n \quad (n = 0, 1, 2, \cdots)$$

form an orthonormal basis for H^2, so (cf. section 1) C_φ is a Hilbert–Schmidt operator on H^2 iff

$$\infty > 2\pi \sum_0^\infty ||C_\varphi e_n||_2^2 = \sum_0^\infty \int_0^{2\pi} |\varphi(e^{it})|^{2n} \, dt$$

$$= \int_0^{2\pi} [1 - |\varphi(e^{it})|^2]^{-1} \, dt,$$

and this completes the proof.

Corollary 3.2. *If φ takes U into a polygon inscribed in the unit circle, then C_φ is a Hilbert–Schmidt operator on H^2.*

Proof. Let P denote the polygon. By Proposition 1.5 we may assume without loss of generality that φ maps U univalently onto P. It is clearly enough to show that the preimage of each vertex of P is the center of an arc of the unit circle over which $(1 - |\varphi|)^{-1}$ is integrable. Without loss of generality we may suppose that our vertex is 1, and that $\varphi(1) = 1$. A standard local mapping argument based on the reflection principle [1, sec. 351, page 104] shows that there exists a neighborhood $\triangle$ of the point 1, a function h analytic and non-vanishing on $\triangle$, and a number $0 < \alpha < 1$ such that

$$1 - \varphi(z) = (1 - z)^\alpha h(z) \tag{3.1}$$

for all z in $\triangle$. Let Γ be a closed arc of the unit circle centered at 1, and contained in $\triangle$. Then h is bounded away from 0 on $\triangle$, and since $|1 - e^{i\theta}|/\theta \to 1$ as $\theta \to 0$, we see that there exists $\delta > 0$ such that

$$|1 - \varphi(e^{i\theta})| \geqq \delta\theta^\alpha$$

for all $e^{i\theta}$ in Γ. This shows that $|1 - \varphi|^{-1}$ is integrable over Γ. But $\varphi(\Gamma)$ approaches the point 1 non-tangentially, so on Γ the function $|1 - \varphi|$ is bounded by a constant multiple of $1 - |\varphi|$; which shows that $(1 - |\varphi|)^{-1}$ is also integrable over Γ. This completes the proof.

In the next section we will use Theorem 3.1 to exhibit univalent maps φ with $\|\varphi\|_\infty = 1$ which induce Hilbert–Schmidt operators on H^2, yet take U onto Jordan domains with smooth boundaries. In section 5 we will show that the hypothesis of Corollary 3.2 actually guarantees that C_φ is in the *trace class*. Right now we are going to use Theorem 3.1 to derive a crude, but useful geometric sufficient condition for C_φ to be a Hilbert–Schmidt operator.

Corollary 3.3. *Suppose R is a simply connected domain in U such that*

$$\iint_R (1 - |z|)^{-3}\, dx\, dy < \infty. \tag{3.2}$$

If φ takes U into R, then C_φ is Hilbert–Schmidt.

Before giving the proof we note that this result makes it easy to exhibit many domains R *not* contained in inscribed polygons for which C_φ is Hilbert–Schmidt whenever $\varphi(U) \subset R$. For example, it is easy to construct a ribbon R in the unit disc which spirals out to the unit circle and satisfies (3.1); or a simply connected domain R with infinitely many spikes terminating at the unit circle, which satisfies (3.1). On the other hand, note that Corollary 3.3 is not adequate to prove Corollary 3.2, since $(1 - |z|)^{-3}$ is not integrable over any inscribed polygon.

Proof of Corollary 3.3. We first introduce some notation. If $f(z) = \sum a_n z^n$ is analytic in U with $f(0) = 0$, let

$$(3.3) \qquad \|f\|_D^2 = \iint_U |f'|^2 \, dA = \sum n\, |a_n|^2,$$

where A is planar Lebesgue measure, normalized so that $A(U) = 1$. Thus if $f \in H^2$ and $f(0) = 0$, then $\|f\|_2 \leqq \|f\|_D$.

Now without loss of generality we may assume that $0 \in R$, $\varphi(0) = 0$, and (Proposition 1.5) φ maps U univalently onto R. Letting $e_n(z) = z^n$, we have

$$\|C_\varphi e_n\|_2 \leqq \|C_\varphi e_n\|_D \quad (n = 1, 2, \cdots),$$

hence

$$\begin{aligned}\sum_0^\infty \|C_\varphi e_n\|_2^2 &\leqq 1 + \sum_1^\infty \|C_\varphi e_n\|_D^2 \\ &= 1 + \sum_1^\infty \|\varphi^n\|_D^2 \\ &= 1 + \iint_U \left(\sum_1^\infty n^2\, |\varphi|^{2(n-1)}\right) |\varphi'|^2 \, dA \\ &\leqq 1 + 2 \iint_U (1 - |\varphi|^2)^{-3}\, |\varphi'|^2\, dA \\ &= 1 + 2 \iint_R (1 - |w|^2)^{-3}\, dA(w)\end{aligned}$$

where the last line follows from the fact that $|\varphi'|^2$ is the Jacobian of the transformation $w = \varphi(z)$. This completes the proof.

The quantity $\|f\|_D^2$ is usually called the *Dirichlet integral* of f, and the space D of functions f analytic in U with $f(0) = 0$ and $\|f\|_D < \infty$ is called the *Dirichlet space*. Clearly D is a Hilbert space with norm $\|\cdot\|_D$. The definition of Hilbert–Schmidt operator given in section 1 generalizes immediately to operators between different Hilbert spaces [14, sec. 2.5, page 40], and it is easy to see that the calculation which occurs in the proof of Corollary 3.3 actually shows that C_φ is a Hilbert–Schmidt operator from H^2 to D iff

$$\iint_U (1 - |\varphi|)^{-3}\, |\varphi'|^2\, dx\, dy < \infty .$$

If we consider operators from D into H^2, and D into itself, then using the orthonormal basis

$$e_n(z) = z^n / n^{1/2} \quad (n = 1, 2, \cdots)$$

for D, we obtain the following result, whose straightforward proof we omit.

Proposition 2.4. *(a) Suppose $\varphi(0) = 0$. Then C_φ is a Hilbert–Schmidt operator on D iff*

$$\iint_U (1 - |\varphi|^2)^{-2} |\varphi'|^2 \, dx \, dy < \infty. \tag{3.4}$$

(b) C_φ is a Hilbert–Schmidt operator from D to H^2 iff

$$\int_0^{2\pi} \log (1 - |\varphi(e^{it})|) \, dt > -\infty. \tag{3.5}$$

These results have amusing consequences. For example, suppose φ is univalent and $\varphi(0) = 0$. Then letting $R = \varphi(U)$ and $w = u + iv = \varphi(z)$, we transform condition (3.5) into

$$\iint_R (1 - |w|^2)^{-2} \, dy \, dv < \infty. \tag{3.6}$$

The quantity on the left side of (3.6) is just the *hyperbolic area* of R [18, Chapter XI, sec. 1, page 509], so we obtain:

Corollary 3.5. *If φ is univalent and fixes the origin, then C_φ is a Hilbert–Schmidt operator on D iff $\varphi(U)$ has finite hyperbolic area.*

On the other hand, Rudin and DeLeeuw have proved that φ is an *extreme point* in the unit ball of H^∞ iff condition (3.5) fails [5, Theorem 7.9, page 125]. Thus Proposition 3.4(b) can be rephrased as follows: *C_φ is a Hilbert–Schmidt operator from D into H^2 iff φ is not an extreme point of the H^∞ unit ball.*

4. Composition operators constructed from the functions $z(-\log z)^\alpha$. In this section we study a family φ_α $(\alpha > 0)$ of univalent maps taking U into itself, which are constructed in a natural way from the functions

$$f_\alpha(z) = z(-\log z)^\alpha \quad (\operatorname{Re} z > 0, \ |z| < 1), \tag{4.1}$$

where both the logarithm and power functions have branch line the negative real axis (note that $-\log z$ has positive real part for $|z| < 1$). We show that each φ_α maps U onto a Jordan domain whose boundary approaches the unit circle smoothly, that each φ_α induces a compact composition operator on H^2, and that φ_α induces a Hilbert–Schmidt operator on H^2 iff $\alpha > 2$. In particular this shows that there exist compact composition operators on H^2 which are not Hilbert–Schmidt; and that there are univalent maps which generate Hilbert–Schmidt composition operators on H^2, yet take U onto domains whose boundaries touch the unit circle smoothly.

The following lemma isolates the properties of f_α which are crucial for our purposes.

Lemma 4.1. *For $\epsilon > 0$ let*

$$H(\epsilon) = \{|z| < \epsilon, \ \operatorname{Re} z > 0\}.$$

For each $\alpha > 0$ *there exists* $0 < \epsilon_\alpha < 1$ *such that:*

(a) *the function* f_α *defined by equation* (4.1) *maps* $H(\epsilon_\alpha)$ *univalently onto a Jordan domain* R_α *contained in the open disc* $|w - 1| < 1$; *and the boundary of* R_α *touches the circle* $|w - 1| = 1$ *only at the origin;*

(b) *the boundary of* R_α *touches the circle* $|w - 1| = 1$ *smoothly, and is convex in a neighborhood of the origin;*

(c) *there exist numbers* $0 < m < M$ *such that*

$$m \leqq (1 - |1 - f_\alpha(iy)|)/y(-\log y)^{\alpha-1} \leqq M \tag{4.2}$$

for all $0 < y \leqq \epsilon_\alpha$.

We defer the proof of this lemma to the end of the section, and proceed immediately to the construction of the maps φ_α. Let g_α be a one-to-one conformal map of U onto $H(\epsilon_\alpha)$, with $g_\alpha(1) = 0$, and let

$$\varphi_\alpha(z) = 1 - f_\alpha(g_\alpha(z)) \quad (z \text{ in } U). \tag{4.3}$$

Thus $\varphi_\alpha(1) = 1$, and it follows from Lemma 4.1 that φ_α maps U univalently onto a Jordan domain in U whose boundary touches the unit circle only at the point 1, does so smoothly, and is convex in a neighborhood of 1.

Theorem 4.2. C_{φ_α} *is a compact operator on* H^2 *for all* $\alpha > 0$, *and is a Hilbert–Schmidt operator iff* $\alpha > 2$.

Proof. Fix $\alpha > 0$. For convenience we will drop the subscript α throughout this proof; hence $f_\alpha = f$, $\varphi_\alpha = \varphi$, $\epsilon_\alpha = \epsilon$, and $R_\alpha = R$. Since the boundary of R touches the unit circle only at the point 1, and is convex in a neighborhood of 1, it follows from Theorem 2.4 that the compactness of C_φ will follow once we verify that

$$\lim_{t\to 0} |\varphi'(e^{it})| = \infty. \tag{4.4}$$

Clearly $|f'(iy)| \to \infty$ as $y \to 0$, and the reflection principal guarantees that g extends conformally to a neighborhood of the point 1. Thus g' is bounded away from 0 near 1, so (4.4) follows from the chain rule, establishing the compactness of C_φ.

Since $\varphi(e^{it})$ is continuous and has modulus <1 for all $e^{it} \neq 1$, it follows from Theorem 3.1 that C_φ is a Hilbert–Schmidt operator on H^2 iff $(1 - |\varphi|)^{-1}$ is integrable over an arc of the unit circle centered at 1. Let h be the (composition) inverse of g. By the reflection principle h extends conformally to a neighborhood of the origin, so both h' and its reciprocal are bounded on an interval $[-i\delta, i\delta]$ of the imaginary axis ($\delta > 0$). Choosing δ smaller than the number $\epsilon = \epsilon_\alpha$ of Lemma 4.1, performing the change of variable $\xi + i\eta = g(z)$, and using the boundedness of h' and its reciprocal on $[-i\delta, i\delta]$, we see that C_φ is Hilbert–Schmidt iff

$$\int_{-\delta}^{\delta} [1 - |1 - f(i\eta)|]^{-1}\, d\eta < \infty.$$

Now $f(-i\eta) = \overline{f(i\eta)}$, so by inequality (4.2) of Lemma 4.1, the convergence of the integral above is equivalent to

$$\int_0^\delta \eta^{-1}(-\log \eta)^{-\alpha+1}\, d\eta < \infty,$$

which holds precisely when $\alpha > 2$. This completes the proof.

Corollary 4.3. *There exist compact composition operators on H^2 which are not Hilbert–Schmidt.*

By rounding off the corners that $H(\epsilon_\alpha)$ has at the points $\pm i\epsilon_\alpha$ in such a way that the new domain has a smooth boundary and still contains an interval of the imaginary axis centered at the origin, we can modify the maps φ_α so that they satisfy Theorem 4.2, and now take U onto Jordan domains with smooth boundaries. Thus we obtain:

Corollary 4.4. *There exist univalent maps φ such that C_φ is a Hilbert–Schmidt operator on H^2, yet $\|\varphi\|_\infty = 1$ and $\varphi(U)$ has a smooth boundary.*

In the next section we will show that Corollary 4.4 remains true with "trace class" replacing "Hilbert–Schmidt". Note that if φ is one of the maps mentioned in Corollary 4.4, then $\varphi' \in H^p$ for all $p < \infty$ by Theorem 1.2(b). This shows that Corollary 1.2 fails if the hypothesis "$\varphi' \in H^\infty$" is replaced by "$\varphi' \in H^p$ for all $p < \infty$." In addition, it is easy to check that φ_1 maps U onto a domain whose boundary, near the origin, has an equation of the form (2.3) with $\alpha = 1$. Thus Corollary 2.3 fails for $\alpha = 1$.

We now turn to the proof of Lemma 4.1. Although the analysis of f_α is completely elementary, the factor $(-\log z)^\alpha$ seems to complicate matters when $\alpha \neq 1$, and makes the proof rather lengthy.

Proof of Lemma 4.1. As in the proof of Theorem 4.2 we drop the subscript α, and write $f = f_\alpha$, *etc.* For $0 < \epsilon < 1$ formula (4.1) defines f on the closure of $H(\epsilon)$, except at the origin. By defining $f(0) = 0$ we extend f continuously to the closure of $H(\epsilon)$.

We are going to show that the conclusion of Lemma 4.1 holds with $\epsilon = \epsilon_\alpha$ chosen so that

$$r(-\log r)^{\alpha+1} \leqq \min(2^{-\alpha}, 2^{-\alpha-1}), \tag{4.6}$$

and

$$-\log r \geqq 2\alpha + (\pi/2), \tag{4.7}$$

whenever $0 < r \leqq \epsilon$. For the rest of the proof we assume ϵ has been so chosen. We write $H = H(\epsilon)$, and denote the closure of H (with respect to the plane) by Cl (H). We denote the argument of $-\log z$ by $\gamma(z)$. Note that $|\gamma(z)| < \pi$ for z in the complex plane slit along the negative real axis. For z in Cl (H) we have the following important estimate on $\gamma(z)$:

$$|\gamma(z)| < \min(1, \pi/4\alpha) \quad (z \text{ in Cl } (H)). \tag{4.8}$$

To prove this, note that

$$|\gamma(z)| \leqq |\tan \gamma(z)| = |(\arg z)/\log |z| \,| \leqq (\pi/2)/(2\alpha + \pi/2),$$

where the last inequality follows from (4.7). This immediately yields (4.8).

Proof of part (*a*). We are going to show that f is one-to-one on Cl (H), hence f maps H univalently onto a Jordan domain. For this it is enough to show that Re $f' > 0$ on H. For once this last inequality has been established, then given z_1 and z_2 distinct points in Cl (H) $-$ $\{0\}$ we write

$$z(t) = tz_2 + (1 - t)z_1 \quad (0 \leqq t \leqq 1).$$

and get

$$\begin{aligned} |f(z_2) - f(z_1)| &= \left| \int_0^1 f'(z(t))z'(t)\,dt \right| \\ &= |z_2 - z_1| \left| \int_0^1 f'(z(t))\,dt \right| \\ &\geqq |z_2 - z_1| \int_0^1 \operatorname{Re} f'(z(t))\,dt. \end{aligned}$$

Now if $0 < t < 1$, then $z(t) \in H$, hence Re $f'(z(t)) > 0$. Thus the last integral is not zero, so $f(z_2) \neq f(z_1)$, as desired. Clearly $f(z) = 0$ iff $z = 0$, so f is univalent on Cl (H). To see that Re $f' > 0$ on H we compute f', and obtain

$$\arg f'(z) = (\alpha - 1)\gamma(z) + \arg(-\alpha - \log z). \tag{4.9}$$

It follows from (4.7) that

$$\operatorname{Re}(-\alpha - \log z) \geqq \pi/2 > \operatorname{Im}(-\alpha - \log z),$$

so that

$$\arg(-\alpha - \log z) < \pi/4.$$

If $\alpha < 1$, then $(\alpha - 1)\gamma(z)$ and $\arg(-\alpha - \log z)$ have different signs, and since they are both less than $\pi/2$ in magnitude we have $|\arg f'(z)| < \pi/2$. On the other hand, if $\alpha \geqq 1$, then

$$|\arg f'(z)| < |(\alpha - 1)\gamma(z)| + \pi/4 < \pi/4 + \pi/4 = \pi/2,$$

where the last inequality follows from (4.8). Thus for all $\alpha > 0$ we have $|\arg f'| < \pi/2$, that is, Re $f' > 0$ on H.

We complete the proof of part (a) by showing that f maps Cl (H) $-$ $\{0\}$ into the open disc $|w - 1| < 1$. First we show that the intervals $[-i\epsilon, 0)$ and $(0, i\epsilon]$ of the imaginary axis go into this disc. Since $f(\bar{z}) = \overline{f(z)}$, it is enough to consider the point $f(iy)$ for $0 < y \leqq \epsilon$. For such y we employ the simplified notation $u = \operatorname{Re} f(iy)$, $v = \operatorname{Im} f(iy)$, and $\gamma = \gamma(iy)$. Now

$$\begin{aligned} u &= -y\,|\log iy|^{\alpha} \sin \alpha\gamma \\ v &= y\,|\log iy|^{\alpha} \cos \alpha\gamma, \end{aligned} \tag{4.10}$$

and we note that $\gamma < 0$, so $u > 0$. We claim that

$$(4.11) \qquad 1 \geqq u \geqq 2v^2.$$

From this it follows easily that

$$(4.12) \qquad |1 - w|^2 \leqq (1 - u)^2 + u/2 < (1 - (u/4))^2 < 1,$$

hence $f(iy)$ lies in the desired disc.

To prove (4.11) recall that $x \geqq \sin x \geqq 2x/\pi$ for $0 \leqq x \leqq \pi/2$. Since $|\alpha\gamma| < \pi/4$ by (4.8), we have

$$(4.13) \qquad \sin \alpha\, |\gamma| \geqq (2/\pi)\alpha\, |\gamma| \geqq (2/\pi)\alpha \sin |\gamma|.$$

But

$$(4.14) \qquad \sin |\gamma| = (\pi/2)/|\log iy|,$$

so from (4.10), (4.13), and (4.14) we obtain

$$u/v^2 \geqq \sin \alpha\, |\gamma|/y\, |\log iy|^\alpha.$$

Now (4.7) implies that

$$(4.15) \qquad |\log iy| \leqq 2(-\log y),$$

and the last two inequalities yield

$$u/v^2 \geqq \alpha/2^{\alpha+1}y(-\log y)^{\alpha+1}.$$

By (4.6), the right side of this inequality exceeds 2, which proves the second inequality in (4.11). To prove the first, note that $u \leqq y\, |\log iy|^\alpha$ by (4.10), so by (4.15),

$$u \leqq 2^\alpha y(-\log y)^\alpha \leqq 2^\alpha y(-\log y)^{\alpha+1},$$

where the last inequality follows from the fact that $-\log y \geqq 1$, which in turn follows from (4.7). By (4.6) the last quantity in the above inequality is $\leqq 1$, which completes the proof of (4.11).

So far we have shown that f maps the part of the boundary of H which lies on the imaginary axis, except for the origin, into the disc $|w - 1| < 1$. We now show that for all z in Cl $(H) - \{0\}$ the point $f(z)$ lies in this disc. Because $f(\bar{z}) = \overline{f(z)}$ we may assume that Im $z \leqq 0$. Write $z = re^{-i\theta}$, where $0 \leqq \theta < \pi/2$. We will show that $|f(z)| \leqq |f(-ir)|$ and $\arg f(z) \geqq \arg f(-ir)$. Since both $f(z)$ and $f(-ir)$ lie below the real axis, and $f(-ir)$ is in the disc $|w - 1| < 1$, so is $f(z)$, and we will be done. Now

$$|f(z)| = r\, |(-\log r) + i\theta|^\alpha \leq r\, |(-\log r) + i\pi/2|^\alpha$$
$$= |f(-ir)|,$$

which proves the assertion about the moduli. To study the arguments it is helpful to consider the right triangles with vertices $\{0, -\log r, -\log(-ir)\}$

and $\{0, -\log r, -\log z\}$, noting that $-\log z$ lies on the side joining $-\log r$ and $-\log(-ir)$. Since $\gamma(z) \leqq \gamma(-ir)$, and $\tan x/x$ increases with x for $0 \leqq x < \pi/2$, we see that

$$\tan \gamma(z)/\gamma(z) \leqq \tan \gamma(-ir)/\gamma(-ir).$$

From the triangles we have

$$\tan \gamma(z) = \theta/(-\log r) \quad \text{and} \quad \tan \gamma(-ir) = (\pi/2)/(-\log r),$$

so

$$\theta/\gamma(z) \leqq (\pi/2)/\gamma(-ir).$$

Thus

$$[\gamma(-ir) - \gamma(z)]/\gamma(-ir) \leqq (\pi/2 - \theta)/(\pi/2),$$

and it follows easily from this and the fact that $\alpha\gamma(-ir) \leqq \pi/4$ (inequality (4.8)) that

$$\alpha[\gamma(-ir) - \gamma(z)] \leqq (\pi/2) - \theta,$$

that is,

$$\alpha\gamma(-ir) - (\pi/2) \leqq \alpha\gamma(z) - \theta.$$

But the left side of the last inequality is $\arg f(z)$, and the right side is $\arg f(-ir)$, so the proof of part (a) is complete.

Proof of part (b). Let $R = f(H)$. Since the line from the origin to $f(iy)$ has slope $-\text{ctn}\,\gamma(iy)$ for $0 < |y| < \epsilon$, and since $\gamma(iy) \to 0$ as $y \to 0$, it follows that the boundary of R has a vertical tangent at the origin. We are going to show that $\arg f'(iy)$ increases to zero as y decreases to zero. Since the angle between the horizontal and the line tangent to ∂R at $f(iy)$ is just $\pi/2 + \arg f'(iy)$, and since R is symmetric about the horizontal axis, this will show that the boundary of R is convex and touches the circle $|w - 1| = 1$ smoothly at the origin. Let

$$\psi(iy) = \arg(-\alpha - \log iy),$$

so by (4.9) we have

$$\arg f'(iy) = \alpha\gamma(iy) + [\psi(iy) - \gamma(iy)] \tag{4.16}$$

for $0 < y < \epsilon$. Write $\gamma = \gamma(iy)$ and $\psi = \psi(iy)$; and note that both γ and ψ are negative. Now consider the triangle whose vertices are 0, $-\log iy$, and α. Clearly the angles at the vertices 0 and $-\log iy$ decrease to zero as y decreases to zero. But these angles are just $-\gamma$ and $\gamma - \psi$ respectively, hence by (4.16) we have $\arg f(iy) \uparrow 0$ as $y \downarrow 0$, which completes the proof of part b.

Proof of part (c). As in the proof of part (a) we use the notation $u = \text{Re}\, f(iy)$, $v = \text{Im}\, f(iy)$, and $\gamma = \gamma(iy)$ $(0 < y \leqq \epsilon)$. In addition we write $f = f(iy)$. Finally, if F and G are positive functions on $(0, \epsilon]$, we write $F \sim G$ to denote the fact that both F/G and G/F are bounded on $(0, \epsilon]$. Note that the relation $\sim$ is reflexive, symmetric and transitive. We want to show that

$$1 - |1 - f| \sim y(-\log y)^{\alpha-1}.$$

First we claim that

$$1 - |1 - f| \sim u. \tag{4.17}$$

Indeed, we saw in (4.12) that

$$|1 - f| < 1 - u/4 < 1.$$

The first of these inequalities yields

$$u/4 < 1 - |1 - f|,$$

while the fact that $|1 - f| < 1$ gives

$$|1 - f| > |1 - f|^2 > 1 - 2u,$$

which proves (4.17). Now (4.13), (4.14), and (4.15) imply that

$$\sin \alpha\gamma \sim \alpha \sin \gamma = \alpha\pi/2 \, |\log iy| \sim 1/(-\log y),$$

so from (4.10), (4.15), and (4.17) we obtain the relations

$$1 - |1 - f| \sim u \sim y(-\log y)^{\alpha} \sin \alpha\gamma \sim y(-\log y)^{\alpha-1},$$

which completes the proof of the lemma.

5. Trace class composition operators on $\mathbf{H}^2$. In section 3 we saw that every analytic function taking U into a polygon inscribed in the unit circle induces a Hilbert–Schmidt composition operator on H^2 (Corollary 3.2). The main result of this section (Theorem 5.1) is that such a function actually induces a *trace class* operator. In addition we prove an analogue of Corollary 3.3 which gives a geometric sufficient condition for a composition operator to be in the trace class, and we exhibit a univalent φ such that C_φ is in the trace class, but $\partial\varphi(U)$ touches the unit circle smoothly.

Theorem 5.1. *If φ takes U into a polygon inscribed in the unit circle, then C_φ is a trace class operator on H^2.*

Proof. Suppose P is the polygon. We are going to find one-to-one conformal maps χ and ψ of U into polygons inscribed in the unit circle such that $P = \chi(\psi(U))$. Suppose for the moment this has been done. Then it follows from Corollary 3.2 that C_ψ and C_χ are Hilbert–Schmidt operators on H^2, so $C_{\chi\circ\psi} = C_\psi C_\chi$ is in the trace class by Theorem 1.3(b), hence so is C_φ by Proposition 1.5.

To find χ and ψ we proceed as follows. For each pair of adjacent vertices of P choose a point on the unit circle strictly between them, and let Q be the inscribed polygon whose vertices are those of P and the points just chosen. Thus Q has twice as many vertices as P, and except for its vertices the closure of P lies in (the interior of) Q. Let χ be a one-to-one conformal map of U onto Q, and let $R = \chi^{-1}(P)$. Then R is a Jordan domain contained in the unit disc whose boundary touches the unit circle only at the inverse images by χ of the vertices of P. We are going to show that R lies in a polygon inscribed in the unit circle. Then we will be done, since letting ψ be a Riemann map of U onto R we will have

$$\chi(\psi(U)) = \chi(R) = P,$$

where χ and ψ map U into inscribed polygons.

To prove that R lies in an inscribed polygon we need only show that whenever ∂R touches the unit circle, it does so non-tangentially. Suppose ∂R touches the unit circle at z_0, and let $\beta\pi$ be the interior angle of Q at the vertex $\chi(z_0)$, so $0 < \beta < 1$. Since χ takes U conformally onto Q, a standard local mapping argument based on the reflection principle shows that at z_0 the map χ multiplies all angles by β [1, sec. 351, page 104]. Now at $\chi(z_0)$ the sides of P make strictly positive angles α and γ with the sides of Q, so the boundary curves of R at z_0 make strictly positive angles α/β and γ/β with the unit circle. This completes the proof.

The next result strengthens Corollary 4.4 and shows that there exist simply connected domains R in the unit disc *not* contained in any inscribed polygon, for which C_φ is in the trace class whenever $\varphi(U) \subset R$.

Proposition 5.2. *There exists a one-to-one conformal map of U into itself which induces a trace class composition operator on H^2, yet takes U onto a Jordan domain whose boundary touches the unit circle smoothly.*

Proof. Fix $\alpha > 2$, let $\psi = \varphi_\alpha \circ \varphi_\alpha$ (where φ_α is the map defined in section 4), and let $R = \psi(U)$. Then ψ is a one-to-one conformal map of U into itself, ∂R touches the unit circle only at the point 1, and $C_\psi = C_{\varphi_\alpha}{}^2$ is in the trace class by Theorems 1.3 and 4.2. So we will be done if we show that the boundary of R is smooth in a neighborhood of the point 1. Since φ_α is analytic in a deleted neighborhood of the point 1, so is ψ; hence ∂R is smooth in a deleted neighborhood of 1, and we will be done if we can show that its tangent turns continuously at 1. But this follows quickly from the fact that arg φ_α' can be extended continuously to $|z| \leqq 1$ by defining it to be zero at the point 1; which in turn follows from the fact that arg f_α' extends continuously to the closure of $H(\epsilon_\alpha)$ (notations as in the statement of Lemma 4.1) by defining arg $f_\alpha'(0) = 0$. This completes the proof.

Our next result is an analogue of Corollary 3.3 for trace class operators.

Proposition 5.3. *Suppose R is a simply connected domain in U such that*

$$\iint_R (1 - |z|)^{-\delta}\, dx\, dy < \infty$$

for some $\delta > 4$. If $\varphi(U) \subset R$, then C_φ is a trace class operator on H^2.

Proof. By Theorem 1.3 we will be done if we can show that $\sum ||C_\varphi e_n||_2 < \infty$, where $e_n(z) = z^n$. Given $\delta > 4$ set $\delta = 2\alpha + 3$, so $\alpha > 1/2$. Then

$$\sum_1^\infty ||C_\varphi e_n||_2 = \sum_1^\infty ||\varphi^n||_2 \leqq \left(\sum_1^\infty n^{2\alpha} \, ||\varphi^n||_2^2\right)^{1/2} \left(\sum_1^\infty n^{-2\alpha}\right)^{1/2}$$

by the Cauchy-Schwartz inequality. By the reasoning used in the proof of Corollary 3.3, the first sum on the right is majorized by

$$\sum_1^\infty n^{2\alpha} \, ||\varphi^n||_D^2 = \iint_U \left(\sum_1^\infty n^{2(\alpha+1)} \, |\varphi|^{2(n-1)}\right) |\varphi'|^2 \, dx \, dy$$

where $||\cdot||_D^2$ is the Dirichlet integral; while the second sum converges because $\alpha > 1/2$. As in the proof of Corollary (3.3), the last integral converges iff

$$\iint_U (1 - |\varphi|^2)^{-2\alpha-3} \, |\varphi'|^2 \, dx \, dy < \infty,$$

so the change of variable $w = \varphi(z)$ finishes the proof.

For example, we could take R to be a ribbon in the unit disc which spirals out to $|z| = 1$ in such a way that $(1 - |z|)^{-5}$ is integrable over it. Then any φ taking U into R would induce a trace class operator on H^2. On the other hand it is clear that Proposition 5.3 is too crude to prove Theorem 5.1.

We do not know if there exist Hilbert-Schmidt composition operators which are not in the trace class. We suspect that such operators do exist, and that the required examples can be found among the maps φ_α of section 4. More specifically we conjecture that φ_α is in the trace class iff $\alpha > 4$.

6. Generalizations to H^p. In this section we prove analogues of the previous results for general H^p spaces. We show that C_φ is compact on *every* H^p space as soon as it is compact on one of them (Theorem 6.1); that condition (0.2) characterizes the *absolutely p-summing* operators on H^p, at least for $2 \leqq p < \infty$ (Theorem 6.2); and that an analytic function which takes U into a polygon inscribed in the unit circle induces a *nuclear* composition operator on H^p $(1 \leqq p < \infty)$.

We do not consider composition operators on H^∞, since from our point of view they have been satisfactorily classified by Schwartz [17, Theorem 2.8, page 28], who shows that C_φ is compact on H^∞ iff $||\varphi||_\infty < 1$. Thus if C_φ is compact on H^∞, then it is actually nuclear.

Theorem 6.1. *If C_φ is compact on H^p for some $p < \infty$, then it is compact on H^p for all $p < \infty$.*

This theorem shows that the results of section 2 are valid for all $p < \infty$. For example, if φ has an angular derivative at some point of the unit circle, then C_φ is not compact on any H^p; while if φ satisfies the hypothesis of Theorem 2.4, then C_φ is compact on every H^p $(0 < p < \infty)$. In particular the maps φ_α discussed in section 4 induce compact composition operators on H^p for all

$p < \infty$. We remark that these results can also be proved directly by simple modification of the proofs given for $p = 2$.

For the proof of Theorem 6.1 we need the following two theorems. The first is an important factorization result for H^p functions, while the second is a standard result from measure theory.

Riesz Factorization Theorem [5, *Theorem* 2.5, *page* 20]. *If $f \in H^p$ $(0 < p < \infty)$, then $f = BF$ where $B \in H^\infty$, $|B(e^{it})| = 1$ a.e.; $F \in H^p$ with $\|F\|_p = \|f\|_p$, and F has no zeroes in U.*

Vitali Convergence Theorem [3, *Theorem* III, 6.15, *page* 150], [9, *page* 108, *Theorem* C]. *Let $0 < p < \infty$, let μ be a finite (positive) measure, and let (f_n) be a sequence in $L^p(\mu)$ which converges a.e. $[\mu]$. Then (f_n) converges in $L^p(\mu)$ iff*

$$\lim_{\mu(E)\to 0} \sup_n \int_E |f_n|^p \, d\mu = 0. \tag{6.1}$$

In [9] this theorem is given for $p = 1$, and in [3] for $1 \leqq p < \infty$. However its extension to $0 < p < 1$ is immediate.

Proof of Theorem 6.1. Fix $0 < p, q < \infty$, and suppose C_φ is a compact operator on H^p. Let (f_n) be a sequence in the H^q unit ball which converges to zero uniformly on compact subsets of U. By Proposition 1.5(a) we need only show that $\|f_n \circ \varphi\|_q \to 0$. Now $f_n = B_n F_n$, where B_n and F_n satisfy the conclusion of the Riesz Factorization Theorem. In particular both sequences (B_n) and (F_n) lie in the H^q unit ball, so by Theorem 1.1 they are normal families. Thus there exists a sequence of positive integers $n_i \uparrow \infty$ such that both (F_{n_i}) and (B_{n_i}) converge uniformly on compact subsets of U. Since F_{n_i} never vanishes on U, it has an analytic q/p-th power G_i in U. Clearly the sequence (G_i) lies in the H^p unit ball and converges at each point of U.

Since C_φ is compact on H^p, we have $|\varphi(e^{it})| < 1$ *a.e.* by Proposition 1.5(b), so the sequence $(G_i \circ \varphi)$ converges *a.e.* on the unit circle. Since C_φ is compact on H^p, it follows by Proposition 1.5(a) that $(G_i \circ \varphi)$ actually converges in the norm of H^p, so by the Vitali Convergence Theorem it satisfies condition (6.1), where μ is Lebesgue measure on the unit circle. Since $|G_i \circ \varphi|^p = |F_{n_i} \circ \varphi|^q$, and $|B_{n_i}(e^{it})| = 1$ *a.e.*, the sequence $(f_{n_i} \circ \varphi)$ also satisfies condition (6.2) with q replacing p. Since $f_n \to 0$ uniformly on compact subsets of U, and $|\varphi(e^{it})| < 1$ *a.e.*, we see in addition that $f_n \circ \varphi \to 0$ *a.e.* on the unit circle, so another application of Vitali's Theorem shows that $\|f_n \circ \varphi\|_q \to 0$, which completes the proof.

It would be of interest to know if results of this sort are true for other classes of composition operators. For example, is C_φ nuclear on every H^p space as soon as it is nuclear on one of them $(1 \leqq p < \infty)$?

Our next result generalizes Theorem 3.1. To place it in its proper setting, recall that the Theorem of Pełczynski and Pietsch mentioned in section 1 implies that a linear operator on Hilbert space is absolutely 2-summing iff it is Hilbert–Schmidt.

Theorem 6.2. *If φ satisfies condition* (0.2) *then C_φ is absolutely p-summing on H^p for $1 \leqq p < \infty$, and nuclear on H^1. Conversely, if C_φ is absolutely p-summing on H^p for some $2 \leqq p < \infty$, then φ satisfies condition* (0.2).

Thus for $2 \leqq p < \infty$, C_φ is absolutely p-summing on H^p iff $(1 - |\varphi|)^{-1}$ is integrable over the unit circle. In particular, all the results of sections 3 and 4 remain true when H^p replaces H^2 and "absolutely p-summing" replaces "Hilbert–Schmidt," at least when $2 \leqq p < \infty$. We do not know if condition (0.2) is necessary for C_φ to be absolutely p-summing on H^p when $1 \leqq p < 2$. This may be too much to expect, since (0.2) implies that C_φ is actually nuclear on H^1. However we have not been able to show that there are absolutely p-summing composition operators on H^p which are not nuclear, even in the case $p = 1$. For example, we do not know if the operators C_{φ_α} discussed in section 4 are nuclear on H^p for large enough α, or if they fail to be absolutely p-summing on H^p for $1 \leqq p, \alpha \leqq 2$.

Proof of Theorem 6.2. Suppose C_φ is absolutely p-summing on H^p for some $2 \leqq p < \infty$. Let $e_n(z) = z^n$ $(n = 0, 1, 2, \cdots)$, and suppose that λ is a continuous linear functional on H^p of norm $\leqq 1$. By the Hahn–Banach and Riesz theorems there exists h in L^q $(p^{-1} + q^{-1} = 1)$ such that $||h||_q \leqq 1$ and

$$\lambda(f) = \int fh\, d\sigma \qquad (f \text{ in } H^p),$$

where σ is normalized Lebesgue measure on the unit circle. Thus $\lambda(e_n)$ is the $-n^{\text{th}}$ Fourier coefficient of h $(n = 0, 1, 2, \cdots)$, and since $1 \leqq q < 2$, the Hausdorff–Young Inequality yields

$$\sum_0^N |\lambda(e_n)|^p \leqq ||h||_q^p \leqq 1 \qquad (N = 0, 1, 2, \cdots).$$

Since C_φ is absolutely p-summing on H^p, this implies that there exists $A > 0$ such that

$$A \geqq \sum_0^\infty ||C_\varphi e_n||_p^p = \sum_0^\infty ||\varphi^n||_p^p = \int (1 - |\varphi|^p)^{-1}\, d\sigma$$

Since the function $(1 - x)/(1 - x^p)$ is bounded away from zero for $0 < x < \infty$, this shows that C_φ satisfies condition (0.2).

Conversely suppose $1 \leqq p < \infty$, and C_φ satisfies condition (0.2). Then, except for a subset E of the unit circle having measure zero, we have $|\varphi(e^{it})| < 1$. Thus by Theorem 1.1 we have

$$|C_\varphi f(e^{it})| \leqq 2^{1/p}[1 - |\varphi(e^{it})|]^{-1/p}\, ||f||_p \tag{6.2}$$

for all $e^{it} \notin E$ and all $f \in H^p$. Let

$$\gamma(e^{it}) = 2^{1/p}[1 - |\varphi(e^{it})|]^{-1/p},$$

and for $e^{it} \notin E$ let

$$\lambda_t(f) = C_\varphi f(e^{it})/\gamma(e^{it}) \quad (f \in H^p).$$

Condition (6.2) implies that λ_t is a bounded linear functional on H^p with norm

$\leqq 1$; so given $f_1, \cdots, f_N$ in H^p we have for $e^{it} \notin E$:

$$\sum_{n=1}^{N} |C_\varphi f_n(e^{it})|^p = \gamma(e^{it})^p \sum_{n=1}^{N} |\lambda_t(f_n)|^p$$
$$\leqq \gamma(e^{it})^p \sup \sum_{n=1}^{N} |\lambda(f_n)|^p,$$

where the supremum is extended over all λ in the unit ball of the dual of H^p. By condition (0.2) we have $\gamma \in L^p$, so integrating both sides of the above inequality over the unit circle we obtain

$$\sum_{n=1}^{N} ||C_\varphi f_n||_p^p \leqq ||\gamma||_p^p \sup \sum_{n=1}^{N} |\lambda(f_n)|^p,$$

which shows that C_φ is absolutely p-summing.

To see that C_φ is nuclear on H^1 when φ satisfies (0.2) note that formally

$$C_\varphi f = \sum_{n=0}^{\infty} \lambda_n(f)\varphi^n,$$

where λ_n is the n^{th} Taylor coefficient of f. A simple estimate shows that for all n, the linear functional λ_n is bounded on H^1 and has norm 1, so

$$\sum_{0}^{\infty} ||\lambda_n||\ ||\varphi^n||_1 = \int (1 - |\varphi|)^{-1}\, d\sigma < \infty,$$

which completes the proof.

We remark that the proof that C_φ is absolutely p-summing on H^p whenever φ satisfies (0.2) is really a special case of the proof of the following result of A. Shields, L. Wallen, and J. Williams:

Theorem. *Suppose F is a normed space, μ is a measure and $T : F \to L^p(\mu)$ is a linear operator. If there exists $\gamma \in L^p(\mu)$ such that $\gamma \geqq 0$ and for each f in F,*

$$|Tf(x)| \leqq \gamma(x)\ ||f||\ a.e.\ [\mu]$$

(where the exceptional set may depend on f), then T is absolutely p-summing.

A proof of this theorem is given in [20, Corollary 11]. and depends on a rather lengthy development. The original proof (unpublished) is more elementary, and forms the basis of the argument we have just presented. Indeed, the fact that C_φ is absolutely p-summing whenever φ satisfies (0.2) follows quickly from Theorem F and estimate (6.2).

Our next result generalizes Theorem 5.1 to arbitrary H^p spaces ($1 \leqq p < \infty$).

Theorem 6.3. *If φ takes U into a polygon inscribed in the unit circle, then C_φ is nuclear on H^p ($1 \leqq p < \infty$).*

Proof. Suppose $\varphi(U)$ lies in an inscribed polygon P. Then, as we saw in the proof of Corollary 3.2, the function φ satisfies condition (0.2), so by Theorem 6.2, C_φ is nuclear on H^1 and absolutely p-summing on H^p for $1 < p < \infty$. Now Theorem 1.4 and the argument used to prove Theorem 5.1 show that $\varphi(U) \subset P$

implies C_φ is absolutely $q/2$-summing on H^p as soon as it is absolutely q-summing, and is nuclear as soon as it is absolutely 2-summing. But C_φ is already absolutely p-summing on H^p, so iterating this last observation we see that it is actually nuclear, which completes the proof.

We comment that the method used to prove Proposition 5.2 shows that given $1 \leqq p < \infty$, there exists a univalent φ such that C_φ is nuclear on H^p, yet $\varphi(U)$ is a Jordan domain whose boundary touches the unit circle smoothly.

Finally we remark that the proof of Theorem 6.3 shows that whenever φ takes U into an inscribed polygon, then C_φ, viewed as an operator on H^2, belongs to every class c_p (see [10, page 251] or [4, Defn. XI.9.1, page 1089] for the definition of c_p).

References

1. C. Caratheodory, *Theory of functions*, Vol. II, Chelsea, New York, 1960.
2. F. Carlson, *Quelques inégalitiés concernant les fonctions analytiques*, Ark. for Matematik, Astron. och Fysik **29** No. 11 (1943), 1–5.
3. N. Dunford & J. T. Schwartz, *Linear operators, part I, General theory*, Wiley (Interscience), New York, 1958.
4. N. Dunford & J. T. Schwartz, *Linear operators, part II, Spectral theory*, Wiley (Interscience), New York 1963.
5. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
6. R. M. Gabriel, *Some results concerning the moduli of regular functions along curves of certain types*, Proc. London Math. Soc. **28** (1928), 121–127.
7. I. M. Gel'fand & N. Ya. Vilenkin. *Generalized functions, vol.* **4**, *Applications to harmonic analysis*, Academic Press, New York, 1964.
8. G. M. Goluzin, *Geometric theory of functions of a complex variable*. American Math. Society, Providence, 1969.
9. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950.
10. C. McCarthy, c_p, Israel J. Math **5** (1967), 249–271.
11. R. Nevanlinna, *Analytic functions*, Springer-Verlag, New York, 1970.
12. A. Pełczynski, *A characterization of Hilbert-Schmidt operators*, Studia Math. **28** (1967), 355–360.
13. A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, Studia Math. **28** (1967), 333–353.
14. A. Pietsch, *Nukleare lokalkonvexe Räume*, Akademie-Verlag, Berlin, 1969.
15. J. V. Ryff, *Subordinate H^p functions*, Duke Math. J. **33** (1966), 347–354.
16. R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin 1960.
17. H. J. Schwartz, *Composition operators on H^p*, Thesis, University of Toledo, 1969.
18. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.
19. S. Warchawski, *On differentiability at the boundary in conformal mapping*, Proc. Amer. Math. Soc. **12** (1961), 614–620.
20. T. K. Wong, *On a class of absolutely p-summing operators*, Studia Math. **39** (1971), 181–189.

The research of the first author was partially supported by National Science Foundation Grant GP-33695. The research of the second author was partially supported by N.R.C. (Canada) Grant A8145.

Michigan State University
Queen's University, Canada
Date communicated: June 19, 1972