On Convexity and Compactness in F-Spaces with Bases

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Introduction. This paper deals mainly with non-locally convex F-spaces, and contains variations on the following theme: *every closed, bounded, convex subset of* ℓ^p $(0 < p < 1)$ *is compact.* More generally (Theorem 1), if E is an F-space with an absolute basis (e_n), and the set $\{te_n : t > 0\}$ is unbounded in the E-metric for each n, then the following three statements are equivalent:

(a) If (f_k) is a block basis for (e_n) which is bounded away from zero, then the series $\sum t_n e_n$ diverges for some sequence (t_n) in ℓ^1.

(b) No infinite dimensional subspace of E is locally convex.

(c) Every closed, metrically bounded, convex subset of E is compact.

A related result (Theorem 2) which depends only on the topology of E, and not on the particular metric that produces it, is the following: if E has a boundedly complete basis satisfying condition (a) above, then every closed, (topologically) bounded, convex subset of E is compact.

We apply these results to some sequence spaces related to the ℓ^p spaces $(0 < p < 1)$. Specifically we consider the spaces $\bigcap_{p>p_0} \ell^p$ $(0 \leqq p_0 < 1)$; the spaces ℓ^φ of complex sequences f with

$$\|f\| = \sum \varphi(|f(n)|) < \infty, \tag{0.1}$$

where φ is a continuous, unbounded, subadditive, increasing function on $[0, \infty)$, with $\varphi(t) = 0$ iff $t = 0$; and the spaces $\ell(p_n)$ of complex sequences f with

$$\|f\| = \sum |f(n)|^{p_n} < \infty, \tag{0.2}$$

where (p_n) is a sequence of numbers with $0 < p_n \leqq 1$. Using Theorem 1 we show that whenever the function $t^{-p}\varphi(t)$ is monotone decreasing on $(0, \infty)$ for some $0 < p < 1$, then every closed, norm bounded, convex subset of ℓ^φ is compact; and that the same conclusion holds for $\ell(p_n)$ iff $\lim\sup p_n < 1$. We show that if $\lim p_n = 1$, then every infinite dimensional subspace of $\ell(p_n)$ has a further one isomorphic to a dense subspace of ℓ^1. This is of interest because

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$\ell(p_n)$ is not locally convex when p_n tends rather slowly to 1; which yields an example of a non-locally convex F-space with the property that every infinite dimensional subspace contains a further infinite dimensional locally convex subspace.

We use Theorem 2 to show that every closed, bounded, convex subset of $\bigcap_{p>p_0} \ell^p$ $(0 \leqq p_0 < 1)$ is compact. In particular, no infinite dimensional subspace of $\bigcap_{p>p_0} \ell^p$ is isomorphic to a normed space. We show that, in spite of this, every infinite dimensional subspace contains a further infinite dimensional locally convex subspace.

This last result, along with the corresponding one for $\ell(p_n)$, depends on the well known selection theorem of Bessaga and Pełczyński [1, Theorem 3], set in the context of locally pseudo-convex spaces (Theorem B). In fact, throughout the paper we use standard techniques related to this theorem; in particular, the "sliding hump" argument used in its proof provides the crucial step (Lemma 3) in the proofs of Theorems 1 and 2.

The body of the paper is organized into four sections, the first of which contains background material. The main theorems are stated and applied in the second section, and proved in the third. In the last section we comment on some related topics and open problems.

This investigation was motivated by the following result of W. J. Stiles [24, Theorem 4]: *every continuous linear mapping of a normed space into* ℓ^p $(0 < p < 1)$ *is compact.* We wish to thank Professor Stiles for providing us with a preprint of his paper.

1. Background. Let E be a real or complex linear space. By an F-*norm* on E we mean a non-negative, subadditive functional $\|\cdot\|$ with the following additional properties:

(1.1)
- (a) $\|e\| = 0$ iff $e = 0$.
- (b) $\|te\| \leqq \|e\|$ for all scalars t with $|t| \leqq 1$.
- (c) $\lim_n \|e/n\| = 0$ for each e in E.
- (d) the metric $d(e, f) = \|e - f\|$ is complete.

We remark that this definition is slightly more restrictive than the usual one (cf. [3, p. 52]). Two F-norms on E are *equivalent* if they induce the same topology. We call a linear space equipped with an F-norm an F-*space.* Every F-space is a complete, metrizable, linear topological space, and conversely every such space can be regarded as an F-space [9, p. 51, Problem C]. Note, however, that in this paper we regard different (but equivalent) F-norms on E as defining different (but isomorphic) F-spaces.

Let E be an F-space and $\|\cdot\|$ its F-norm. A subset B of E is *norm bounded* if $\sup \{\|f\|: f \text{ in } B\} < \infty$, and (*topologically*) *bounded* if for each neighborhood U of zero there exists a positive number t such that $tB \subset U$. Every bounded set is norm bounded, but the converse may fail: for example, the formula

$$\|t\| = \min(|t|, 1) \quad (t \text{ real})$$

defined an F-norm on the real line for which every subset is norm bounded.

If (e_n) is a basis for an F-space E, then we denote its coordinate functionals by (e_n'), so each element f in E has a unique expansion $f = \sum e_n'(f)e_n$, where the series converges in the topology of E. It is well known that the coordinate functionals are continuous [11, Chapter IX, Section 5, Theorem 2]; in particular an F-space with a basis has enough continuous linear functionals to separate points. The basis (e_n) is called *regular* if $\inf \|e_n\| > 0$ [8, Definition 1.1]; *absolute* if

$$\|f\| = \sum \|e_n'(f)e_n\| \tag{1.2}$$

for each f in E; and *boundedly complete* if the series $\sum t_n f_n$ (t_n scalars) converges whenever its partial sums form a bounded set (cf. [11, Chapter III] for Banach spaces). A sequence (f_k) is *basic* if it is a basis for its closed linear span, and *block basic for* (e_n) (cf. [11, Chapter IV, Section 3, Definition 5]) if

$$f_k = \sum_{n \in I_k} s_n e_n \qquad (k = 1, 2, \cdots), \tag{1.3}$$

where $I_k = \{n_k + 1, \cdots, n_{k+1}\}$, (n_k) is a strictly increasing sequence of positive integers, and (s_n) is a scalar sequence. It is easy to see that every block basic is basic. A basic sequence is *regular* (resp. *absolute, boundedly complete*) if it is a regular (resp. absolute, boundedly complete) basis for its closed linear span. We denote the linear span of a sequence (f_k) by sp (f_k), and its closure by $\overline{\text{sp}}\ (f_k)$.

Two basic sequences (f_k) and (g_k) are called *equivalent* if for (t_k) a scalar sequence,

$$\sum t_k f_k \quad \text{converges iff} \sum t_k g_k \text{ converges.}$$

It follows from the Uniform Boundedness Principle [3, II.1.11] that (f_k) and (g_k) are equivalent iff there is an isomorphism of $\overline{\text{sp}}\ (f_k)$ onto $\overline{\text{sp}}\ (g_k)$ taking f_k to g_k ($k = 1, 2, \cdots$). In this paper, *isomorphism* always means *linear homeomorphism.*

The following material occurs in the applications, but is needed for neither the statements nor the proofs of Theorems 1 and 2. A non-negative, subadditive functional S on a linear space E is called a *p-seminorm* ($0 < p \leqq 1$) if $S(te) = |t|^p S(e)$ for all scalars t and vectors e; and a *p-norm* if, in addition, $e = 0$ whenever $S(e) = 0$. A linear topological space E is called *locally bounded* if there is a bounded neighborhood of the origin. It is well known [17] that E is locally bounded iff its topology is induced by a p-norm. If the topology of E is induced by a family (S_i) of p_i-seminorms, then E is called *locally pseudo-convex* [25, Section 1] (or *semiconvex* [15, Definitions 1.4]). In this case E is metrizable iff (S_i) can be taken to be countable.

2. Main results and applications.

Theorem 1. Let E be an F-space with an absolute basis (e_n) *such that*

$$\sup \{\|te_n\| : t > 0\} = \infty \qquad (n = 1, 2, \cdots). \tag{2.1}$$

Then the following are equivalent:

(a) *If* (f_k) *is a regular block basis for* (e_n), *then* $\sum \|t_k f_k\| = \infty$ *for some* (t_k) *in* ℓ^1.

(b) *No infinite dimensional subspace of E is locally convex.*

(c) *Every closed, norm bounded, convex subset of E is compact.*

Note that condition (2.1) is essential: if it fails for some integer N, then the one dimensional subspace spanned by the vector e_N is closed, norm bounded, and convex, but not compact.

Theorem 1 depends entirely on the norm of E. The next result depends only on the topology.

Theorem 2. *Let E be an F-space with a boundedly complete basis* (e_n). *Suppose that for each regular block basis* (f_k) *for* (e_n) *the series* $\sum t_k f_k$ *diverges for some* (t_k) *in* ℓ^1. *Then every closed, bounded, convex subset of E is compact.*

We prove these results in the next section, and devote the rest of this one to applications.

The spaces ℓ^φ. These are special cases of spaces previously studied by Mazur and Orlicz [12], and Rolewicz [18]. It is not difficult to see that ℓ^φ is an F-space in the norm (0.1).

Corollary 1. *Suppose the function* $t^{-p}\varphi(t)$ *is monotone decreasing on* $(0, \infty)$ *for some* $0 < p < 1$. *Then every closed, norm bounded, convex subset of* ℓ^φ *is compact.*

Proof. Let e_n denote the n^{th} *standard unit vector*: $e_n(m) = \delta_{mn}$ (Kronecker delta). Then (e_n) is an absolute basis in ℓ^φ and satisfies (2.1) since φ is unbounded. Thus we need only verify (a) of Theorem 1 for $E = \ell^\varphi$. Let (f_k) be a regular block basis for (e_n), given in terms of (e_n) by (1.3). Let $d = \inf \|f_k\| > 0$. Since $t^{-p}\varphi(t)$ is decreasing on $(0, \infty)$ we have $\varphi(at) \geqq t^p\varphi(a)$ whenever $a > 0$ and $0 \leqq t \leqq 1$. Choose a non-negative scalar sequence (t_k) with $\sum t_k = 1$ and $\sum t_k^p = \infty$. Then for each k,

$$\|t_k f_k\| \geqq \sum_{n \in I_k} t_k^p \varphi(|s_n|) \geqq d t_k^p ,$$

so $\sum \|t_k f_k\| = \infty$.

If $\varphi(t) = t^p$, then $\ell^\varphi = \ell^p$, and a subset is bounded iff it is norm bounded.

Corollary 2. *Every closed, bounded, convex subset of* ℓ^p $(0 < p < 1)$ *is compact.*

The spaces $\ell(p_n)$ $(0 < p_n \leqq 1)$. Equation (0.2) defines an F-norm on $\ell(p_n)$, for (p_n) a sequence with $0 < p_n \leqq 1$. These spaces have been studied by various authors [2], [10], [14], [19], [23]. Of particular interest to us is the following result of S. Simons [23, Theorems 3 and 5].

Theorem A. *The following are equivalent:*

(a) $\ell(p_n) = \ell^1$.

(b) $\ell(p_n)$ *is locally convex.*
(c) $\sum N^{p_n/(p_n-1)} < \infty$ *for some integer* $N > 1$.

This result and Theorem 1 yield the following:

Corollary 3. *The following statements are equivalent:*
(a) $\limsup p_n < 1$
(b) *No infinite dimensional subspace of* $\ell(p_n)$ *is locally convex.*
(c) *Every closed, norm bounded, convex subset of* $\ell(p_n)$ *is compact.*

Proof. The standard unit vectors (e_n) defined in the proof of Corollary 1 form an absolute basis for $\ell(p_n)$ which satisfies (2.1). Thus we need only show that statement (a) of Theorem 1 holds for $E = \ell(p_n)$ iff $\limsup p_n < 1$.

Suppose $\limsup p_n < 1$. Then for some $N > 0$ we have $p = \sup\{p_n : n > N\} < 1$. Suppose (f_k) is a regular block basis for (e_n) satisfying (1.3). Then $d = \inf ||f_k|| > 0$, so choosing (t_k) as in the proof of Corollary 1 and using the inequality

$$||t_k f_k|| \geqq |t_k|^p \, ||f_k|| \geqq d \, |t_k|^p,$$

we obtain $\sum ||t_k f_k|| = \infty$, so (a) of Theorem 1 holds.

Conversely, if $\limsup p_n = 1$, then by Theorem A there is a sequence (n_k) such that $\ell(p_{n_k}) = \ell^1$. Let $f_k = e_{n_k}$ $(k = 1, 2, \cdots)$. Then (f_k) is a regular block basis for (e_n), and $\sum ||t_k f_k|| < \infty$ whenever $(t_k) \in \ell^1$, so (a) of Theorem 1 fails.

Our next result requires the selection theorem of Bessaga and Pełczyński [1, Theorems 3 and 3′] in the context of locally bounded spaces. For later use we state it in a more general setting.

Theorem B. *Let E be a locally pseudo-convex F-space with a basis (e_n). Suppose (g_n) is a sequence in E with* $\inf ||g_n|| > 0$, *and*

$$\lim_n e'_m(g_n) = 0 \qquad (m = 1, 2, \cdots).$$

Then there exists a basic subsequence (g_{n_k}) equivalent to a block basis for (e_n).

A proof of this theorem is sketched in Section 3. The result was first proved in [1] for Banach spaces, and extensions to locally bounded and locally convex F-spaces were stated without proof. Kalton [8, Theorem 4.3] gives a proof for locally convex F-spaces, and in section 3 this proof is carried over almost *verbatim* to the locally pseudo-convex case.

Theorem 3. *If* $\lim p_n = 1$, *then every infinite dimensional subspace of $\ell(p_n)$ contains a further one isomorphic to a dense subspace of ℓ^1.*

Proof. Suppose $\lim p_n = 1$. Then it is easy to see that $\ell(p_n)$ is locally bounded (in fact, Simons shows that $\ell(p_n)$ is locally bounded iff $\inf p_n > 0$ [23, Theorem 6]), so Theorem B applies. Suppose G is an infinite dimensional subspace of $\ell(p_n)$. Then there exists a sequence (g_n) in G such that for each n,

$$g_n(k) = 0 \quad (k = 1, 2, \cdots, n-1)$$

while $||g_n|| = 1$ (cf. [1, p. 157, Corollary 2]). We are going to show that (g_n) has a basic subsequence (g_{n_k}) equivalent to the standard unit vector basis of ℓ^1. From this it follows that there is an isomorphism of $\overline{\text{sp}}\ (g_{n_k})$ onto ℓ^1 taking g_{n_k} onto e_k $(k = 1, 2, \cdots)$. Thus sp (g_{n_k}) is a subspace of G isomorphic to a dense subspace of ℓ^1, which completes the proof.

Using Theorem B we find that some subsequence of (g_n) is equivalent to a block basis (f_k) for (e_n), and since $||g_n|| = 1$ for all n, we must have

$$0 < m = \inf ||f_k|| \leqq \sup ||f_k|| = M < \infty. \tag{2.2}$$

It is enough to show that (f_k) has a subsequence equivalent to the standard basis for ℓ^1. Suppose f_k has the form (1.3), and let

$$r_k = \inf \{p_n : n \in I_k\} \qquad (k = 1, 2, \cdots). \tag{2.3}$$

Replacing (f_k) by a suitable subsequence, we may assume that

$$\sum_k 2^{r_k/(r_k-1)} < \infty. \tag{2.4}$$

We claim that $\sum t_k f_k$ converges in E iff $(t_k) \in \ell^1$. Suppose $\sum t_k f_k$ converges. Then the sequence $(t_k f_k)$ tends to zero in E, and since (f_k) is bounded away from zero we must have $\lim t_k = 0$. In particular, we may assume that $|t_k| \leqq 1$ for all k. A computation like the one used to prove Corollary 3 shows that

$$||f|| = \sum ||t_k f_k|| \geqq \sum |t_k|\ ||f_k|| \geqq m \sum |t_k|;$$

hence $(t_k) \in \ell^1$.

Conversely, if $\sum |t_k| \leqq 1$, then $\sum |t_k|^{r_k} < \infty$ by Theorem A and (2.4). Since $|t_k| \leqq 1$, we have from (2.3):

$$\sum ||t_k f_k|| \leqq \sum |t_k|^{r_k}\ ||f_k|| \leqq M \sum |t_k|^{r_k} < \infty,$$

so the series $\sum t_k f_k$ converges in E. Thus (f_k) is equivalent to the standard unit vector basis of ℓ^1.

If (p_n) converges to 1 too rapidly, then according to Theorem A, $\ell(p_n) = \ell^1$, and Theorem 3 is just a special case of [1, Corollary 5, p. 157]. However when (p_n) tends to 1 and fails to satisfy (c) of Theorem A, then a new phenomenon occurs:

Corollary 4. There exist F-spaces which are not locally convex, but have the property that every infinite dimensional subspace contains a further infinite dimensional locally convex subspace.

This situation occurs again in the next class of examples.

The spaces $\bigcap_{p>p_0} \ell^p$ $(0 \leqq p_0 < 1)$. Fix $0 \leqq p_0 < 1$, and let $E = \bigcap_{p>p_0} \ell^p$, taken in the natural upper bound topology. Thus E is a complete, metrizable, linear topological space (a sequence converges to zero in E iff it converges to zero in ℓ^p for all $p > p_0$), and it is easy to see that E is neither locally convex nor locally bounded. However, unlike the previous examples, E has no "natural"

F-norm. For example, if $1 \geqq p_n \downarrow p_0$, and $||f||_n = \sum_k |f(k)|^{p_n}$, then the equation

$$||f|| = \sum_{n=1}^{\infty} 2^{-n} \min(||f||_n, 1)$$

defines an F-norm on E which induces the correct topology, but makes every subset norm bounded. In fact we will shortly see that E contains infinite dimensional locally convex subspaces, so no F-norm on E can render Theorem 1 applicable. However the topological result, Theorem 2, applies readily.

Corollary 5. Every closed, bounded, convex subset of $\bigcap_{p>p_0} \ell^p$ $(0 \leqq p_0 < 1)$ *is compact.*

Proof. A subset of $E = \bigcap_{p>p_0} \ell^p$ is bounded iff it is bounded in ℓ^p for each $p > p_0$. From this it follows easily that the standard basis vectors (e_n) form a boundedly complete basis for E. Suppose (f_k) is a regular block basis for (e_n). Then (f_k) is bounded away from 0 in ℓ^p for some $p_0 < p < 1$. Choosing (t_k) in ℓ^1 with $\sum |t_k|^p = \infty$, and performing a simple calculation, we see that $\sum t_k f_k$ diverges in ℓ^p; hence in E. Thus the basis (e_n) satisfies the hypothesis of Theorem 2.

The last result shows that $\bigcap_{p>p_0} \ell^p$ contains no infinite dimensional subspace isomorphic to a normed space when $0 \leqq p_0 < 1$. By analogy with ℓ^p $(0 < p < 1)$ one might guess that *no* infinite dimensional subspace could be locally convex, but this is not the case.

Theorem 4. Every infinite dimensional subspace of $\bigcap_{p>p_0} \ell^p$ $(0 \leqq p_0 < 1)$ *contains a further infinite dimensional subspace which is locally convex.*

Proof. Suppose G is an infinite dimensional subspace of $E = \bigcap_{p>p_0} \ell^p$. Then as in the proof of Theorem 3 there is a sequence (g_k) in G which converges to zero coordinatewise, but not in E. Since E is a locally pseudo-convex F-space, Theorem B applies, and by passing to a suitable subsequence we may assume that (g_k) is equivalent to a regular block basis (f_k) for (e_n).

Let T be the resulting isomorphism of $\overline{\text{sp}}\ (f_k)$ onto $\overline{\text{sp}}\ (g_k)$ taking f_k to g_k $(k = 1, 2, \cdots)$. We are going to show that there is a block basis (h_j) for (f_k) such that $\overline{\text{sp}}\ (h_j)$ is locally convex. Suppose for the moment that this has been done. Then (Th_j) is a block basis for (g_k) equivalent to (h_j) and contained in G. Thus $\overline{\text{sp}}\ (Th_j)$ is a subspace of G isomorphic to $\overline{\text{sp}}\ (h_j)$; hence infinite dimensional and locally convex; which completes the proof.

It remains to find (h_j). For $0 < p \leqq 1$, let $||f||_p = \sum |f(n)|^p$. Normalize (f_k) so that $||f_k||_1 = 1$. Let (I_j) be a sequence of consecutive disjoint blocks of positive integers, with 2^j elements in I_j $(j = 1, 2, \cdots)$. Let

$$h_j = 2^{-j} \sum_{k \in I_j} f_k \qquad (j = 1, 2, \cdots), \tag{2.5}$$

so (h_j) is block basic for (f_k) and $||h_j||_1 = 1$ for each j; and let $H = \overline{\text{sp}}\ (h_j)$. We claim that H is locally convex.

To see this, let W be the space of sequences (t_i) such that

$$S_p(t_i) = \sup_j |t_j| \, ||h_j||_p^{1/p} < \infty \tag{2.6}$$

for $p_0 < p \leqq 1$, taken in the topology induced by the semi-norms $\{S_p : p_0 < p \leqq 1\}$. Clearly W is locally convex. Since for each fixed j, the numbers $||h_j||_p^{1/p}$ increase as p decreases, the same is true for $S_p(t_i)$ for each sequence (t_i); and it follows readily that W is an F-space.

We complete the proof by showing that H is isomorphic to W. Suppose $h \in H$, so $h = \sum t_j h_j$, where

$$||h||_p = \sum |t_j|^p \, ||h_j||_p < \infty \quad (p_0 < p \leqq 1).$$

Let $Th = (t_j)$. The last equation shows that

$$S_p^p(Th) \leqq ||h||_p \quad (p_0 < p \leqq 1),$$

so T is a one-to-one continuous linear map taking H into W. It is enough to show that T maps H *onto* W; the continuity of T^{-1} then follows from the Interior Mapping Principle [3. II. 2.1].

Suppose $(t_j) \in W$. Fix $p_0 < p \leqq 1$, and choose q with $p_0 < q < p$. Since $||f_k||_1 = 1$, every coordinate of f_k is $\leqq 1$, so $||f_k||_p \leqq ||f_k||_q$ $(k = 1, 2, \cdots)$. From this and (2.5) we have for each j:

$$||h_j||_p \leqq 2^{-jp} \sum_{k \in I_j} ||f_k||_q = 2^{-j(p-q)} \, ||h_j||_q \,. \tag{2.8}$$

Now $|t_j|^p \, ||h_j||_p \leqq S_p^p(t_j)$, and $||h_j||_p \geqq ||h_j||_1 = 1$ for all j, so (t_j) is bounded. Thus we may assume without loss of generality that $|t_j| \leqq 1$ for all j. Combining this with (2.8) we get

$$|t_j|^p \, ||h_j||_p \leqq 2^{-j(p-q)} \, |t_j|^q \, ||h||_q \leqq S_q^q(t_j) 2^{-j(p-q)}$$

$(j = 1, 2, \cdots)$; hence $\sum |t_j|^p \, ||h_j||_p < \infty$. Thus the series $\sum t_j h_j$ converges to an element h of H, and $Th = (t_j)$; so T maps H onto W.

We remark that in the last part of the proof we are really showing that the space of sequences (t_j) such that

$$\sum_j |t_j|^p \, ||h_j||_p < \infty \qquad (p_0 < p \leqq 1),$$

in the topology induced by the above p-seminorms, is locally convex. Similar results have already been obtained by Fenske and Schock (see [7, Section 3.2, p. 344, Beispiel 1]).

3. Proofs of Theorems 1 and 2. Before proving these theorems we require some preliminary results. If E is an F-space with a basis (e_n), then we call the locally convex topology induced on E by the seminorms

$$f \to |e_n'(f)| \quad (n = 1, 2, \cdots)$$

the *topology of coordinatewise convergence* (with respect to (e_n)); and denote it by κ, or $\kappa(e_n)$ if we wish to emphasize the basis. Clearly κ is Hausdorff, metrizable, and weaker than the original topology of E. We call a κ-convergent sequence *coordinatewise convergent,* since $\kappa\text{-}\lim f_n = 0$ if $\lim_n e_m'(f_n) = 0$ for each m. Embedding the space (E, κ) into a countable product of scalar fields, we obtain the following useful fact: if (f_n) is a κ-bounded sequence in E, then there is a subsequence (f_{n_k}) such that

$$\lim_k e_m'(f_{n_k}) = t_m \quad \text{exists for each } m. \tag{3.1}$$

Lemma 1. *Let E be an F-space with a basis (e_n).*

(a) *If (e_n) is absolute and obeys condition* (2.1), *then every norm bounded subset of E is relatively κ-compact.*

(b) *The basis (e_n) is boundedly complete iff every bounded subset of E is relatively κ-compact.*

Proof. (a): Suppose (e_n) is absolute and obeys (2.1). Let B be a norm bounded subset of E. It follows from (1.2) and (2.1) that B is κ-bounded. Since κ is metrizable, it is enough to show that every sequence in B has a coordinatewise convergent subsequence. If (f_n) is a sequence in B, then, from the remarks preceding the statement of Lemma 1, there is a subsequence (f_{n_k}) satisfying (3.1). The continuity of the F-norm yields

$$\begin{aligned}\sum_1^N \|t_m e_m\| &= \lim_k \sum_1^N \|e_m'(f_{n_k})e_m\| \\ &= \lim_k \left\| \sum_1^N e_m'(f_{n_k})e_m \right\| \\ &\leqq \sup_k \|f_{n_k}\|\end{aligned}$$

for $N = 1, 2, \cdots$, so it follows that $\sum \|t_m e_m\| < \infty$; hence the series $\sum t_m e_m$ converges to an element f of E. Since $e_m'(f) = t_m$ for each m, we see that (f_{n_k}) converges coordinatewise to f, which completes the proof of (a).

(b): Suppose (e_n) is boundedly complete. Let B be a bounded subset of E. Then B is bounded in the weaker topology κ, so if (f_n) is a sequence in B, then there is a subsequence (f_{n_k}) satisfying (3.1). We claim that the partial sums of the series $\sum t_m e_m$ form a bounded set. It is enough to show that the set

$$B_0 = \left\{ \sum_{n=1}^N e_n'(f)e_n : f \in B; N = 1, 2, \cdots \right\}$$

is bounded, because the partial sums in question belong to its closure by (3.1); and the closure of a bounded set is bounded.

To see that B_0 is bounded, we first note that E can be equivalently renormed so that

$$\|f\| = \sup_N \left\| \sum_1^N e_n'(f)e_n \right\| \tag{3.2}$$

for each f in E [11, Chapter IX, Section 5, proof of Theorem 2]. Suppose $\epsilon > 0$ is given. Since B is bounded,

$$\sup \{||tf|| : f \text{ in } B\} < \epsilon$$

for some $t > 0$; hence by (3.2),

$$\left\|t \sum_1^N e_n'(f)e_n\right\| = \left\|\sum_1^N e_n'(tf)e_n\right\| \leqq ||tf|| < \epsilon$$

for each f in B and each N, so B_0 is bounded.

Thus the series $\sum t_m e_m$ has bounded partial sums. It therefore converges to an element f in E, since (e_n) is boundedly complete. Moreover, $e_m'(f) = t_m$ for each m, so (f_{n_k}) converges coordinatewise to f; hence B is relatively κ-compact.

Conversely, suppose every bounded subset of E is relatively κ-compact. Then whenever the partial sums of the series $\sum t_m e_m$ form a bounded sequence, they must have a subsequence which converges coordinatewise to some f in E. In particular, $e_m'(f) = t_m$ for each n. But $f = \sum e_m'(f)e_m$, where the series converges in E; hence $\sum t_m e_m$ converges in E. Thus the basis (e_n) is boundedly complete.

Lemma 2. Let E be an F-space with a basis (e_n).

(a) *If (e_n) is absolute, then so is any block basis.*

(b) *If (e_n) is boundedly complete, then so is any block basis.*

Proof. Part (a) is immediate; indeed it has already been used several times in Section 2. To prove (b), suppose (e_n) is boundedly complete, and let (f_k) be a block basis for (e_n). Set $F = \overline{\text{sp}}\ (f_k)$, and suppose B is a bounded subset of F. Then B is bounded in E; hence relatively $\kappa(e_n)$-compact by Lemma 1(b). But $\kappa(f_k)$ is weaker than the restriction to F of $\kappa(e_n)$, and F is $\kappa(e_n)$-closed in E. It follows easily from this that B is relatively $\kappa(f_k)$-compact, so (f_k) is boundedly complete by Lemma 1(b).

The following well known "sliding hump" result is crucial to the proofs of Theorems 1 and 2.

Lemma 3. Let E be an F-space with a basis (e_n), and suppose (g_n) is a sequence in E which converges to zero coordinatewise. Then for each positive sequence (ϵ_k) there exists a subsequence (g_{n_k}) and a block basis (f_k) for (e_n) such that $||g_{n_k} - f_k|| < \epsilon_k$ $(k = 1, 2, \cdots)$.

Proof. Choose a positive integer p_1 such that

$$\left\|\sum_{p_1+1}^{\infty} e_i'(g_1)e_i\right\| < \epsilon_1 ,$$

and let $n_1 = 1$ and $f_1 = \sum_{i=1}^{p_1} e_i'(p_1)e_1$. Then $||g_{n_1} - f_1|| < \epsilon_1$. Choose $n_2 > n_1$ such that

$$\left\|\sum_{i=1}^{p_1} e_i'(g_{n_2})e_i\right\| < \epsilon_2/2,$$

and choose $p_2 > p_1$ such that

$$\left\|\sum_{p_2+1}^{\infty} e_i'(g_{n_2})e_i\right\| < \epsilon_2/2.$$

Let $f_2 = \sum_{p_1+1}^{p_2} e_i'(g_{n_2})e_i$, so $\|f_2 - g_{n_2}\| < \epsilon_2$; and continue in this manner.

Before proving Theorem 1, we record an elementary fact about F-norms.

Lemma 4. Suppose $\|\cdot\|$ *is an F-norm on a linear space E. Then* $\|te\| \geqq |t|\,\|e\|/2$ *for all e in E,* $0 \leqq |t| \leqq 1$.

Proof. The inequality is trivial for $|t| = 0, 1$. For each positive integer n we have $\|n^{-1}e\| \geqq n^{-1}\,\|e\|$ by subadditivity. If $0 < |t| < 1$, then $(n+1)^{-1} \leqq |t| < n^{-1}$ for some positive integer n, so using the above inequality and (1.1b) we get

$$\|te\| \geqq \|(n+1)^{-1}e\| \geqq (n+1)^{-1}\,\|e\| \geqq (|t|/2)\,\|e\|.$$

Proof of Theorem 1. (c) $\Rightarrow$ (b): Suppose (c) holds and F is a subspace of E which is locally convex in the relative topology. Then the neighborhood $\{f \in F\colon \|f\| \leqq 1\}$ contains a closed, convex neighborhood of zero, which must, by (c), be compact. Thus F must be finite dimensional [9, Theorem 7.8, p. 62].

(b) $\Rightarrow$ (a): Suppose (a) fails, so there is a regular block basis (f_k) such that $\sum t_k f_k$ converges for each sequence (t_k) in ℓ^1. We claim that (f_k) is equivalent to the standard unit vector basis of ℓ^1. Suppose (t_k) is a scalar sequence, and $\sum t_k f_k$ converges in E. Then $t_k f_k \to 0$ in E, and since $d = \inf \|f_k\| > 0$ we must have $t_k \to 0$. Thus we may assume without loss of generality that $|t_k| \leqq 1$ for all k. From Lemmas 2 and 4 we have

$$\|f\| = \sum \|t_k f_k\| \geqq (1/2) \sum |t_k|\,\|f_k\| \geqq (d/2) \sum |t_k|,$$

so $(t_k) \in \ell^1$. Thus (f_k) is equivalent to the standard basis for ℓ^1; so (b) also fails.

(a) $\Rightarrow$ (c): Suppose (a) holds and B is a closed, norm bounded, convex subset of E. If B is not compact, then it contains a sequence (h_k) having no convergent subsequence. Using Lemma 1(a) and passing to a subsequence we may assume that (h_k) converges coordinatewise to some element h in E; hence the vectors $g_k = h_k - h$ $(k = 1, 2, \cdots)$ form a sequence in $B - h$ which converges to zero coordinatewise, but has no norm convergent subsequence. In particular, $d = \inf \|g_k\| > 0$. Using Lemma 3 and extracting another subsequence, we obtain a block basis (f_k) for (e_n) such that

$$\|f_k - g_k\| < d/2^k \quad (k = 1, 2, \cdots). \tag{3.3}$$

In particular, $\inf \|f_k\| > d/2$, so (f_k) is regular; and it follows from (a) that there is a sequence (t_n) in ℓ^1 such that $\sum \|t_k f_k\| = \infty$. We see from (1.1b) that (t_k) can be chosen to be non-negative; moreover it can be normalized so that

$\sum t_k = 1$. Thus $0 \leqq t_k \leqq 1$ for all k; so

$$\left\| \sum_1^N t_k g_k \right\| \geqq \left\| \sum_1^N t_k f_k \right\| - \left\| \sum_1^N t_k (f_l - g_k) \right\|$$
$$\geqq \sum_1^N \|t_k f_k\| - \sum_1^N \|f_k - g_k\|$$
$$\geqq \sum_1^N \|t_k f_k\| - d$$

for $N = 1, 2, \cdots$, where the second inequality follows from (1.1b), the triangle inequality, and the fact that (f_k) is an absolute basic sequence (Lemma 2a, Section 3). Thus the convex hull of (g_k) is unbounded in norm, so the same is true of the convex set $B - f$ which contains it. But this contradicts the norm boundedness of B, so B must be compact.

This proof shows, in addition, that conditions (a), (b), and (c) in the statement of Theorem 1 are also equivalent to:

(b′) No infinite dimensional subspace of E is isomorphic to ℓ^1.

Proof of Theorem 2. Suppose E is an F-space with a boundedly complete basis (e_n) such that for every regular block basis (f_k) for (e_n) the series $\sum t_k f_k$ diverges for some (t_k) in ℓ^1. As in the proof of Theorem 1, it is enough to show that every bounded sequence with no convergent subsequence has unbounded convex hull. Using Lemma 1b, translating, and passing to a subsequence, we see that it suffices to show that the convex hull of every sequence which converges to zero coordinatewise, but not in norm, is unbounded. So suppose (g_k) converges coordinatewise to zero, but $d = \inf \|g_k\| > 0$. As before, using Lemma 3 and passing to a subsequence, we find a block basis (f_k) for (e_n) satisfying (3.3), and a sequence (t_k) in ℓ^1 such that $\sum t_k f_k$ diverges. Examining real and imaginary parts of (t_k), then positive and negative subsequences of these, we see that (t_k) can be chosen to be non-negative. Moreover it can be normalized so that $\sum t_k = 1$. Since (f_k) is boundedly complete (Lemma 2) the partial sums of the series $\sum t_k f_k$ form an unbounded set, so there exists an $\epsilon > 0$ and a strictly increasing sequence (K_j) of positive integers such that

$$\left\| j^{-1} \sum_{k=1}^{K_j} t_k f_k \right\| > \epsilon \qquad (j = 1, 2, \cdots). \tag{3.4}$$

We are going to use (3.3) and (3.4) to show that there is a sequence (h_j) of convex combinations of (g_k) such that $\inf \|j^{-1} h_j\| > 0$; hence the convex hull of (g_k) is unbounded. Choose a positive integer p such that

$$\sum_{k=p}^{\infty} \|g_k - f_k\| < \epsilon/4, \tag{3.5}$$

and choose $q > p$ so that whenever $j > q$, we have

$$\left\| q^{-1} \sum_{k=1}^{p-1} t_k f_k \right\| < \epsilon/2, \tag{3.6}$$

and $K_j > p$. Then it follows from (3.4), (3.6), and (1.1b) that whenever $j > q$ we have

$$\left\| j^{-1} \sum_{k=p}^{K_j} t_k f_k \right\| > \epsilon/2,$$

which, along with (3.5) yields

$$\begin{aligned} \left\| j^{-1} \sum_{k=p}^{K_j} t_k g_k \right\| &\geqq \left\| j^{-1} \sum_{k=p}^{K_j} t_k f_k \right\| - \left\| j^{-1} \sum_{k=p}^{K_j} t_k (f_k - g_k) \right\| \\ &\geqq \epsilon/2 - \sum_{k=p}^{K_j} \|f_k - g_k\| \\ &\geqq \epsilon/2 - \epsilon/4 = \epsilon/4. \end{aligned}$$

Let $\alpha_j = \sum_{k=p}^{K_j} t_k$, so $0 < \alpha < 1$, and let

$$h_j = \alpha_j^{-1} \sum_{k=p}^{K_j} t_k g_k \qquad (j = 1, 2, \cdots).$$

Then each h_j belongs to the convex hull of (g_k), so from the last inequality and (1.1b) we have $\inf \|j^{-1} h_j\| \geqq \epsilon/4$, which completes the proof.

We conclude this section with a proof of Theorem B, closely following Kalton [8, Proposition 4.1 and Theorem 4.3]. We require the following result.

Stability Theorem *[8, Proposition 4.1]. Let E be a locally pseudo-convex F-space, with $(S_n : n \geqq 0)$ a sequence of p_n-seminorms $(0 < p_n \leqq 1)$ inducing its topology. Suppose (f_k) is a basic sequence in E with coordinate functionals (f_k') defined on* $\overline{\text{sp}}$ (f_k)*, and suppose that*

$$\sup_k |f_k'(e)|^{p_0} \leqq S_0(e) \tag{3.7}$$

for all e in $\overline{\text{sp}}$ (f_k)*. Let (g_k) be a sequence in E with*

$$\sum_{k=1}^{\infty} S_n(f_k - g_k) = K_n < \infty \qquad (n \geqq 0), \tag{3.8}$$

and $K_0 < 1$. Then (g_k) is basic and equivalent to (f_k).

With the exception of a few exponents of the form p_i/p_0, the proof is identical with that of [8, Prop. 4.1], and we omit it.

Proof of Theorem B (cf. [8, Theorem 4.3]). We are assuming that (g_n) converges to zero coordinatewise, and that $\inf \|g_n\| > 0$. Let $(S_n : n \geqq 1)$ be a sequence of p_n-seminorms inducing the topology of E. Recall that E has an F-norm obeying (3.2). Using this and the fact that (g_n) is bounded away from zero, we can arrange that

$$\inf_n S_1(g_n) = \epsilon > 0, \tag{3.9}$$

and

$$S_1(e) = \sup_N S_1\left(\sum_1^N e_n'(e)e_n\right) \tag{3.10}$$

for all e in E. For we can clearly take S_1 to satisfy (3.9). If it does not already satisfy (3.10), then denote the right hand side of (3.10) by $S(e)$. Clearly S is a p_1-seminorm on E, and $S \geqq S_1$; in particular, S satisfies (3.9). We claim that S is continuous on E. It is enough to show continuity at the origin. Given $\epsilon > 0$, choose $\delta > 0$ such that $S_1(e) < \epsilon$ whenever $\|e\| < \delta$ (by the continuity of S_1 on E). It follows from (3.2) that whenever $\|e\| < \delta$, then $\|\sum_1^N e_n'(e)e_n\| < \delta$ for all N, hence

$$S_1\left(\sum_1^N e_n'(e)e_n\right) < \epsilon \qquad (N = 1, 2, \cdots),$$

and therefore $S(e) \leqq \epsilon$. This proves the continuity of S. Renaming S as S_1 we see that $(S_n : n \geqq 1)$ is still a sequence of p_n-seminorms which induces the topology of E, and now S_1 satisfies (3.9) and (3.10).

Now the seminorms S_n are all continuous, so it follows from Lemma 3 that there exists a subsequence (g_{n_k}) and a block basis (f_k) for (e_n) such that

$$\max_{1 \leqq j \leqq k} S_j(g_{n_k} - f_k) < (\tfrac{1}{2})^{k+3} \qquad (k = 1, 2, \cdots); \tag{3.11}$$

hence (3.8) is satisfied (with g_{n_k} instead of g_k). Letting (f_k') be the coordinate functionals for (f_k), and using (3.10) and the argument in [8, Theorem 4.3], we find that for each e in $\overline{\text{sp}}(f_k)$,

$$\sup |f_k'(e)|^{p_1} \leqq (4/\epsilon)S_1(e).$$

Let $p_0 = p_1$ and $S_0 = (4/\epsilon)S_1$. Then S_0 is a continuous p_0-seminorm on E satisfying (3.7); and

$$K_0 = \sum_k S_0(f_{n_k} - g_k) \leqq \tfrac{1}{2}.$$

Thus, with respect to the seminorms $(S_n : n \geqq 0)$, the sequences (f_k) and (g_{n_k}) satisfy the hypotheses of the stability theorem, so (g_{n_k}) is basic and equivalent to (f_k).

4. Further remarks and open problems.

Completely non-convex spaces. We call a linear topological space *completely non-convex* if it has no infinite dimensional locally convex subspace. It follows from the results of Section 2 that the spaces ℓ^p $(0 < p < 1)$, ℓ^φ with $t^{-p}\varphi(t)$ decreasing for some $0 < p < 1$, and $\ell(p_n)$ with $\limsup p_n < 1$ are all completely non-convex; while $\bigcap_{p>p_0} \ell^p$ $(0 \leqq p_0 < 1)$ and $\ell(p_n)$ with $\limsup p_n = 1$ are not.

We call a linear topological space *r-convex* $(0 < r < 1)$ if its topology is induced by a family of r-seminorms [10, p. 104 and 106]. Note that an r-convex space is also p-convex for each $0 < p < r$; for if S is an r-seminorm, and $0 < p \leqq r$, then $S^{p/r}$ is a p-seminorm.

The *type* $t(E)$ of a linear topological space E is the supremum of the numbers r such that E is r-convex [22, p. 178]. It is known that $t(\ell^p) = p$, and more generally that $t(\ell(p_n)) = \liminf p_n$ [10, satz 4, 5]. In particular, whenever p_n tends to 1 slowly enough, then it follows from Theorem A that $\ell(p_n)$ is not locally convex, but has type 1. It is not difficult to see that $t(\bigcap_{p>p_0}\ell^p) = p_0$ $(0 \leqq p_0 < 1)$, and $t(\ell^\varphi) \leqq p$ if $x^{-p}\varphi(x)$ is decreasing $(0 < p < 1)$.

Question 1. Suppose $t(\ell^\varphi) < 1$. Is ℓ^φ completely non-convex? What happens when $t(\ell^\varphi) = 1$?

We will see shortly that the Hardy spaces H^p $(0 < p < 1)$ of functions f analytic in the open unit disc with

$$||f|| = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty \tag{4.1}$$

are locally bounded F-spaces of type p which, while not completely non-convex themselves, have closed subspaces which *are* completely non-convex. These examples suggest the following question:

Question 2. Does every locally bounded F-space of type <1 have a completely non-convex closed subspace?

Note that the spaces $\bigcap_{p>p_0}\ell^p$ $(0 \leqq p_0 < 1)$ have type <1, but are not locally bounded; while the spaces $\ell(p_n)$ $(\lim p_n = 1)$ are locally bounded, but have type 1. By Theorems 3 and 4, neither of these spaces has a completely non-convex subspace.

The Hardy spaces H^p $(0 < p < 1)$. With the norm (4.1) these spaces are clearly locally bounded, and are known to be F-spaces [4, p. 37, Corollary 2] of type p [10, Satz 3]. The next result, suggested to us by Allen Shields, shows that they contain completely non-convex closed subspaces.

Proposition 1. H^p $(0 < p < 1)$ *contains a complemented subspace isomorphic to* ℓ^p.

Proof. It follows from [4, Section 9.3, p. 153] that when $0 < p < 1$ there is a sequence (z_k) of points in the open unit disk, and functions (h_k) in H^p such that (i) $h_j(z_k) = \delta_{jk}$ for all j, k; (ii) for each sequence (w_k) in ℓ^p the series

$$S(w_k) = \sum w_k h_k (1 - |z_k|^2)^{-1/p}$$

converges in the H^p norm; and (iii) the linear map

$$T(f) = (f(z_k)(1 - |z_k|^2)^{1/p} : k \geqq 1)$$

takes H^p continuously onto ℓ^p. It follows from these facts that the composition map $P = ST$ is a projection taking H^p onto a subspace isomorphic to ℓ^p.

Thus H^p $(0 < p < 1)$ contains a completely non-convex closed subspace (the same method works for the spaces of analytic functions considered in [20, Sections 3 and 4]). The next result shows that H^p is not itself completely non-convex.

Theorem C *[5, Theorem 12, p. 51]. Let (n_k) be a sequence of positive integers with* $\inf n_{k+1}/n_k > 1$. *Let $H^p(n_k)$ denote the collection of H^p functions f whose Taylor expansions have the form*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}.$$

Then the norm

$$\|f\|_2 = \left(\sum |a_k|^2\right)^{1/2} \quad (f \text{ in } H^p(n_k))$$

is equivalent to (4.1) *on $H^p(n_k)$. In particular, $H^p(n_k)$ is isomorphic to ℓ^2.*

The Hahn-Banach theorem. The Hahn-Banach theorem fails in every non-locally convex F-space with a basis. More precisely, every such space contains a closed subspace which supports a continuous linear functional having no continuous linear extension to the whole space [21, Theorem 1]. The null space of this non-extendable functional is closed in E, but not weakly closed (cf. [5, Theorem 17]), so the last statement can be rephrased as follows:

Theorem E. *A non-locally convex F-space with a basis cannot have enough continuous linear functionals to separate points from closed subspaces.*

It is not known if this phenomenon occurs in every non-locally convex F-space (see [16] for related results). The arguments in [21, p. 645] and Theorem B show that it does occur in every closed, non-locally convex subspace of a locally pseudo-convex F-space with a basis.

Since all the non-locally convex spaces of Section 2 have bases, none of them have enough continuous linear functionals to separate points from closed subspaces. However the next result shows that an amusing vestige of the Hahn-Banach theorem may still remain.

Proposition 3. *The spaces ℓ^p $(0 < p < 1)$ have enough continuous linear functionals to separate points from closed, bounded, convex subsets.*

Proof. By Corollary 2, every closed, bounded, convex subset of ℓ^p is compact; hence weakly compact (the weak topology is Hausdorff and weaker than the norm topology). The result now follows upon applying the Hahn-Banach theorem to ℓ^p, taken in the weak topology.

The proof shows, for example, that any space satisfying the hypotheses of Theorem 2 has enough continuous linear functionals to separate points from closed, bounded, convex subsets. In fact, all we are really saying is that if the dual of a linear topological space separates points from each other, then it separates points from compact, convex sets.

Additional notes. After this paper was submitted for publication, we learned from Professor N. J. Kalton that the hypothesis of bounded completeness is not needed in Theorem 2. Kalton uses Lemma 3 (Section 3) to show that the condition:

(C) for each regular block basis (f_k) for (e_n) the series $\sum t_k f_k$ diverges for some $(t_k) \in \ell^1$,

implies that every coordinatewise convergent sequence whose convex hull is bounded must also be norm convergent. Thus the norm topology agrees with κ on every closed, norm bounded, convex set; so every such set is compact. Kalton also pointed out that condition (C) above is the non-locally convex analogue of Definition 5.2 of [8].

We also learned that Theorem 3 has been proved independently by W. J. Stiles in his paper *On local convexity in F-spaces* [Colloq. Math., to appear].

Added in proof. Philippe Turpin has answered Question 1. He shows that any space ℓ^φ of type 1 has a subspace isomorphic to ℓ^1. In addition he shows that there exist ℓ^φ spaces of type 0 (*i.e.* not locally bounded) which also contain subspaces isomorphic to ℓ^1. Question: can this happen when $0 < t(\ell^\varphi) < 1$?

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