Symbolic Extensions and Smooth Dynamical Systems

Tomasz Downarowicz and Sheldon Newhouse

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Abstract

Let $f : X \to X$ be a homeomorphism of the compact metric space X. A *symbolic extension* of (f, X) is a subshift on a finite alphabet (g, Y) which has f as a topological factor. We show that a generic C^1 non-hyperbolic (i.e.,non-Anosov) area preserving diffeomorphism of a compact surface has no symbolic extensions. For $r > 1$, we exhibit a residual subset $\mathcal{R}$ of an open set $\mathcal{U}$ of C^r diffeomorphisms of a compact surface such that if $f \in \mathcal{R}$, then any possible symbolic extension has topological entropy strictly larger than that of f. These results complement the known fact that any C^∞ diffeomorphism has symbolic extensions with the same entropy. We also show that C^r generically on surfaces, homoclinic closures which contain tangencies of stable and unstable manifolds have Hausdorff dimension two.

1 Introduction

Dynamical systems are studied on three major levels: measure-theoretic, topological, and smooth, where the amount of structure increases from left to right. Connections between properties appearing on different levels of structure have always gained a high interest.

In this paper, by (f, X) we will mean a topological dynamical system, i.e., a homeomorphism f of a compact metric space X to itself. However, most of our attention will be focused on the smooth case of a C^r diffeomorphism acting on a compact Riemannian manifold M. We investigate how the existence and precision of certain "good" topological models of these systems depend on the degree of smoothness.

A well studied class of systems is the collection of subshifts over finite alphabets defined as follows: Let Σ be a finite set viewed as a discrete topological space. We equip the countable product $\Sigma^{\mathbf{Z}}$ of bi-infinite sequences of elements of Σ with the product topology. The *left shift map* $\sigma : \Sigma^{\mathbf{Z}} \to \Sigma^{\mathbf{Z}}$ given by

$$\sigma(\mathbf{a})(i) = a(i+1) \;\; \forall i, \quad \mathbf{a} = (\dots a(i), a(i+1), \dots).$$

is a homeomorphism. Any pair (σ, X), where X is a closed σ-invariant subset of $\Sigma^{\mathbf{Z}}$ is called a *symbolic system* (or *subshift*). Subshifts possess a number of good properties such as expansiveness (recall that (f, X) is expansive if there exists a constant $\epsilon > 0$ such that if $x \neq y \in X$ then $d(f^n x, f^n y) > \epsilon$ for some $n \in \mathbf{Z}$), uppersemicontinuity of the entropy function $\mu \mapsto h_\mu(f)$, and finiteness of the topological entropy $h_{top}(f)$. Due to the convenient "digital" form these systems allow an abundance of applications in more practical areas such as information theory, signal processing, and computer science. The same form makes them also relatively easy for abstract studies. For these reasons building a symbolic model has been a key tool in the investigation of dynamical systems since the beginning of the 20th century. Classical examples of such approach are:

1) describing a homotopy class of a trajectory of a geodesic flow on a surface of negative curvature by a sequence of labels of certain closed curves (Hadamard, Morse),

2) parameterizing a unimodal map on $[0, 1]$ by the *kneading sequence*, obtained by labeling the trajectory of the critical point c with respect to the partition into $[0, c]$ and $(c, 1]$ – the key notion in the study of chaos, bifurcations, etc.

The technique relying on labeling trajectories of points usually leads to symbolic measurable *factors* of the system. Sometimes such factors provide full description on the measure-theoretic level. For instance, Krieger's generator theorem says that every ergodic measure-preserving invertible transformation with finite entropy has a finite generating partition $\mathcal{P}$, i.e., it is measure-theoretically isomorphic to the symbolic system represented by the shift map on the $\mathcal{P}$-names (with an appropriate measure).

Symbolic modeling on the topological level is obviously a much more subtle task. A classical result by Hedlund states that a system (f, X) is topologically conjugate to a subshift if and only if f is expansive and X is zero dimensional. It is also clear that systems defined on connected spaces (for example on manifolds) do not even admit symbolic topological factors. In order to symbolically represent such systems at the same time respecting the topology one has to weaken the notion of topological conjugacy. Since factors are impossible, we go in the opposite direction (i.e., we consider extensions).

Recall that (g, Y) is a *topological extension* (or, for brevity, an *extension*) of (f, X) if there exists a continuous surjection $\pi : Y \to X$ such that $f\pi = \pi g$.

We agree to measure the "imprecision" of the model obtained as an extension by the amount of entropy added to each invariant measure μ on X. Namely, with the notation as above, we define

$$h^{\pi}_{ext}(\mu) = \sup\{h_\nu(g) : \pi_*\nu = \mu\},$$

and we will be interested in extensions minimizing this function. An extension for which $h^{\pi}_{ext}(\mu) = h_\mu(f)$ for every μ (or equivalently $h_\nu(g) = h_{\pi_*\nu}(f)$ for every invariant measure ν on Y) is considered a particularly good model and is called a *principal extension*. One can say that such model preserves the entire *information theory* of the original system.

We are interested in finding the best symbolic extensions which might exist. Accordingly, let us introduce some appropriate concepts. For a topological dynamical system (f, X), let $\mathcal{M}(f)$ denote the space of $f-$invariant Borel probability measures on X. Let $S(f)$ denote the collection of all possible symbolic extensions (g, Y, π) of (f, X) (we take $S(f) = \emptyset$ if there is no such extension).

Define

1) the *symbolic extension entropy function*

$$h_{sex}(\mu) = \begin{cases} \inf\{h^{\pi}_{ext}(\mu) : (g, Y, \pi) \in S(f)\} & \text{if } S(f) \neq \emptyset \\ \infty & \text{if } S(f) = \emptyset \end{cases}$$

2) the *symbolic extension entropy of the system*

$$h_{sex}(f) = \begin{cases} \inf\{h_{top}(g) : (g, Y, \pi) \in S(f)\} & \text{if } S(f) \neq \emptyset \\ \infty & \text{if } S(f) = \emptyset \end{cases},$$

3) the *residual entropy of the system*

$$h_{res}(f) = h_{sex}(f) - h_{top}(f).$$

(In some papers the function $h_{res}(\mu) = h_{sex}(\mu) - h_\mu(f)$ is also considered.) Clearly, the above concepts determine topological conjugacy invariants. From our point of view it is important to place the systems (f, X) into three categories:

- PSYM = {systems which admit a principal symbolic extension} (which is equivalent to $h_{sex}(\mu) \equiv h_\mu(f)$, see discussion below),
- SYM = {systems which admit symbolic extensions but none of them principal} (i.e., with $h_\mu(f) \not\equiv h_{sex}(\mu) < \infty$ for some μ), and
- NSYM = {systems with no symbolic extensions at all} (i.e., with $h_{sex}(f) = \infty$).

If a system has a symbolic extension, it obviously must have finite topological entropy. In 1988, J. Auslander asked the converse question: Does every topological system with finite topological entropy have a symbolic extension? In 1990 M. Boyle answered this question negatively. He produced an example of a (zero-dimensional) finite entropy system with no symbolic extension. He also coined the notion of residual entropy $h_{res}(f)$, mentioned the possibility of constructing examples with this parameter strictly between 0 and ∞, and he proved that topological entropy zero implies residual entropy zero. In 2000 the first author of this paper provided a formula allowing one to evaluate the residual entropy of any zero-dimensional system [7]. Using this formula, one can construct systems with an arbitrary pair of values $h_{top}(f) > 0$ and $h_{res}(f) \geq 0$. The formula also implies that all zero-dimensional *asymptotically h-expansive* systems (as defined by M. Misiurewicz in [21]) have residual entropy zero. In 2002 M. Boyle published his early examples and results in a joint paper with D. Fiebig and U. Fiebig [3]. They also gave a new example on a connected space (the two dimensional disc) admitting no symbolic extensions, and where the derivative exists on a residual (in the sense of Baire) set. In the same paper they proved that asymptotic h-expansiveness (with no further restrictions) is equivalent to the existence of a principal symbolic extension. As remarked in [6] this also holds for continuous maps (not just homeomorphisms). Using a result of J. Buzzi [4], that every C^∞ map of a compact C^∞ manifold is asymptotically h-expansive, this gives the following striking result:

Every C^∞ map on a compact C^∞ manifold has a principal symbolic extension.

In a recent paper [2], M. Boyle and the first author found direct formulas for the evaluation of the function $h_{sex}(\cdot)$ in terms of certain functional-analytic properties of the so called *entropy structure*, a sequence of entropy functions evaluated with respect to appropriately chosen shrinking partitions. Among other things, it is shown that $h_{sex}(f) = \sup_\mu h_{sex}(\mu)$. Also, a criterion for attainability of $h_{sex}(\cdot)$ as $h^\pi_{ext}(\cdot)$ in a symbolic extension is provided – it occurs if and only if the former function is affine. It follows immediately that the existence of a principal symbolic extension (and hence asymptotic h-expansiveness) is equivalent to the condition $h_{sex}(\mu) \equiv h_\mu(f)$. Theorem 4.3 below gives a summary of some of the results in [2], especially those which are relevant here. In [6], alternative methods (more topological) of presenting entropy structures are given.

The above mentioned results give a fairly complete description of the kinds of symbolic extensions which exist under various topological conditions (or in the C^∞ case). It is natural to ask about the situation for C^r systems with $1 \leq r < \infty$, and the present paper is the first to consider this question.

We obtain two main results:

(A) There exist C^1 diffeomorphisms admitting no symbolic extensions. Such maps are typical among non-Anosov area preserving diffeomorphisms of surfaces.

(B) For $2 \leq r < \infty$, there exist C^r diffeomorphisms with positive residual entropy. Such maps are typical in certain open sets of C^r diffeomorphisms of surfaces having a homoclinic tangency (see section 6 for the definition and related properties).

Remark. One may ask why (**A**) requires the area preserving property while (**B**) does not. The answer is that the proofs involve the following statement.

(C) *If there is an open set $\mathcal{U}$ of diffeomorphisms so that each $f \in \mathcal{U}$ has a persistently non-hyperbolic homoclinic closure $\Lambda(f)$, then there is a sequence of diffeomorphisms g_i converging to f so that $\Lambda(g_i)$ is defined for all i and has associated periodic points with intervals of homoclinic tangencies.*

The situation is that statement (**C**) is known to be true in the C^1 complement of the closure of Anosov area preserving diffeomorphisms and for C^r open sets of diffeomorphisms with persistent homoclinic tangencies ($r \geq 2$). In fact, this is a significant step in the proofs of the results in [25], [15]. As an aside, we mention that the work of Pujals-Samborino [35] probably implies this statement for C^1 non-area-preserving surface diffeomorphisms, but, at present, we make no definite assertion to this effect.

In the proof of result (**B**) mentioned above we obtain a specific lower bound for $h_{res}(f)$ which may also be an upper bound in many cases.

Regarding general systems with finitely many derivatives, we also present the following conjectures.

Conjecture 1.1 *For $2 \leq r < \infty$, every C^r self-map f of a compact Riemannian manifold has a symbolic extension.*

Remark. This conjecture is of course equivalent to the symbolic extension entropy $h_{sex}(f)$ being finite. There is, in fact, a natural candidate to be an upper bound for $h_{sex}(f)$ as follows. Given the map $f : M \to M$ of the compact Riemannian manifold M, and a positive integer n, let $Lip(f^n)$ denote the Lipschitz constant (maximum norm of the derivative) of f^n. The sequence $\log Lip(f^n)$ is subadditive, so the quantity

$$R(f) = \lim_{n \to \infty} \frac{1}{n} \log Lip(f^n) = \inf_n \frac{1}{n} \log Lip(f^n)$$

is well-defined and finite. It is also independent of the choice of Riemannian metric on M. The quantity $\frac{\ell R(f)}{r}$ is the maximum local volume growth of $\ell-$dimensional C^r disks by f as studied by Gromov and Yomdin in [41], [40], and [11]. In section 6 below, we will see that certain aspects of these volume growths are relevant to the study of symbolic extensions. The estimates in Sections 5 and 6 below suggest the following

Conjecture 1.2 *For a C^r map $f : M \to M$ with $2 \leq r < \infty$, we have*

$$h_{sex}(f) \leq \frac{R(f) r (dim\ M)}{r - 1}.$$

We now proceed to precise statements of our main theorems. To begin, let us recall a few notions in smooth dynamics.

Let f be a C^1 diffeomorphism of the compact Riemannian manifold M, and let Λ be a compact $f-$invariant set; i.e., $f(\Lambda) = \Lambda$. We say that Λ is a *hyperbolic set* if, for each $x \in \Lambda$ there is a splitting $T_xM = E^s_x \oplus E^u_x$ of the tangent bundle to M at x and constants $\lambda > 1, C > 0$ such that,

- $Df(E^s_x) = E^s_{fx}$, $Df(E^u_x) = E^u_{fx}$, and
- for $n \geq 0$, $\mid Df^n_x(v) \mid \geq C\lambda^n \mid v \mid$ for $v \in E^u_x$, and $\mid Df^{-n}_x(v) \mid \geq C\lambda^n \mid v \mid$ for $v \in E^s_x$.

Thus, $Df \mid E^u$ and $Df^{-1} \mid E^s$ are eventually expanding on Λ in the norms induced by the Riemannian metric. It is known that the conditions above are independent of the choice of Riemannian metric and that, if they hold, then there is a metric with $C = 1$. Also, the subspaces E^u_x, E^s_x depend continuously on $x \in \Lambda$.

If the whole mainfold M is a hyperbolic set, then f is called an *Anosov* diffeomorphism. Such diffeomorphisms are clearly very special, and, since the tangent bundles of their underlying manifolds have non-trivial continuous subbundles, these manifolds also are very special. In particular, a surface which has an Anosov diffeomorphism must be the two dimensional torus.

Given a compact C^∞ manifold M, and a positive integer $1 \leq r < \infty$, let $\mathcal{D}^r(M)$ denote the space of C^r diffeomorphisms of M with the uniform C^r topology. When we consider the space $\mathcal{D}^\infty(M)$ of C^∞ diffeomorphisms, we take the usual inverse limit topology induced by the inclusions $\mathcal{D}^\infty(M) \to \mathcal{D}^r(M)$. All of these spaces are *Baire* spaces; i.e, countable intersections of dense open sets are dense. We frequently consider *residual sets*; i.e. those that contain a countable intersection of dense open sets.

Let $\mathcal{A}(M)$ denote the (possibly empty) subset of $\mathcal{D}^1(M)$ consisting of Anosov diffeomorphisms. It is known that if $\mathcal{A}(M)$ is not empty, then it is an open set in $\mathcal{D}^1(M)$ whose complement has non-empty interior. In fact, in a certain sense even if $\mathcal{A}(M) \neq \emptyset$, one should think of $\mathcal{A}(M)$ as being a rather small open set in $\mathcal{D}^1(M)$. A very important property of $f \in \mathcal{A}(M)$ is *structural stability*. If g is C^1 close to f, there is a homeomorphism $h : M \to M$ such that $hgh^{-1} = f$. Since the set of C^∞ diffeomorphisms is dense in $\mathcal{D}^1(M)$, it follows that every Anosov diffeomorphism is topologically conjugate to a C^∞ Anosov diffeomorphism, and, hence, has a principal symbolic extension. This also follows immediately from standard results concerning Markov Partitions. Indeed, an Anosov diffeomorphism is a boundedly finite-to-one factor of a subshift of finite type.

Henceforth, for the most part, we restrict ourselves to two dimensional manifolds. Thus, let $M = M^2$ be a compact orientable surface, let ω be a symplectic form on M, and let $\mathcal{D}^1_\omega(M)$ denote the set of C^1 diffeomorphisms of M preserving ω. We give $\mathcal{D}^1_\omega(M)$ the standard uniform C^1 topology which it inherits as a subspace of $\mathcal{D}^1(M)$. It is well known that $\mathcal{D}^1_\omega(M)$ is itself a Baire space.

We will prove the following theorems.

Theorem 1.3 *There is a residual subset $\mathcal{R} \in \mathcal{D}^1_\omega(M)$ such that if $f \in \mathcal{R}$, then either f is Anosov or $h_{sex}(f) = \infty$.*

For non-area preserving diffeomorphisms, we have

Theorem 1.4 *Fix $2 \leq r < \infty$. There is a residual subset $\mathcal{R}$ of the space $D^r(M)$ of C^r diffeomorphisms of M such that if $f \in \mathcal{R}$ and f has a homoclinic tangency, then f has no principal symbolic extension. Further, there exist an open set $\mathcal{U}$ in the space $D^r(M)$ of C^r diffeomorphisms of M and a constant $c = c(\mathcal{U}) > 0$ such that if $f \in \mathcal{R} \bigcap \mathcal{U}$, then $h_{sex}(f) > h_{top}(f) + c$.*

Remark. It is not true that $h_{sex}(f) = h_{top}(f)$ implies that f has a principal symbolic extension. For a class of examples dealing with this issue, see Example 1 in [7]. On the other hand, we note that simple examples can be constructed as follows. Let f be a system whose non-wandering set (see [37] for the definition) consists of two disjoint sets Λ_1 and Λ_2 such that $f \mid \Lambda_1$ is asymptotically $h-$expansive, $f \mid \Lambda_2$ is not asymptotically $h-$expansive, and $h_{top}(f \mid \Lambda_1) > h_{sex}(f \mid \Lambda_2)$. Then, since f is not asymptotically $h-$expansive, it cannot have a principal symbolic extension. On the other hand, $h_{sex}(f) = h_{top}(f)$ since the supremum of the minimal superenvelope $E\mathcal{H}$ occurs on Λ_1.

Using standard techniques embedding C^r surfaces as normally contracting invariant manifolds in higher dimensional manifolds (e.g., see the first few lines of Section 3.1 in [15]), one gets the following result:

Corollary 1.5 *Let M be any compact C^∞ manifold of dimension greater than one. Then, there is a C^1 diffeomorphism f of M for which $h_{sex}(f) = \infty$. For $2 \leq r < \infty$ there is a residual subset $\mathcal{R}$ of an open set $\mathcal{U} \subset \mathcal{D}^r(M)$, such that if $f \in \mathcal{R}$, then, $h_{sex}(f) > h_{top}(f)$.*

Combining the techniques of the proof of Theorem 1.4 with theorems in [26], [19], and [34], we obtain an interesting result concerning the Hausdorff dimension of certain invariant sets in the presence of a homoclinic tangency.

Theorem 1.6 *Fix $2 \leq r \leq \infty$. There is a residual subset $\mathcal{R}$ of the space $D^r(M)$ of C^r diffeomorphisms of a compact two dimensional manifold M such that if $f \in \mathcal{R}$ and f has a homoclinic tangency, then f has compact invariant topologically transitive sets of Hausdorff dimension two.*

It is interesting to ask if the conclusion in Theorem 1.6 can be strengthened by replacing the words "Hausdorff dimension two" by "positive Lebesgue measure." At present we have no strong opinions about such a result.

Let us conclude this section by mentioning an interesting relation between our results and a weak form of a conjecture of Palis. We will state this conjecture using Conley's notion of chain recurrence. Given a diffeomorphism f on a compact manifold M, an $\epsilon-$chain is a finite sequence $x_0, x_1, \ldots, x_n$ in M such that $d(f(x_{i+1}), x_i) < \epsilon$ for $0 \leq i < n$. A point x is *chain recurrent* if, for any $\epsilon > 0$ there is an $\epsilon-$chain starting and ending at x. The set of all chain recurrent points, denoted $\mathcal{R}(f)$, is a compact $f-$invariant set. We call a diffeomorphism f *hyperbolic* if its chain recurrent set is a hyperbolic set. It is known [9] that the set of hyperbolic diffeomorphisms coincides with those satisfying Smale's Axiom A and the no cycle property (the referenced article is for flows, but carries over to diffeomorphisms by the standard technique of taking suspensions [39]). They form an open set, and even coincide with the *chain stable* diffeomorphisms. Here, we say that a C^1 diffeomorphism is *chain stable* if there is a neighborhood $\mathcal{N} \subset \mathcal{D}^1(M)$ of f such that if $g \in \mathcal{N}$, then $(g, \mathcal{R}(g))$ is topologically conjugate to $(f, \mathcal{R}(f))$. Let $\mathcal{H}(M)$ denote the set of hyperbolic diffeomorphisms in M. Using chain stability and C^∞ approximations, it follows that any hyperbolic diffeomorphism f has a principal symbolic extension.

The weak Palis conjecture is the following:

Conjecture 1.7 *For $1 \leq r \leq \infty$ there is a residual subset $\mathcal{B}$ of C^r diffeomorphisms on a compact surface such that if $f \in \mathcal{B}$, then either $f \in \mathcal{H}(M)$ or f has a homoclinic tangency.*

We note that if we replace the word "residual" by the word "dense", then this conjecture has been proved for $r = 1$ by Pujals and Samborino [35].

However, it is still not known whether hyperbolicity is actually C^1 dense on surfaces; i.e., whether $\mathcal{D}^1(M^2) \setminus \mathcal{H}(M^2)$ has non-empty interior.

Observe that Conjectures 1.1 and 1.7 imply the following type of classification for surface diffeomorphisms:

$$\begin{array}{cccccc} \text{hyperbolic} & \text{or} & r = \infty & \Longrightarrow & \text{PSYM} \\ \text{non-hyperbolic} & \text{and} & 2 \leq r < \infty & \Longrightarrow & \text{SYM (generically)} \\ \text{non-hyperbolic} & \text{and} & r = 1 & \Longrightarrow & \text{NSYM (generically)} \end{array}$$

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2 An overview of the proofs of Theorems 1.3, 1.4, and 1.6

The proofs of these results will combine the (topological) theory in [2] and the (smooth) theory of homoclinic tangencies. We give here a very rough description of how we use these theories (we use some concepts which are defined later in the paper). For typical smooth systems, one can find a non-decreasing generating sequence of partitions $\alpha_1 \leq \alpha_2 \leq \dots$ whose boundaries are $\mu-$null sets for every $\mu \in \mathcal{M}(f)$ (we call this an *essential sequence* of partitions). In this case, the sequence of entropies $h_\mu(\alpha_k, f)$ determines an entropy structure in the sense of [6]. From [2], the rate of convergence of $h_\mu(\alpha_k, f)$ to $h_\mu(f)$ for $\mu \in \mathcal{M}(f)$ gives a criteria for

(a) when a map has a symbolic extension, and

(b) how close (in the metric and topological entropy senses) a symbolic extension can be to a given system

For example, at one extreme (see Theorem 4.3), a map f has a *principal symbolic extension* if and only if

$$h_\mu(\alpha_k, f) \to h_\mu(f) \quad (\text{as} \quad k \to \infty) \tag{1}$$

uniformly in μ.

At the opposite extreme (Proposition 4.4), a sufficient condition for a map f to have no symbolic extension at all is that the convergence in (1) is highly non-uniform in the following sense.

There exist $\rho_0 > 0$ and a compact set $\mathcal{E} \subset \mathcal{M}(f)$ such that for every $\mu \in \mathcal{E}$ and every $k > 0$, (2)

$$\limsup_{\nu \to \mu, \quad \nu \in \mathcal{E}} h_\nu(f) - h_\nu(\alpha_k, f) > \rho_0.$$

For Theorem 1.3, suppose we are given a C^1 symplectic surface diffeomorphism f in the complement of the closure of the Anosov diffeomorphisms. Using statement **(C)** in section 1, we can C^1 perturb f to a symplectic f_1 having an interval I of homoclinic tangencies between the stable and unstable manifolds $W^u(p(f_1))$ and $W^s(p(f_1))$ for some hyperbolic saddle periodic point $p(f_1)$. We choose symplectic coordinates in which I is flat and then take a further symplectic perturbation to a map f_2 in which $I \subset W^s(p(f_2))$ and $W^u(p(f_2))$ oscillates rapidly near I (see Figure 1 in Section 5). Using methods of hyperbolic dynamics, we next take more perturbations to finally obtain a symplectic map g for which an approximate version of statement (2) holds persistently (see Lemma 5.1). Then, we employ methods of residual sets to get the full statement (2) and complete the proof of Theorem 1.3.

For Theorems 1.4 and 1.6, we again start with statement **(C)**, and take C^r perturbations ($r \geq 2$) to get a C^r diffeomorphism with an interval of homoclinic tangencies and then one with a lot of unstable oscillations near the interval of tangencies. To proceed for Theorem 1.4, we make use of a transfinite characterization of the minimal superenvelope entropy function $E\mathcal{H}$, methods of hyperbolic dynamics, and a special technique (Lemma 6.2) to get measures with a certain non-uniformity in the convergence in statement (1). Restrictions imposed by the C^r topology only allow us to obtain positive residual entropy. Finally, for Theorem 1.6, we first perturb a system with an interval of homoclinic tangencies to get a new system with an invariant zero dimensional hyperbolic set meeting its stable and unstable manifolds in "thick" Cantor sets. We apply methods first given in Manning-McCluskey [19], and later extended by Palis, Takens, Viana in [33], [34]) to relate these Cantor sets to Hausdorff dimension. Again, we complete the proof with methods involving residual sets.

3 Preliminaries

We first make some definitions. Fix $f \in D^r(M)$ with $r \geq 1$.

Fix a Riemannian metric on M. The induced Finsler structure $|\cdot|$ on M will be called a *Riemannian norm* on M. Let d be the associated topological metric (i.e. distance function) on M.

Given a point $x \in M$ we define the stable and unstable sets of x by

$$W^s(x) = \{y \in M : d(f^n y, f^n x) \to 0 \text{ as } n \to \infty\}$$

and

$$W^u(x) = \{y \in M : d(f^n y, f^n x) \to 0 \text{ as } n \to -\infty\}.$$

It is clear that these are *invariant* in the sense that $f(W^\sigma(x)) = W^\sigma(x)$ for $\sigma = s, u$.

For a point $x \in M$, let $O(x)$ denote the orbit of x; i.e., the set $\{f^n(x) : n \in \mathbf{Z}\}$.

We define, for $\sigma = u, s$,

$$W^\sigma(O(x)) = \bigcup_{n \in \mathbf{Z}} W^\sigma(f^n x).$$

Let Λ be a hyperbolic set for f. We call Λ a *hyperbolic basic set* if

1. there is a neighborhood U of Λ such that

$$\bigcap_{n \in \mathbf{Z}} f^n(U) = \Lambda$$

   and

2. f has a dense orbit in Λ.

A neighborhood U as in the preceding definition is called an *adapted neighborhood* for Λ.

The following theorems about hyperbolic basic sets are well-known; e.g., see Theorem 3.2 in [13].

Theorem 3.1 *(Stable Manifold Theorem) Let Λ be a hyperbolic basic set for the C^r diffeomorphism f on the compact C^∞ manifold M. Let $T_xM = E^s_x \oplus E^u_x$ be the hyperbolic splitting of the tangent space to M at $x \in \Lambda$.*

Then, for $\sigma = s, u$, the set $W^\sigma(x)$ is a C^r injectively immersed copy of E_x^σ which is tangent at x to E_x^σ. Moreover, the submanifolds $W^\sigma(x)$ depend C^r continuously on compact sets as x varies in Λ.

Given a compact subset $E \subset M$, let $Emb(E, M)$ denote the space of continuous embeddings from E to M; i.e., the space of injective continuous maps from E into M. We give $Emb(E, M)$ the standard metric

$$d(h_1, h_2) = \sup_{x \in E} d(h_1 x, h_2 x).$$

For a subset $E \subset M$, let $i_E : E \to M$ denote the inclusion map.

Theorem 3.2 *(Persistence of hyperbolic basic sets) Let $\Lambda = \Lambda_f$ be a hyperbolic basic set for the C^1 diffeomorphism f on M with adapted neighborhood U. Given $\epsilon > 0$, there is a neighborhood $\mathcal{N}_\epsilon$ of f in $D^1(M)$ such that if $g \in \mathcal{N}_\epsilon$, then $\Lambda_g = \bigcap_n g^n(U)$ is a hyperbolic basic set for g and there is a unique continuous embedding $h_g : \Lambda_f \to M$ such that $h_g(\Lambda_f) = \Lambda_g$, $gh_g = h_g f$ and $d(h_g, i_{\Lambda_f}) < \epsilon$. Moreover, the map $g \to h_g$ from $\mathcal{N}_\epsilon$ to $Emb(\Lambda_f, M)$ is continuous and $h_f = i_{\Lambda_f}$.*

A periodic point p of f is a point such that there is a positive integer $n > 0$ such that $f^n(p) = p$. The least such positive integer $\tau(p)$ is called the *period* of p. The periodic point p is *hyperbolic* if the eigenvalues of the derivative $Df^{\tau(p)}(p)$ have modulus different from 1. We call the eigenvalues of $Df^{\tau(p)}(p)$ the *eigenvalues of* p. They are the same for all points in the orbit of p. Thus, a periodic point p is hyperbolic if and only if the orbit of p is a hyperbolic basic set. The hyperbolic periodic point p is called a *hyperbolic saddle point* if it has eigenvalues of modulus greater than one and less than one. In the case of a surface diffeomorphism, this of course implies that the eigenvalues are real and of multiplicity one.

More generally, we say that an $f-$invariant set Λ is *periodic* if there are a subset $\Lambda_1 \subset \Lambda$ and a positive integer $\tau > 0$ such that

1. $f^\tau(\Lambda_1) = \Lambda_1$,
2. $f^j(\Lambda_1) \cap \Lambda_1 = \emptyset$ for $0 \le j < \tau$, and
3. $\Lambda = \bigcup_{0 \le j < \tau} f^j(\Lambda_1)$.

In this case, we call τ the *period* of Λ, and we call Λ_1 a *base* of Λ. If Λ is a periodic hyperbolic basic set with base Λ_1 and period τ, then there is a neighborhood U_1 of Λ_1 such that

1. $f^j(U_1) \bigcap U_1 = \emptyset$ for $0 \leq j < \tau$, and

2. if $U = \bigcup_{0 \leq j < \tau} f^j(U_1)$ then U is an adapted neighborhood of Λ.

In this case, we call U_1 an adapted base neighborhood for the base Λ_1 of Λ.

Let $\mathcal{M}(f)$ denote the space of $f-$invariant probability measures on M. Let $\mathcal{M}_e(f)$ denote the ergodic elements of $\mathcal{M}(f)$.

A measure $\mu \in \mathcal{M}(f)$ is called a *hyperbolic* measure for f if its topological support $supp(\mu)$ is contained in a hyperbolic basic set for f. Note that this differs from some current usage in which measures with non-zero characteristic exponents are frequently called hyperbolic measures.

Let ρ be a metric on $\mathcal{M}(f)$ giving the topology of weak convergence: e.g. let $\phi_1, \phi_2, \ldots$ be a countable dense subset of the unit ball in $C(M, \mathbf{R})$ and set

$$\rho(\mu, \nu) = \sum_{i \geq 1} \frac{1}{2^i} | \mu(\phi_i) - \nu(\phi_i) |.$$

In the remainder of this section we consider compact $f-$invariant subsets Λ, Λ_1, and Λ_2 of M. No hyperbolicity conditions are assumed.

If Λ is such a set, let $\mathcal{M}(\Lambda)$ denote the set of $f-$invariant probability measures supported in Λ. Then, $\mathcal{M}(\Lambda)$ is clearly a compact subset of $\mathcal{M}(f)$.

Let ρ_H denote the Hausdorff metric on the collection of compact subsets of $\mathcal{M}(f)$.

For two compact $f-$invariant subsets Λ_1, Λ_2, let

$$\bar{\rho}(\Lambda_1, \Lambda_2) = \rho_H(\mathcal{M}(\Lambda_1), \mathcal{M}(\Lambda_2)).$$

Thus, $\bar{\rho}(\Lambda_1, \Lambda_2) < \epsilon$ iff for each $\mu \in \mathcal{M}(\Lambda_1)$ there is a $\nu \in \mathcal{M}(\Lambda_2)$ such that $\rho(\mu, \nu) < \epsilon$ and vice versa.

Below the term *partition* means a finite Borel measurable partition.

Given $\mu \in \mathcal{M}(f)$ and two partitions α, β, and a positive integer n, we set

$$\alpha \bigvee \beta = \{A \bigcap B : A \in \alpha, B \in \beta, A \bigcap B \neq \emptyset\},$$

$$\alpha^n = \bigvee_{i=0}^{n-1} f^{-i}(\alpha),$$

$$H_\mu(\alpha) = -\Sigma_{A\in\alpha}\mu(A)\log\mu(A),$$

$$H_\mu(\beta \mid \alpha) = H_\mu(\alpha \bigvee \beta) - H_\mu(\alpha),$$

$$h_\mu(\alpha) = h_\mu(\alpha, f) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha^n) = \inf_{n>0} \frac{1}{n} H_\mu(\alpha^n),$$

$$h_\mu(\beta \mid \alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\beta^n \mid \alpha^n) = \inf_{n>0} \frac{1}{n} H_\mu(\beta^n \mid \alpha^n). \tag{3}$$

We remark that the fact the we have the "inf" in (3) follows from [8].

Observe that

$$h_\mu(\alpha \bigvee \beta) = h_\mu(\alpha) + h_\mu(\beta \mid \alpha). \tag{4}$$

Suppose that Λ is a periodic invariant set for f with base Λ_1 and $\alpha = \{A_1, A_2, \ldots, A_s\}$ is a finite partition of M. We say that Λ is *subordinate* to α if for every positive integer n, there is an element $A_{i_n} \in \alpha$ such that $f^n(\Lambda_1) \subseteq A_{i_n}$. Observe that in this case, if n is any positive integer, then there is a unique element $B \in \alpha^n$ so that $\Lambda_1 \subseteq B$. Hence, if μ is an invariant probability measure with $\mu(\Lambda) = 1$, then $h_\mu(\alpha) = 0$.

4 Entropy Structures

We recall some elements from the theory of entropy structures as developed in Boyle-Downarowicz [2].

Let $f : M \to M$ be a homeomorphism of the compact metric space M.

An increasing sequence $\alpha_1 \leq \alpha_2 \leq \ldots$ of partitions of M is called *essential* (for f) if

1. $diam(\alpha_k) \to 0$ as $k \to \infty$, and
2. $\mu(\partial \alpha_k) = 0$ for every $\mu \in \mathcal{M}(f)$. Here $\partial \alpha_k$ denotes the union of the boundaries of elements in the partition α_k.

Note that essential sequences of partitions may not exist (e.g., for the identity map on the unit interval). However, for any finite entropy system (f, M) it follows from the work of Lindenstrauss and Weiss [17], [18] that the product $f \times R$ with R an irrational rotation has essential sequences of partitions. It is known that f has a (principal) symbolic extension if and only if $f \times R$ has one, so we may replace f by $f \times R$ in considering the questions of symbolic extensions, and assume there are essential sequences of partitions (this follows from Lemma 7.9 in [2] and statement 2 in Theorem 4.3). This allows us to define a sequence of uppersemicontinuous functions on $\mathcal{M}(f)$ whose properties completely determine the existence of symbolic extensions and their entropy functions.

Alternatively, in the present case, since we are dealing with elements of residual sets of smooth diffeomorphisms, we can easily obtain essential sequences of partitions without changing the space. Indeed, let M be a compact C^∞ manifold. Let $\alpha_1 \leq \alpha_2 \ldots$ be an increasing sequence of partitions whose diameters tend to 0 such that each α_k is the partition into simplexes given by a smooth triangulation $\mathcal{T}_k$ of M. We call $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \}$ a *simplicial* sequence of partitions on M.

Proposition 4.1 *Let $\mathcal{A}$ be a simplicial sequence of partitions on M. Then, for each $1 \leq r \leq \infty$, there is a residual subset $\mathcal{R}^r_{\mathcal{A}} \subset \mathcal{D}^r(M)$ such that if $f \in \mathcal{R}^r_{\mathcal{A}}$, then $\mathcal{A}$ is an essential sequence of partitions for f.*

Proof. Let $m = dim\ M$. Fix k, and consider the partition

$$\alpha_k = \{\Delta_i\}$$

where Δ_i is a smooth simplex in M. Thus, Δ_i is the image of a smooth map ψ_i from an open $\ell-$disk in $\mathbf{R}^\ell$ for some $0 \leq \ell \leq m$. Let β_k denote the $(m-1)-$skeleton of the triangulation $\mathcal{T}_k$. Thus, β_k consists of all the simplexes in $\mathcal{T}_k$ of dimension less than m.

For a subset $B \subset M$, let $dim\ B$ denote its topological dimension.

Using standard transversality techniques (e.g., see [14]), we find a residual subset $\mathcal{R}_1$ of $\mathcal{D}^r(M)$ such that if $f \in \mathcal{R}_1$, $n > 0$, and a_1, a_2 are simplexes in β_k such that $f^n(a_1) \bigcap a_2 \neq \emptyset$, then $f^n(a_1)$ meets a_2 transversely. This, of course, implies that $a_1 \bigcap f^{-n} a_2$ is a countable union of open smooth disks of dimension less than $dim\ a_1$. Repeating this transversality construction using the open smooth disks just mentioned covering the intersections $a_1 \bigcap f^{-n} a_2$ instead of a_1, we see that there is a residual subset $\mathcal{R}_2$ of $\mathcal{D}^r(M)$ such that if $f \in \mathcal{R}_2$, $0 < n_1 < n_2$, and a_1, a_2, a_3 are three simplexes in β_k such that $a_1 \bigcap f^{-n_1} a_2 \bigcap f^{-n_2} a_3 \neq \emptyset$, then,

$$dim\ \left(a_1 \bigcap f^{-n_1} a_2 \bigcap f^{-n_2} a_3\right) < dim\ \left(a_1 \bigcap f^{-n_1} a_2\right).$$

Continuing with this construction yields a residual subset $\mathcal{R} \subset \mathcal{D}^r(M)$ such that if $f \in \mathcal{R}$, then for any $x \in M$, the forward orbit of x meets the elements of β_k at most m times. Hence, if $\mathcal{B}_k$ is the union of the elements of β_k we have that

1. $\partial \alpha_k \subset \mathcal{B}_k$, and
2. the orbit of any point of M meets $\mathcal{B}_k$ at most finitely many times.

By the Poincare recurrence theorem, we have that $\mu(\mathcal{B}_k) = 0$ for any $\mu \in \mathcal{M}(f)$. Hence, $\mathcal{A}$ is essential for $f \in \mathcal{R}$. QED.

Now, fix an essential sequence of partitions $\mathcal{A} = \{\alpha_k\}$ for f.

Since $\mu(\partial A) = 0$ for each $A \in \alpha_k$, it follows that the function $\mu \to \mu(E)$ is continuous for each $E \in \alpha_k^n$ for any $n > 0$. Thus, for fixed k, the function

$$h_k = h_k(\mu) = h_\mu(\alpha_k)$$

is an infimum of continuous functions on $\mathcal{M}(f)$. Hence, it is uppersemicontinuous. Likewise, by (3) and (4), for each $k \geq 1$, $h_{k+1}(\mu) - h_k(\mu)$ is non-negative and also uppersemicontinuous. Thus, the essential sequence $\mathcal{A}$

of partitions for f gives us a non-decreasing sequence of uppersemicontinuous functions with uppersemicontinuous differences $\mathcal{H} \stackrel{\text{def}}{=} \{h_k\}$ on $\mathcal{M}(f)$ as studied in [2].

We define a superenvelope ϕ of the sequence $\mathcal{H} = \{h_k\}$ to be a function $\phi : \mathcal{M}(f) \to \mathbf{R}$ such that $\phi - h_k$ is non-negative and uppersemicontinuous for each k. Additionally, we admit one unbounded superenvelope - the constant infinity function.

The pointwise infimum of any collection of superenvelopes is again a superenvelope, so it follows that there is a unique minimal superenvelope of $\mathcal{H}$ which we denote by $E\mathcal{H} = E\mathcal{H}(\cdot, f)$.

Note that

$$h_\mu = \sup_k h_k(\mu),$$

so, $E\mathcal{H}(\mu) \geq h_\mu$ for each $\mu \in \mathcal{M}(f)$.

The following transfinite inductive formula [2] for $E\mathcal{H}$ will be useful in the proof of Theorem 1.4.

For a given bounded function $g : \mathcal{M}(f) \to \mathbf{R}$, let $\tilde{g}$ denote the uppersemicontinuous envelope of g; that is,

$$\tilde{g}(\mu) = \limsup_{\nu \to \mu} g(\nu).$$

We also let $\tilde{g} \equiv \infty$ for any unbounded g.

It is easy to see that $\tilde{g}$ is the pointwise infimum of all continuous functions ϕ such that $\phi(\mu) \geq g(\mu)$ for all $\mu \in \mathcal{M}(f)$. Write h for the entropy function $h(\mu) = h_\mu(f)$.

We define a family of functions $u_\zeta : \mathcal{M}(f) \to \mathbf{R}$ for all ordinals ζ.

Set $u_0 = 0$. Having defined u_ζ, set

$$u_{\zeta+1} = \lim_{k \to \infty} (\widetilde{u_\zeta + h - h_k}). \tag{5}$$

This defines u_ζ for successor ordinals. For a limit ordinal η, let

$$u_\eta = \widetilde{\sup_{\zeta < \eta} u_\zeta}.$$

The following Proposition is a consequence of Theorem 3.3 in [2].

Proposition 4.2 *Let $\{u_\zeta\}$ be the family of functions defined above and assume that $E\mathcal{H}$ is bounded. Then $u_\zeta = u_{\zeta+1}$ if and only if $E\mathcal{H} = h + u_\zeta$. Moreover, this occurs at a countable ordinal.*

Given an arbitrary homeomorphism $f : M \to M$ of the compact metric space M, let us recall some definitions from section 1. Let $S(f)$ denote the set of symbolic extensions (g, Y, π) of f. For $(g, Y, \pi) \in S(f)$, $\mu \in \mathcal{M}(f)$ set

$$S(\mu, g) = \{\nu \in \mathcal{M}(g) : \pi_\star \nu = \mu\}.$$

Define $h_{sex}(\mu) = h_{sex}(\mu, f)$ by

$$h_{sex}(\mu) = \begin{cases} \inf_{(g,Y,\pi)\in S(f)} \sup_{\nu \in S(\mu,g)} h_\nu(g) & \text{if } S(f) \neq \emptyset \\ \infty & \text{if } S(f) = \emptyset \end{cases}$$

and

$$h_{sex}(f) = \begin{cases} \inf_{(g,Y,\pi)\in S(f)} h_{top}(g) & \text{if } S(f) \neq \emptyset \\ \infty & \text{if } S(f) = \emptyset \end{cases}$$

Note that $h_{sex}(\cdot, f)$ is either a bounded real-valued function or identically equal to ∞.

In the following, we let $\epsilon > 0$ and $\delta > 0$ denote real numbers and n denote a positive integer.

Following Katok [16], we define the $d_{f,n}$ metric on M, by

$$d_{f,n}(x, y) = \max_{0 \le j < n} d(f^j x, f^j y).$$

A set $E \subset M$ is $(n, \delta) -$ *separated* if whenever $x, y \in E$ and $x \neq y$ we have $d_{f,n}(x, y) > \delta$. Let $B(x, n, \epsilon)$ denote the closed ϵ-ball about x in the $d_{f,n}$ metric.

For an arbitrary subset K of M we define $r(n, \delta, K)$ to be the maximal cardinality of an (n, δ)-separated subset of K.

Given $x \in M$, and fixed $\epsilon > 0$, we set

$$h_x(\epsilon) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \delta, B(x, n, \epsilon)).$$

This measures the topological entropy of the set of points whose forward orbits remain $\epsilon-$close to that of x for $n \ge 0$.

We will say that a homeomorphism f on M is *asymptotically* $h-$expansive if

$$\lim_{\epsilon \to 0} \sup_{x \in M} h_x(\epsilon) = 0.$$

By Proposition (20.8) in [5] or Corollary 2.1(b) in [21], this definition is equivalent to the original one given by Misiurewicz in terms of open covers in [21](see also [6] for other equivalent conditions).

The following structure theorem for symbolic extensions, proved in [2], is essential for our work here.

Theorem 4.3 *Given (f, M) as above, the following statements are true.*

1. $h_{sex}(f) = \sup_{\mu \in \mathcal{M}(f)} E\mathcal{H}(\mu, f)$,
2. $h_{sex}(\mu, f) = E\mathcal{H}(\mu, f)$ *as functions on* $\mathcal{M}(f)$,
3. f *has a principal symbolic extension iff*
$$E\mathcal{H} = h \text{ iff}$$
   f *is asymptotically* $h-$*expansive iff*
$$h_k \to h \text{ uniformly },$$
4. f *has a symbolic extension iff* $\sup E\mathcal{H} < \infty$,
5. $h_{sex}(\cdot, f)$ *is realized by a symbolic extension iff* $E\mathcal{H}$ *is affine, and*
6. $h_{sex}(f)$ *is realized by a symbolic extension iff there is an affine superenvelope* ϕ *of* $\mathcal{H}$ *such that*
$$sup\ \phi = sup\ E\mathcal{H}.$$

Next, we present a sufficient condition for f to have $h_{sex}(f) = \infty$. That is, f will have no symbolic extension.

Proposition 4.4 *Suppose $\mathcal{E}$ is a compact subset of $\mathcal{M}(f)$ such that there is a positive real number ρ_0 such that for each $\mu \in \mathcal{E}$ and each $k > 0$,*

$$\limsup_{\nu \in \mathcal{E}, \nu \to \mu} [h_\nu(f) - h_k(\nu)] > \rho_0. \tag{6}$$

Then,

$$h_{sex}(f) = \infty. \tag{7}$$

Proof.

Suppose $E\mathcal{H} < \infty$. Consider the restricted functions $\bar{h}_k = h_k \mid \mathcal{E}$, $\bar{h} = h \mid \mathcal{E}$, $\bar{\mathcal{H}} = (\bar{h}_k)$, and let $E\bar{\mathcal{H}}$ denote the minimal superenvelope of $\bar{\mathcal{H}}$ on the compact set $\mathcal{E}$.

Now, (6) and the fact that $E\bar{\mathcal{H}} - \bar{h}_k$ is uppersemicontinuous on $\mathcal{E}$ give

$$E\bar{\mathcal{H}}(\mu) - h_k(\mu) \geq \rho_0 \ \forall k, \forall \mu \in \mathcal{E} \tag{8}$$

yielding $E\bar{\mathcal{H}}(\mu) - h_\mu(f) \geq \rho_0$ on $\mathcal{E}$.

Also, since $E\bar{\mathcal{H}} \leq E\mathcal{H} \mid \mathcal{E} < \infty$, this contradicts the following modification of Proposition 3.1 in [2].

Lemma 4.5 *If $E\bar{\mathcal{H}} < \infty$ on $\mathcal{E}$ then there is a dense subset of $\mathcal{E}$ on which $E\bar{\mathcal{H}} = h$.*

Proof of Lemma 4.5:

For convenience, let us drop the "bars" and restrict everything to $\mathcal{E}$.

Let us write USC for *uppersemicontinuous* and LSC for *lowersemicontinuous.*

If the Lemma is false, then there is an non-empty open set $\mathcal{U}_1$ on which $E\mathcal{H} - h > 0$. Since $h_k(\mu) \nearrow h_\mu$, we have

$$E\mathcal{H} - h_k \searrow E\mathcal{H} - h.$$

Hence, $E\mathcal{H} - h$ is USC, so

$$\mathcal{U}_1 = \bigcup_{\ell \in \mathcal{N}} \{\mu \in \mathcal{U}_1 : E\mathcal{H}(\mu) - h_\mu \geq \frac{1}{\ell}\}$$

is a countable union of relatively closed sets. Then, one must have interior, so we get that there exists an open set $\mathcal{U}_2 \subset \mathcal{U}_1$ and an $\epsilon > 0$ with

$$E\mathcal{H} - h > \epsilon \text{ on } \mathcal{U}_2.$$

Now, the characteristic function $\chi_{\mathcal{U}_2}$ is LSC, so

$E\mathcal{H} - \epsilon\chi_{\mathcal{U}_2}$ is USC, and

$E\mathcal{H} - \epsilon\chi_{\mathcal{U}_2} - h_k = (E\mathcal{H} - h_k) + (-\epsilon\chi_{\mathcal{U}_2})$ is the sum of USC functions, so it is USC. Further, it is greater than or equal to $E\mathcal{H} - \epsilon\chi_{\mathcal{U}_2} - h$, so it is non-negative.

Hence, we get that $E\mathcal{H} - \epsilon\chi_{\mathcal{U}_2}$ is a superenvelope below $E\mathcal{H}$, contradicting the assumption that $E\mathcal{H}$ was minimal.

5 Proof of Theorem 1.3

As above M is a compact C^∞ surface with a given symplectic form ω, and $\mathcal{D}^1_\omega(M)$ is the space of C^1 diffeomorphisms of M preserving ω.

Let $\{\alpha_k\}, k \geq 1$, be a fixed increasing sequence of simplicial partitions with $diam(\alpha_k) \to 0$ as above.

Given a hyperbolic ergodic invariant measure $\mu \in \mathcal{M}_e(f)$ let $\chi(\mu)$ denote the positive characteristic exponent of μ.

By the Oseledec theorem and the Ruelle inequality (Theorems S.2.9 and S.2.13 in [12]), we then have that, for $\mu-$almost every point x,

$$\lim_{n\to\infty} \frac{1}{n} \log \mid Df^n_x \mid = \chi(\mu),$$

and

$$h_\mu(f) \leq \chi(\mu).$$

For a hyperbolic periodic point p of f with period $\tau(p)$, , we let μ_p denote the orbit measure given by

$$\mu_p = \frac{1}{\tau(p)} \sum_{x\in O(p)} \delta_x$$

where $O(p)$ denotes the orbit of p and δ_x is the point mass at x. Let $\chi(p)$ denote the positive characteristic exponent of p.

For a given diffeomorphism f, let $H_n(f)$ denote the collection of hyperbolic periodic points of f of period less than or equal to n, and let $H(f) = \bigcup_n H_n(f)$. Note that for a given n, $H_n(f)$ might be empty, but it is known that the set $\mathcal{R}_1$ of diffeomorphisms f in $\mathcal{D}^1_\omega(M)$ for which $H(f) \neq \emptyset$ is dense and open in $\mathcal{D}^1_\omega(M)$ (see e.g. [28] which contains further references).

For $f \in \mathcal{R}_1$, let $\tau(f)$ be the minimal period of elements in $H(f)$, and let $\mathcal{R}_{1,m}$ be the subset of $\mathcal{R}_1$ of diffeomorphisms f with $\tau(f) = m$.

Clearly, the sets $\mathcal{R}_{1,m}$ are open in $\mathcal{D}^1_\omega$ and if $n \neq m$, then $\mathcal{R}_{1,n} \bigcap \mathcal{R}_{1,m} = \emptyset$. Thus, we have the representation of $\mathcal{R}_1$ as a disjoint union of open sets

$$\mathcal{R}_1 = \bigsqcup_m \mathcal{R}_{1,m}.$$

Let

$$\chi(f) = \inf\{\chi(p) : p \in H(f) \text{ and } \tau(p) = \tau(f)\}$$

Then, $\chi(f) > 0$ and depends continuously on $f \in \mathcal{R}_1$.

As above, let $\mathcal{A}(M^2)$ denote the (possibly empty) open set of Anosov diffeomorphims on M^2, and let $Cl\mathcal{A}(M^2)$ denote its closure in $\mathcal{D}^1_\omega(M^2)$.

Let

$$\mathcal{R}_{2,m} = \mathcal{R}_{1,m} \setminus Cl\mathcal{A}(M^2)).$$

Thus, each $\mathcal{R}_{2,m}$ is a non-empty open subset of $\mathcal{D}^1_\omega(M^2)$, and we have

$$\mathcal{R}_1 \setminus Cl\mathcal{A}(M^2)) = \bigsqcup_m \mathcal{R}_{2,m}.$$

Given a positive integer n, let us say that a diffeomorphism f satisfies property $\mathcal{S}_n$ if, for each $p \in H_n(f)$,

1. *there is a zero dimensional periodic hyperbolic basic set* $\Lambda(p,n)$ *for* f *such that*

$$\Lambda(p,n) \bigcap \partial\alpha_n = \emptyset, \tag{9}$$

2. $$\Lambda(p,n) \textit{ is subordinate to } \alpha_n, \tag{10}$$

3. *there is an ergodic* $\mu \in \mathcal{M}(\Lambda(p,n))$ *such that*

$$\mid h_\mu(f) - \chi(p) \mid < \frac{1}{n}\chi(p), \tag{11}$$

   and

4. *for every ergodic* $\mu \in \mathcal{M}(\Lambda(p,n))$, *we have*

$$\rho(\mu, \mu_p) < \frac{1}{n} \textit{ and } \mid \chi(\mu) - \chi(p) \mid < \frac{1}{n}\chi(p). \tag{12}$$

Given positive integers $m \le n$, let $\mathcal{D}_{m,n}$ denote the subset of $\mathcal{R}_{2,m}$ consisting of diffeomorphisms f satisfying property $\mathcal{S}_n$.

Lemma 5.1 *For every positive integers* $m \le n$ *the set* $\mathcal{D}_{m,n}$ *is dense and open in* $\mathcal{R}_{2,m}$.

Remark. Part of Lemma 5.1 is related to constructions which were given previously in [24]. However, the proof of hyperbolicity of the set in [24] corresponding to $\Lambda(p, N)$ was only sketched and the reader was referred to somewhat complicated estimates in [31]. Here will give a simpler proof of hyperbolicity. Furthermore, the more detailed properties described in (9)–(12) were not needed and not presented in [24]. They are new results.

Assuming Lemma 5.1 we can prove Theorem 1.3 as follows.

Let

$$\mathcal{R}_2 = \bigcup_{m \geq 1} \bigcap_{n \geq m} \mathcal{D}_{m,n},$$

and let $\mathcal{R} = \mathcal{R}_2 \bigcup \mathcal{A}(M^2)$.

Then, clearly $\mathcal{R}_2$ is residual in $\mathcal{D}^1_\omega(M^2) \setminus Cl\mathcal{A}(M^2)$, and $\mathcal{R}$ is residual in $\mathcal{D}^1_\omega(M^2)$.

If $f \in \mathcal{R}$ is not Anosov, then $f \in \mathcal{R}_2$.

Now, for $f \in \mathcal{R}_2$, we set

$$\mathcal{E}_1 = \mathcal{E}_1(f) = \{\mu_p : p \in H(f) \text{ and } \chi(p) > \frac{\chi(f)}{2}\},$$

and we let

$$\mathcal{E} = \mathcal{E}(f) = Closure\ \mathcal{E}_1(f).$$

We claim:

$$\mathcal{E} \text{ satisfies (6) in Proposition 4.4 with } \rho_0 = \frac{\chi(f)}{2}. \tag{13}$$

Then, applying Proposition 4.4 proves Theorem 1.3.

Proof of (13):

Set $\rho_0 = \frac{\chi(f)}{2}$, and fix an integer $k > 0$. It suffices to prove condition (6) for any $\mu = \mu_p \in \mathcal{E}_1(f)$.

Since $\mu_p \in \mathcal{E}_1(f)$, we clearly have $\chi(p) > \rho_0$.

For each $n \geq \max(k, \tau(p))$, Lemma 5.1 gives us a periodic hyperbolic basic set $\Lambda(p, n)$ subordinate to α_n such that all the ergodic measures in $\mathcal{M}(\Lambda(p, n))$ are $\frac{1}{n}$ close to μ_p and have exponents greater than

$$\left(\frac{n-1}{n}\right) \chi(p). \tag{14}$$

Since $\Lambda(p,n)$ is subordinate to α_n, we have that any measure $\nu \in \mathcal{M}(\Lambda(p,n))$ has $h_\nu(\alpha_n) = 0$. Moreover, there is an ergodic measure $\nu_n \in \mathcal{M}(\Lambda(p,n))$ such that

$$h_{\nu_n}(f) > \left(\frac{n-1}{n}\right)\chi(p) \tag{15}$$

Since $\{\alpha_i\}$ is non-decreasing, we have that

$$h_{\nu_n}(\alpha_n) \geq h_{\nu_n}(\alpha_k),$$

which also gives $h_{\nu_n}(\alpha_k) = 0$.

So, for large n, we have

$$h_{\nu_n}(f) - h_{\nu_n}(\alpha_k) = h_{\nu_n}(f) > \rho_0.$$

Recall that, in K. Sigmund [38], it is proved that every ergodic measure supported on a hyperbolic basic set Λ is a weak-* limit of periodic point measures in $\mathcal{M}(\Lambda)$. This together with (14) gives that, for large n, we have $\nu_n \in \mathcal{E}$. Now, (13) follows by letting $n \to \infty$.

Proof of Lemma 5.1:

We first prove that $\mathcal{D}_{m,n}$ is open in $\mathcal{D}^1_\omega(M^2)$.

So, assume $f \in \mathcal{D}^1_\omega(M^2)$ and $\Lambda(p,n)$ is as in the definition of $\mathcal{D}_{m,n}$.

By Theorem 3.2, if g is C^1 near f, then there is an injective continuous map $h_g : \Lambda(p,n) \to M^2$ which is C^0 near the inclusion $i_{\Lambda(p,n)} : \Lambda(p,n) \to M^2$ such that $gh_g = h_g f$. Let $\Lambda_g(p,n) = h_g(\Lambda(p,n))$.

Then, h_g provides a topological conjugacy between $(f, \Lambda(p,n))$ and $(g, \Lambda_g(p,n))$ such that $\Lambda_g(p,n)$ is close to $\Lambda(p,n)$ in the Hausdorff metric, and, for each $\mu \in \mathcal{M}(\Lambda(p,n))$, the push-forward measure $h_{g\star}\mu$ is $\rho-$close to μ. Now, elementary methods, which will be left to the reader, can be used to prove that, for g close enough to f, (9)-(12) hold for g. This gives that $\mathcal{D}_{m,n}$ is open in $\mathcal{D}^1_\omega(M^2)$.

Next, we prove that $\mathcal{D}_{m,n}$ is dense in $\mathcal{R}_{2,m}$. This is the main technical result of the present paper.

Let $f \in \mathcal{R}_{2,m}$. We want to find g C^1-close to f so that property $\mathcal{S}_n$ holds for g.

We will expand some techniques which were used in [24] and [25].

Let $p \in H_n(f)$. For simplicity, we assume that p is a fixed point of f. The extension to general periodic points is similar and will be left to the reader.

We assume that all maps we now consider will be in $\mathcal{D}^1_\omega(M^2)$.

Using standard approximation techniques with generating functions (e.g. as in [25]), we may find a C^2 $f_1 \in D^1_\omega(M^2)$ which is C^1 close to f so that $f_1(p) = p$ and $Df_1(p) = Df(p)$. Replacing f by f_1 we may therefore assume that f is C^2.

As usual, for any set E and any $z \in E$, we let $C(z, E)$ denote the connected component of E containing z. By a C^r coordinate chart *centered* at a point z, we mean a pair (U, ψ) where U is an open neighborhood of z in M^2 and ψ is a C^r diffeomorphism from U onto an open neighborhood of 0 in $\mathbf{R}^2$ such that $\psi(z) = 0$. Let $\omega_0 = dx \wedge dy$ be the standard symplectic form on $\mathbf{R}^2$, and ω be the given symplectic form on M^2. If the diffeomorphism ψ satifies $\psi^\star\omega_0 = \omega$, then we call the pair (U, ψ) a *symplectic* coordinate chart centered at z. Since we are considering a C^∞ manifold M^2 with a C^∞ symplectic form ω on it, the Darboux theorem guarantees that, for each $z \in M$, there is a C^∞ symplectic coordinate chart (U, ψ) centered at z.

Let (x, y) denote the standard Euclidean coordinates on $\mathbf{R}^2$.

Let $\mathbf{E}^s = \{y = 0\}, \mathbf{E}^u = \{x = 0\}$ be the coordinate lines in $\mathbf{R}^2$.

The next lemma states that, if f is C^r, then we may choose a C^r symplectic coordinate chart (U, ψ) centered at p so that ψ carries the local stable and unstable manifolds of p into $\mathbf{E}^s$ and $\mathbf{E}^u$, respectively. For a neighborhood U of p where p is a hyperbolic fixed point of a diffeomorphism f, let $W^u(p, U) = C(p, W^u(p) \bigcap U)$, $W^s(p, U) = C(p, W^s(p) \bigcap U)$. When we wish to denote the dependence of these on f, we write $W^u(p, U, f)$, etc.

Lemma 5.2 *Fix $r \geq 1$. Let $f \in \mathcal{D}^r(M^2)$ be a C^r symplectic diffeomorphism of M^2, and let p be a hyperbolic fixed point of f. Then, there are neighborhoods U of p in M^2, V of 0 in $\mathbf{R}^2$ and a C^r symplectic diffeomorphism $\psi : U \to V$ such that*

$$\psi(p) = 0, \tag{16}$$

$$\psi(W^s(p, U)) \subseteq \mathbf{E}^s, \tag{17}$$

and

$$\psi(W^u(p, U)) \subseteq \mathbf{E}^u. \tag{18}$$

Proof. We begin by choosing a symplectic coordinate chart (U_1, ψ_1) centered at p. Let E^u_p, E^s_p be the subspaces of T_pM given by hyperbolicity. Following ψ_1 by a rotation, we may assume that $D\psi_1(E^s_p) = \mathbf{E}^s$, and

$D\psi_1(E_p^u) = \{x = ay\}$ for some real constant a. Letting A_1 denote the linear symplectic automorphism $A_1(x, y) = (x - ay, y)$, we have that $\psi_2 = A_1 \circ \psi_1$ is a symplectic coordinate map so that

$$f_2 \stackrel{\text{def}}{=} \psi_2 f \psi_2^{-1}$$

becomes a symplectic diffeomorphism between two neighborhoods of 0 in $\mathbf{R}^2$ having 0 as a hyperbolic fixed point with $\mathbf{E}^u$ and $\mathbf{E}^s$ as expanding and contracting subspaces, respectively.

For a small positive number δ, and a point z in a space, let $B_\delta(z)$ denote the open ball of radius δ about z.

Set

$$B_\delta^s = B_\delta(0) \bigcap \mathbf{E}^s,$$

$$B_\delta^u = B_\delta(0) \bigcap \mathbf{E}^u,$$

and

$$B_\delta = B_\delta^s \times B_\delta^u.$$

Thus, B_δ is a small square centered at 0.

Now, we can use the Hadamard-Perron theorem (stable manifold theorem) for hyperbolic fixed points [13] to conclude that there are a a small $\delta > 0$, and a C^r function $\eta^s : B_\delta^s \to B_\delta^u$ such that $D\eta^s(0) = 0$ and

$$W^s(0, B_\delta, f_2) = \{(x, y) : y = \eta^s(x).\} \tag{19}$$

Letting

$$\psi_3(x, y) = (x, y - \eta^s(x))$$

we have that $f_3 \stackrel{\text{def}}{=} \psi_3 f_2 \psi_3^{-1}$ is a C^r local symplectic diffeomorphism of $(\mathbf{R}^2, 0)$ with 0 as a hyperbolic fixed point such that $\mathbf{E}^u$, $\mathbf{E}^s$ are the expanding and contracting subspaces at 0 and such that

$$W^s(0, B_\delta, f_3) \subseteq \mathbf{E}^s.$$

Again applying the Hadamard-Perron Theorem to the diffeomorphism f_3, we have that there is a C^r function $\eta^u : B_\delta^u \to B_\delta^s$ such that

$$W^u(0, B_\delta, f_3) = \{(x,y) : x = \eta^u(y)\}. \tag{20}$$

Letting $\psi_4(x,y) = (x - \eta^s(y), y)$, the function $f_4 = \psi_4 f_3 \psi_4^{-1}$ is symplectic, has 0 as a hyperbolic fixed point, and satisfies

$$W^s(0, B_\delta, f_4) \subseteq \mathbf{E}^s, \ W^u(0, B_\delta, f_4) \subseteq \mathbf{E}^u.$$

Finally, we can choose suitable neighborhoods U, V such that the diffeomorphism $\psi = \psi_4\psi_3\psi_2$ satisfies the requirements of Lemma 5.2. QED.

Now, we consider the neighborhoods U, V given in Lemma 5.2 and the symplectic diffeomorphism $\psi : U \to V$. Let $f_4 = \psi f \psi^{-1}$ be the local representation given above. Let λ_s, λ_u be eigenvalues of $Df_4(0)$ with

$$\mid \lambda_s \mid < 1, \ \mid \lambda_u \mid > 1.$$

We need a well-known lemma which will imply that if f is C^r with $r \geq 2$, then there is a C^1 linearization of f near p so that the tangents to the coordinate curves are generated by C^1 vector fields. For a proof, see pages 163-165 in [33]

Lemma 5.3 *Assume that f_4 is C^r with $r \geq 2$ and the neighborhood V above is small. Let $z_1 \in W^u(0, V) \setminus \{0\}$ and $z_2 \in W^s(0, V) \setminus \{0\}$, and let γ_1, γ_2 be C^r curves in V passing through z_1, z_2, respectively, such that γ_1 is transverse to $W^u(0, V)$ at z_1 and γ_2 is transverse to $W^s(0, V)$ at z_2. Then, there are a small neighborhood V_1 of $W^u(0, V) \bigcup W^s(0, V)$ and two C^1 f_4-invariant non-vanishing vector fields X^u, X^s defined in V_1 such that*

1. *for each $z \in V_1$, $X^u(z)$ is not a multiple of $X^s(z)$,*
2. *$X^u \mid W^u(0, V)$ is tangent to $W^u(0, V)$,*
3. *$X^s \mid W^s(0, V)$ is tangent to $W^s(0, V)$,*
4. *$\gamma_1 \bigcap V_1$ is an integral curve of X^s, and $\gamma_2 \bigcap V_1$ is an integral curve of X^u,*
5. *$Df_{4z}(X^u(z)) = \lambda_u X^u(f_4 z)$ and $Df_{4z}(X^s(z)) = \lambda_s X^s(f_4 z)$ for $z \in V_1 \bigcap f_4^{-1} V_1$.*

Given the two vector fields X^u, X^s just described, we let $\mathcal{F}^u(z), \mathcal{F}^s(z)$ denote the integral curve of X^u, X^s through z respectively.

In the sequel, to simplify notation, we replace V_1 by V and assume that the vector fields X^u, X^s are defined in V.

Then, $\mathcal{F}^u = \{\mathcal{F}^u(z)\}, \mathcal{F}^s = \{\mathcal{F}^s(z)\}$ give two C^1 foliations $\mathcal{F}^u, \mathcal{F}^s$ on the neighborhood V of 0 in $\mathbf{R}^2$ with the following properties. For a foliation $\mathcal{F}$, let $\mathcal{F}(z)$ denote the leaf through the point z.

1. $f_4(\mathcal{F}^u(z)) = \mathcal{F}^u(f_4 z),\ f_4(\mathcal{F}^s(z) = \mathcal{F}^s(f_4 z)$ for $z \in V \cap f_4^{-1}(V)$
2. $W^u(0, V) \subset \mathcal{F}^u(0),\ W^s(0, V) \subset \mathcal{F}^s(0)$
3. the leaves $\mathcal{F}^u(z), \mathcal{F}^s(z)$ are C^r curves and depend continuously on z in the C^r topology
4. for each $z \in V$, $\mathcal{F}^u(z) \bigcap \mathcal{F}^s(z)$ is a unique point, say $\eta(z)$, and the map $z \to \eta(z)$ is a C^1 diffeomorphism from a neighborhood of 0 onto its image
5. $\eta \circ f_4 \circ \eta^{-1} = Df_4(0)$ on a neighborhood of 0.

Thus, the map η gives a local C^1 linearization of the diffeomorphism f_4. Note that, in general, the map η probably cannot always be chosen to be symplectic. Fortunately, we do not need it to be symplectic.

Let $L = Df_4(0)$ be the derivative of f_4 at 0. Assume that V is small enough so that f_4, η, η^{-1} are defined on V and $\eta f_4 \eta^{-1} = L$ on V.

Given two non-zero vectors $v, w \in \mathbf{R}^2$, let $< v, w >$ denote the standard inner product of v and w. We use the notations

$$| v | = \sqrt{< v, v >},\ ang(v, w) = \left| tan \left[\arccos \left(\frac{< v, w >}{| v || w |} \right) \right] \right|,$$

and call $ang(v, w)$ the *angle* between v and w.

Letting

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we define

$$ang(v, \mathbf{E}^s) = ang(v, e_1),\ ang(v, \mathbf{E}^u) = ang(v, e_2).$$

For a point z, a vector $v \in T_z\mathbf{R}^2 \setminus \{0\}$, and a curve γ through z, we set

$$ang(v, \gamma) = ang(v, T_z\gamma).$$

Given a positive integer k we set

$$V_k^u = V \cap f_4 V \cap f_4^2 V \cap \ldots \cap f_4^k V$$

and

$$V_k^s = V \cap f_4^{-1} V \cap f_4^{-2} V \cap \ldots \cap f_4^{-k} V.$$

For large k, V_k^u is a smooth 2-disk which is a slight thickening of $W^u(0, V)$, and V_k^s is a smooth 2-disk which is a slight thickening of $W^s(0, V)$. Below the expression $dist(E, F)$ denotes the Hausdorff distance between the two sets E and F.

Lemma 5.4 *Assume the definitions given above. Then, there are constants $0 < K_1 < K_2$ such that for every $k \geq 0$, the following properties hold.*

1. *If $z \in V_k^u \setminus V_{k+1}^u$ and $v \in T_z\mathbf{R}^2 \setminus T_z\mathcal{F}^u(z)$,*

   then
$$K_1 | \lambda_s |^k \leq dist(\mathcal{F}^u(z), \mathbf{E}^u) \leq K_2 | \lambda_s |^k, \tag{21}$$

$$ang(Df_{4z}^{-k}(v), f_4^{-k}\mathcal{F}^s(z)) \leq K_2 | \lambda_u |^{-k} | \lambda_s |^k \frac{1}{ang(v, \mathcal{F}^u(z))}. \tag{22}$$

   If, in addition, $ang(Df_{4z}^{-k}(v), f_4^{-k}\mathcal{F}^u(z)) \geq 1$, then
$$K_1 | \lambda_s |^{-k} \, ang(v, \mathcal{F}^u(z)) | v | \leq | Df_{4z}^{-k}(v) | \leq K_2 | \lambda_s |^{-k} | v |. \tag{23}$$

2. *If $z \in V_k^s \setminus V_{k+1}^s$ and $v \in T_z\mathbf{R}^2 \setminus T_z\mathcal{F}^s(z)$, , then*

$$K_1 | \lambda_u |^{-k} \leq dist(\mathcal{F}^s(z), \mathbf{E}^s) \leq K_2 | \lambda_u |^{-k}, \tag{24}$$

   and
$$ang(Df_{4z}^{k}(v), f_4^{k}\mathcal{F}^u(z)) \leq K_2 | \lambda_u |^{-k} | \lambda_s |^k \frac{1}{ang(v, \mathcal{F}^s(z))}. \tag{25}$$

   If, in addition, $ang(Df_{4z}^{k}(v), f_4^{k}\mathcal{F}^s(z)) \geq 1$, then
$$K_1 | \lambda_u |^{k} \, ang(v, \mathcal{F}^s(z)) | v | \leq | Df_{4z}^{k}(v) | \leq K_2 | \lambda_u |^{k} | v |. \tag{26}$$

Proof. The estimates (21), (22), (24), and (25) are obvious for the linear map L. Since $\eta f_4 \eta^{-1} = L$ implies that $\eta f_4^k \eta^{-1}(z) = L^k(z)$ for any

$$k \in \mathbf{Z} \text{ and } z \in \eta \left(\bigcap_{0 \le j \le k} f_4^{-j} V \right),$$

the constants K_1, K_2 can be found depending only on the C^1 size of η and verifying the corresponding estimates for f_4.

Now, let us proceed to prove (23).

As above, we first work with the linear map L.

Write $v = (v_1, v_2)$ with $v_1 \in \mathbf{E}^u, v_2 \in \mathbf{E}^s$, and let

$$w = (w_1, w_2) = (\lambda_u v_1, \lambda_s v_2) = Lv.$$

We consider the maximum norm

$$| v |' = \max(| v_1 |, | v_2 |).$$

and the standard norm

$$| v | = \sqrt{v_1^2 + v_2^2}.$$

Then, of course

$$| v |' \le | v | \le \sqrt{2} | v |'. \tag{27}$$

Let us use $a \sim b$ to mean that there are constants $0 < C_1 < C_2$ independent of k such that

$$C_1 \le \frac{a}{b} \le C_2.$$

The assumptions of (23) give

$$ang(v, \mathcal{F}^u(z)) \sim \frac{| v_1 |}{| v_2 |}, \; ang(Df_4^{-k} v, f_4^{-k} \mathcal{F}^u(z)) \sim \frac{| \lambda_s |^{-k} | v_1 |}{| \lambda_u |^{-k} | v_2 |} \ge 1.$$

If,

$$\frac{| v_1 |}{| v_2 |} \le 1,$$

then,

$$| v |' = | v_2 |,$$

and

$$\begin{aligned} | L^{-k}v |' &= | \lambda_s |^{-k} | v_1 | & (28) \\ &\sim | \lambda_s |^{-k} ang(v, \mathcal{F}^u(z)) | v_2 | & (29) \\ &= | \lambda_s |^{-k} ang(v, \mathcal{F}^u(z)) | v |'. & (30) \end{aligned}$$

Now (23) follows since $| Df_4^{-k}v | \sim | L^{-k}v | \sim | L^{-k}v |'$ and $| v |' \sim | v |$. On the other hand, if

$$\frac{| v_1 |}{| v_2 |} > 1,$$

then $ang(v, \mathcal{F}^u(z))$ is bounded above and below and

$$| v |' = | v_1 |.$$

Then, (28) gives (23).

The proof of (26) is similar.

This completes the proof of Lemma 5.4.

Let us now return to the proof of Lemma 5.1

Fix positive integers $m \leq n$. We want to prove $\mathcal{D}_{m,n}$ is dense in $\mathcal{R}_{2,m}$. For $f \in \mathcal{R}_{2,m}$ we wish to find a $g \in \mathcal{D}_{m,n}$ C^1 close to f satisfying property $\mathcal{S}_n$.

Let $\psi(x, y) = (x_1, y_1)$ be the C^r coordinate system given in Lemma 5.2. Henceforth, all perturbations in this section will be symplectic.

Using methods similar to those in [25], we may find a symplectic C^r g_1 which is C^1-near f so that $g_1(p) = p$ and $W^u(p, g_1) \cap W^s(p, g_1)$ contains an interval $I_1 \subset U$ of tangencies. We can choose $I_1 \subset W^s(p, g_1, U)$, but I_1 is far away from p in $W^u(p, g_1)$.

Then, we shrink and modify U slightly if necessary to arrange that, given a small $\epsilon > 0$ and a large positive integer N, we can take a further perturbation $g = g_{N,\epsilon}$ of f and an interval $I_1' \subset W^u(p, g)$ with the following properties:

1. p is a hyperbolic fixed point of g

2. $g_{N,\epsilon} = f$ on $U \bigcap f(U) \bigcap f^{-1}(U)$

3. $W^u(p,U,g) = W^u(p,U,f)$ and $W^s(p,U,g) = W^s(p,U,f)$

4. $I_1 \subset W^s(p,U,g)$ and $I_1' \subset W^u(p,g)$

5. Letting $I = \psi(I_1)$ and $I' = \psi(I_1')$, we have

$$I = \{a_1 \le x_1 \le a_2, y_1 = 0\} \tag{31}$$

and

$$I' = \left\{a_1 \le x_1 \le a_2, y_1 = A(N)\cos\left(\frac{\pi N(x-c)}{a_2 - a_1}\right)\right\} \tag{32}$$

where

$$A(N) = \frac{\epsilon(a_2 - a_1)}{N},\ 0 < a_1 < a_2 \text{ and } c = \frac{a_1 + a_2}{2}. \tag{33}$$

It is proved in [25] (pages 325-332) that we can arrange for the C^1 distance from $g_{N,\epsilon}$ to f to be no larger than $K_3\epsilon$ for some constant $K_3 > 0$. Also, we can make the angles between I and I' at points in $I \cap I'$ no larger than $K_3\epsilon$.

We only consider large N. Notice that

$$I \bigcap I' \text{ contains precisely } N \text{ points .} \tag{34}$$

Now, we have $g^k(I_1) \subset U$ for all $k \ge 0$, and there is a positive integer T such that, for $k \ge T$, $g^{-k}(I_1') \subset U$. Let $g_1 = \psi g \psi^{-1}$ denote the local coordinate representative of g.

We may assume that

$$I_1 \bigcup I_1' \subset U \setminus g(U),$$

$$g^{-T}(I_1 \bigcup I_1') \subset U \setminus g^{-1}(U),$$

$$\text{the positive orbit } \bigcup_{k\ge 0} g^k(I_1) \text{ is disjoint from } \partial\alpha_n, \tag{35}$$

and

$$\text{the negative orbit } \bigcup_{k \leq 0} g^k(I_1') \text{ is disjoint from } \partial\alpha_n. \tag{36}$$

Figure 1 shows these structures for the map $g_1 = \psi g \psi^{-1}$ carried over to V. The dashed curve is meant to indicate that, while the unstable manifold $W^u(p, g)$ is connected, the part carried over to V by ψ is not connected.

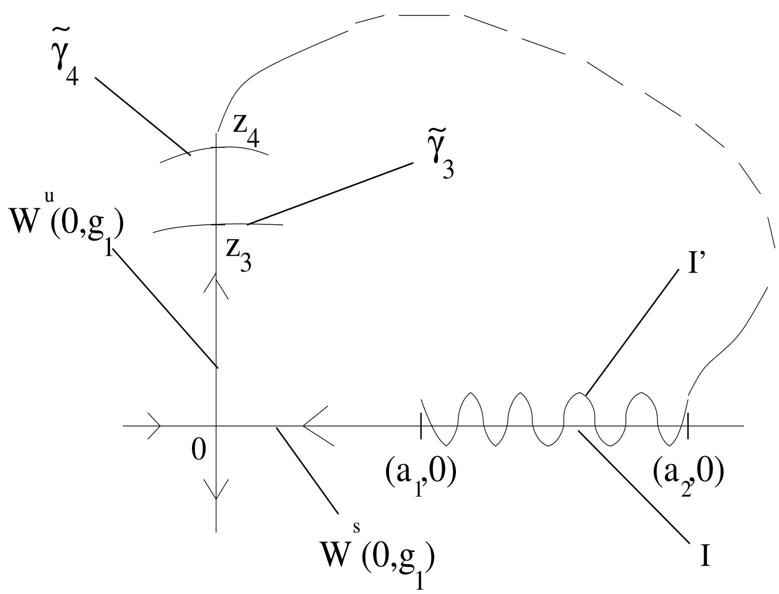

Figure 1: $W^u(0, g_1)$, $W^s(0, g_1)$ and I, I'

Let g_2 be the mapping from $V \setminus g_1^{-1}V$ into $V \setminus g_1(V)$ defined by

$$g_2 = \psi g^T \psi^{-1}.$$

Now, for large N, we will find a positive integer $k = k(N)$ and a curvilinear rectangle D_N near I with the following properties.

P1. The boundary ∂D_N consists of curves which are contained in leaves of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$.

P2. $D_N \subset V_k^s \setminus V_{k+1}^s$

P3. $g_1^k(D_N) \subset V_k^u \setminus V_{k+1}^u$

P4. $g_2(g_1^k(D_N)) \bigcap D_N$ consists of N full-height curvilinear subrectangles of D_N

P5. Set $g_3 = g_2 g_1^k$, and let
$$\Lambda_3 = \bigcap_i g_3^i(D_N)$$
be the largest g_3-invariant set in D_N.

Then, Λ_3 is a hyperbolic set for g_3 and the pair (g_3, Λ_3) is topologically conjugate to the full shift on N symbols.

P6. Let
$$\Lambda(p, N) = \bigcup_{0 \leq i < k+T} g^i(\psi^{-1}\Lambda_3).$$

Then, for N large depending on n, $\Lambda(p, N)$ is a hyperbolic set for $g = g_{\epsilon,N}$ satisfying the conditions of property $\mathcal{S}_n$.

Once these properties have been established, the proof of Lemma 5.1 will be complete.

Figure 2 shows the rectangle D_N and its $g_2 g_1^k-$image.

Let us proceed to construct the rectangle D_N.

Let $J' = g_2^{-1}(I')$. Then, J' is in $W^u(0, g_1) \setminus g_1^{-1}W^u(0, g_1)$.

Let $z_1 = (a_1, 0), z_2 = (a_2, 0)$ be the boundary points of I and let z_3, z_4 be the boundary points of J' chosen so that

$$\mid z_3 \mid < \mid z_4 \mid.$$

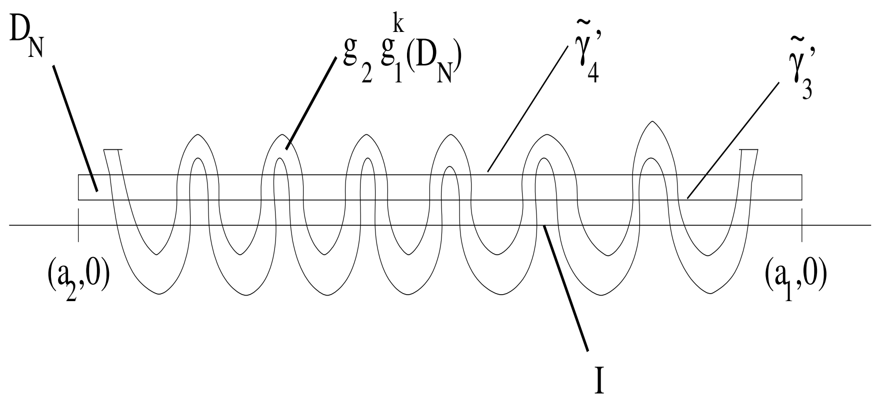

Figure 2: The rectangles D_N and $g_2g_1^k(D_N)$; $D_N \bigcap g_2g_1^k(D_N)$ has N connected components.

By the C^r version of the $\lambda-$Lemma [33], page 155, the forward g_1 orbits of the curves $\mathcal{F}^u(z_1), \mathcal{F}^u(z_2)$ contain curves arbitrarily C^r near $W^u(0, g_1)$, and the backward orbits of the curves $\mathcal{F}^s(z_3), \mathcal{F}^s(z_4)$ contain curves arbitrarily C^r near $W^s(0, g_1)$.

Let $k_0(N)$ be the least positive integer so that for $k \geq k_0(N)$

$$dist(\mathcal{F}^s(g_1^{-k}z_4), 0) \leq \frac{1}{2}\ A(N).$$

Here $A(N)$ is the amplitude defined in (33).

We will choose $k(N) = k_0(N) + n_1$ where n_1 is a positive integer independent of N. The number n_1 will just depend on the constants K_1, K_2 in Lemma 5.4.

Let $\tilde{\gamma}_3 = \mathcal{F}^s(z_3), \tilde{\gamma}_4 = \mathcal{F}^s(z_4)$, and $\tilde{\gamma}_3' = g_1^{-k(N)}\mathcal{F}^s(z_3), \tilde{\gamma}_4' = g_1^{-k(N)}\mathcal{F}^s(z_4)$.

We set D_N to be the rectangle bounded above and below by the arcs in the parts of $\tilde{\gamma}_3', \tilde{\gamma}_4'$ between $\mathcal{F}^u(z_1)$ and $\mathcal{F}^u(z_2)$, and bounded on the left and right by arcs in $\mathcal{F}^u(z_1)$ and $\mathcal{F}^u(z_2)$.

See Figure 2.

The statements below will hold for N large enough, so let us agree that in any case we may increase N without further mention.

Claim 1: The set $\Lambda_3(p, N)$ is hyperbolic for g_3

Proof.

From [30] It suffices to find a cone field $\mathcal{C}^u$ on $\Lambda_3(p, N)$ which is both expanded and co-expanded by g_3. That is, there is a constant $\lambda > 1$ so that, for $z \in \Lambda_3(p, N)$, we have

$$v \in \mathcal{C}^u(z) \Longrightarrow |\ Dg_{3z}(v)\ | \geq \lambda |\ v\ |, \tag{37}$$

and

$$v \in T_z\mathbf{R}^2 \setminus \mathcal{C}^u(z) \Longrightarrow |\ Dg_{3z}^{-1}(v)\ | \geq \lambda |\ v\ |. \tag{38}$$

Let

$$Ang(z) = \frac{1}{2} ang(\mathcal{F}^s(z), g_3(\mathcal{F}^u(g_3^{-1}z))).$$

Define

$$\mathcal{C}^u(z) = \{v \in T_z\mathbf{R}^2 : ang(v, g_3(\mathcal{F}^u(g_3^{-1}z))) \leq Ang(z)\}. \tag{39}$$

For k large, we have that $C(z, \mathcal{F}^s(z) \bigcap D_N)$ is C^2 near the part of $W^s(0)$ between z_1 and z_2, and $C(z, g_3(\mathcal{F}^u(g_3^{-1}z)) \bigcap D_N)$ is C^2 near a connected component of $I' \bigcap D_N$.

Since the angle between the connected components of $I' \bigcap D_N$ and I are $const \cdot \epsilon$, we may assume that $ang(g_3(\mathcal{F}^u(g_3^{-1}z), \mathcal{F}^s(z)) \sim \epsilon$ and less than one.

By (26), for large $k = k(N)$, we get that

$$v \in (\mathcal{C}^u(z) \setminus \{0\}) \Longrightarrow |\ Dg_{3z}(v)\ | \sim |\ \lambda_u\ |^k \epsilon > 2. \tag{40}$$

On the other hand if $v \in T_z\mathbf{R}^2 \setminus \mathcal{C}^u(z)$, then

$$ang(Dg_{2z}^{-1}v, \mathcal{F}^u(g_2^{-1}z)) > const \cdot \epsilon. \tag{41}$$

By (23), this gives

$$v \in (T_z\mathbf{R}^2 \setminus \mathcal{C}^u(z)) \Longrightarrow |\ Dg_{3z}^{-1}(v)\ | \geq const \cdot |\ \lambda_s\ |^{-k} \epsilon > 2.$$

This proves Claim 1.

Claim 2: The pair (g_3, Λ_3) is topologically conjugate to the full shift on N symbols.

Proof. We have already mentioned that our construction gives that $I \bigcap I'$ contains N points. Since D_N is a rectangle whose horizontal curves are C^2 near I, we get that $D_N \bigcap I'$ contains N connected components. Also, the g_3 images of the vertical curves in D_N are C^2 near I'. Hence, $D_N \bigcap g_3(D_N)$ consists of N disjoint full-height subrectangles of D_N. Let us label these components as $A_1, A_2, \ldots, A_N$.

Using hyperbolicity and elementary arguments concerning the g_3 images and pre-images of the boundary arcs of D_N we get that each $g_3^{-1}A_i$ is a full-width subrectangle of D_N. Now, standard induction arguments similar to those in Section 8.4 of Robinson [37] and Sections 2-4 of Moser [22] give that,

1. For each $l \geq 0$,

$$D_N^u \stackrel{\text{def}}{=} D_N \cap g_3(D_N) \cap \ldots \cap g_3^l D_N$$

   consists N^l disjoint full-height subrectangles of D_N whose widths are $const \cdot \mid \lambda_s \mid^{kl}$, and

$$D_N^s \stackrel{\text{def}}{=} D_N \cap g_3^{-1}(D_N) \cap \ldots \cap g_3^{-l} D_N$$

   consists N^l disjoint full-width subrectangles of D_N whose heights are $const \cdot \mid \lambda_s \mid^{kl}$.

2. Each component of $D_N^u \bigcap D_N^s$ is a small rectangle with diameter $const \cdot \mid \lambda_s \mid^{kl}$.

3. For each element

$$\mathbf{a} = (\ldots a(-1)(0)a(l)\ldots) \in \Sigma_N$$

   of the full N shift Σ_N, there is a unique point $\pi(\mathbf{a}) \in \Lambda_3$ such that

$$\bigcap_{l \in \mathbf{Z}} g_3^{-l} A_{a(l)} = \{\pi(\mathbf{a})\}.$$

4. The map π is a topological conjugacy from (σ, Σ_N) to (g_3, Λ_3).

This establishes properties P1-P5.

We now proceed to property P6.

Step 1: $\Lambda(p, N)$ is a zero dimensional hyperbolic basic set for g.

Let us first prove that $\Lambda(p, N)$ is hyperbolic.

Let $\overline{\Lambda} = \psi^{-1}(\Lambda_3)$.

Then, ψ is a smooth conjugacy between $(g^{k+T}, \overline{\Lambda})$ and (g_3, Λ_3). We pull the cone field from Λ_3 over to $\overline{\Lambda}$ by setting

$$\overline{\mathcal{C}^u}(z) = D\psi^{-1}\mathcal{C}^u(\psi(z)).$$

We extend this cone field to $\Lambda(p, N)$ as follows. For $z \in \Lambda(p, N)$, let j be the unique positive integer in $[0, k+T)$ such that $g^{-j}(z) \in \overline{\Lambda}$, and set

$$\overline{\mathcal{C}^u}(z) = Dg^j\overline{\mathcal{C}^u}(g^{-j}z).$$

To show that $\Lambda(p, N)$ is hyperbolic, it suffices to show that $\overline{\mathcal{C}^u}$ is eventually expanded and co-expanded by g.

The smooth conjugacy above and (40),(41) give that for $n_1 = (k+T)l$, $z \in \overline{\Lambda}$ we have

$$v \in \overline{\mathcal{C}^u}(z) \Longrightarrow \mid Dg_z^{(k+T)l}(v) \mid \geq const \cdot 2^l \mid v \mid$$

and

$$v \in T_z\mathbf{R}^2 \setminus \overline{\mathcal{C}^u}(z) \Longrightarrow \mid Dg_z^{(-k-T)l}(v) \mid \geq const \cdot 2^l \mid v \mid.$$

Now, let n_2 be large and write $n_2 = (k+T)l + j$ with l positive and $j \in [0, k+T)$. Then, each $z \in \Lambda(p, N)$ is such that there are integers $j_1 \in [0, k+T)$, $j_2 \in [0, k+T)$ such that $g^{-j_1}(z) \in \overline{\Lambda}$ and $g^{n_2 - j_2}(z) \in \overline{\Lambda}$. It follows that there is a constant C depending only on $k+T$ such that for $z \in \Lambda(p, N)$,

$$v \in \overline{\mathcal{C}^u}(z) \Longrightarrow \mid Dg_z^{n_2}(v) \mid \geq C\ 2^l \mid v \mid$$

and

$$v \in T_z\mathbf{R}^2 \setminus \overline{\mathcal{C}^u}(z) \Longrightarrow \mid Dg_z^{-n_2}(v) \mid \geq C\ 2^l \mid v \mid.$$

This proves that $\Lambda(p, N)$ is $g-$hyperbolic.

Now, it is clear from the construction that $(g, \Lambda(p, N))$ is topologically conjugate to the direct product of a periodic orbit of minimal period $k+T$ and the full $N-$shift. So, $dim\ \Lambda(p, N) = 0$.

We leave it to the reader to show the easy fact that the neighborhood

$$U_1 = \bigsqcup_{i \in [0, k+T)} g^i(\psi^{-1}D_N)$$

is an isolating neighborhood for $\Lambda(p, N)$.

This proves Step 1.

Step 2. $\Lambda(p,N)\bigcap \partial\alpha_n=\emptyset$.

This follows since the set $\Lambda(p,N)$ is can be made arbitrarily close to the orbit $O(g,I\bigcap I')$, and this latter orbit was constructed to be disjoint from $\partial\alpha_n$.

This proves (9) of property $\mathcal{S}_n$.

Step 3. Condition (11) holds.

We first note that, since $(g,\Lambda(p,N)$ is topologically conjugate to the product of a periodic orbit of period $k+T$ and the full $N-$shift, we have that the topological entropy $h_{top}(g,\Lambda(p,N))$ satisfies

$$h_{top}(g,\Lambda(p,N))=\frac{\log N}{k+T}.$$

Let us relate this to λ_u for large N.

We have

$$A(N)\sim\mid\lambda_u\mid^{-k},$$

and, each component of $I'\bigcap\{y\geq 0\}$ has diameter

$$const\cdot\frac{a_2-a_1}{2N}\sim const\cdot\frac{1}{N}.$$

The slope $s(v)$ of a vector v tangent to any connected component of $I'\bigcap D_N$ satisfies

$$s(v)\sim N\cdot A(N).$$

By construction, this slope is a constant times ϵ, so we get

$$\mid\lambda_u\mid^k\sim N. \tag{42}$$

This implies that (since T is bounded as $N\to\infty$)

$$h_{top}(g,\Lambda(p,N))=\frac{\log N}{k+T}\to\log\mid\lambda_u\mid \tag{43}$$

as $N\to\infty$.

As is well-known, the full-shift has a unique invariant probability measure of maximal entropy. Hence, so does its product with a periodic orbit, and, hence, via conjugacy, our map $(g,\Lambda(p,N))$ also has such an invariant measure μ. For large N, the above entropy estimate gives (11).

Step 4. Condition (12) holds.

We know that, for $j \in [0, k+T)$, $g^j(\psi^{-1}(D_N)) \bigcap \partial\alpha_n = \emptyset$. Since $g^j(\psi^{-1}(D_N))$ is connected, it follows that it is in a single element of the partition α. This implies that $\Lambda(p, N)$ is subordinate to α_n as required in (10), and, hence, that any invariant probability measure ν for g supported on $\Lambda(p, N)$ has $h_\nu(\alpha_n) = 0$.

The condition

$$\rho(\mu, \mu_p) < \frac{1}{n} \tag{44}$$

is obtained as follows.

It suffices to show that, given $\zeta > 0$, there is an $N = N(\zeta)$ so large that each sufficiently long orbit in $\Lambda(p, N)$ spends most of its time in the $B_\zeta(p)$, the $\zeta-$ball about p.

We first note that the upper bounds in (21) and (24) and the fact that $\lambda_u = \lambda_s^{-1}$ give that

$$diam(V_k^u \bigcap V_k^s) \leq K_2 | \ \lambda_s \ |^k.$$

This implies that

$$\bigcap_{k \in \mathbf{Z}} f_4^k(V) = \{0\},$$

so,

$$\bigcap_{l \in \mathbf{Z}} g^l U = \psi^{-1}\{0\} = \{p\}.$$

Thus, given $\zeta > 0$, there is a positive integer $n_1 = n_1(\zeta) > T$ such that $n_2 \geq n_1$ implies

$$diam(\bigcap_{-n_2 \leq i \leq n_2} g^i U) < \zeta.$$

Consider a point $z \in \overline{\Lambda}$ and a $k = \ell n_1$ with large ℓ. Since $g^i(z) \in U$ for $i \in [0, k)$, if

$$i \in [n_1, (\ell - 1)n_1),$$

we have

$$g^i z \in \bigcap_{\mid j \mid < n_1} g^j U \subset B_\zeta(p).$$

Hence, the fraction of time in $[0, \ell n_1)$ that $g^i z$ is in $B_\zeta(0)$ is $\frac{\ell-2}{\ell}$.

On the other hand, if $z \in \Lambda(p, N)$, and $\ell_1 > 1$, then there is an integer $j \in [0, k+T)$ such that $g^j(z) \in \overline{\Lambda}$, and, hence, $g^{s(k+T)+j} z \in \overline{\Lambda}$ for $s \in [0, \ell_1)$.

Thus, given $\zeta_1 > 0$, we can choose large integers ℓ and ℓ_1 such that if $k = \ell n_1$, then the fraction of times i such that the orbit segment

$$\{g^i(z) : i \in [0, \ell_1(k+T))\}$$

is in $B_\zeta(p)$ is greater than

$$\frac{(\ell-2)(\ell_1-2)}{\ell\ \ell_1} > 1 - \zeta_1.$$

This gives (44).

Next, we proceed to

$$\mid \chi(p) - \chi(\mu) \mid < \frac{1}{n} \chi(p). \tag{45}$$

First note that $\chi(p) = \log \mid \lambda_u \mid$.

Using (26), for $z \in \overline{\Lambda}$, and $v \in \overline{\mathcal{C}^u}(z) \setminus \{0\}$, we have

$$\mid Dg_z^k(v) \mid \sim \mid \lambda_u \mid^k. \tag{46}$$

Let

$$C_1 = \inf_{\substack{z \in U \setminus g^{-1}U \\ \mid v \mid = 1}} \mid Dg_z^T(v) \mid$$

and

$$C_2 = \sup_{\substack{z \in U \setminus g^{-1}U \\ \mid v \mid = 1}} \mid Dg_z^T(v) \mid.$$

This gives constants K_1, K_2 such that, for $z \in \overline{\Lambda}$, and $v \in \overline{\mathcal{C}^u}(z) \setminus \{0\}$, $q = \ell(k+T)$, we have

$$C_1^\ell K_1 | \lambda_u |^{\ell k} \leq | Dg_z^q(v) | \leq C_2^\ell K_2 | \lambda_u |^{\ell k}.$$

Now, (45) easily follows taking $k >> T$, letting $\ell \to \infty$, and using the fact that the orbit of $z \in \Lambda(p, N)$ passes through $\overline{\Lambda}$ once in every interval $[i, i + k + T)$.

This completes the proof of Lemma 5.1.

6 Proof of Theorem 1.4

We consider $r \geq 2$, and smooth compact C^r surface M. As above, let $H_n(f)$ denote the collection of hyperbolic periodic points of period no larger than n, and let $H(f) = \bigcup_{n \geq 1} H_n(f)$. We assume that $H(f) \neq \emptyset$.

Let $p \in H(f)$. Recall that a homoclinic point for p is a point

$$q \in [W^u(O(p)) \setminus O(p)] \cap [W^s(O(p)) \setminus O(p)]. \tag{47}$$

The homoclinic point q is *transverse* if the intersection in (47) is transverse. Otherwise, we say q is a *homoclinic tangency.*

We extend this definition to hyperbolic basic sets Λ in the obvious way. A homoclinic point for Λ is a point q in

$$(W^u(\Lambda) \setminus \Lambda) \cap (W^s(\Lambda) \setminus \Lambda).$$

A *homoclinic tangency* for Λ is a homoclinic point for Λ which is a tangency of $W^u(x)$ and $W^s(y)$ for some $x, y \in \Lambda$. We also say that Λ has a homoclinic tangency.

There is an equivalence relation $\sim$ on $H(f)$ defined by $p \sim q$ if and only if $W^u(O(p)) \setminus O(p)$ has a non-empty transverse intersection with $W^s(O(q)) \setminus O(q)$ and vice-versa [23]. The closure of an equivalence class is a non-empty closed $f-$invariant topologically transitive set called an $h-$closure or *homoclinic closure*. An $h-$closure is either a single periodic orbit or equals the closure of the transverse homoclinic points of some hyperbolic periodic orbit. An $h-$closure which reduces to a single periodic orbit is called *trivial*, and those which contain at least two periodic orbits (and hence infinitely many) are called *non-trivial.*

We can extend the above equivalence relation to one, also denoted by $\sim$, on the collection of hyperbolic basic sets. We say that $\Lambda_1 \sim \Lambda_2$ for such sets if and only if $W^u(\Lambda_1) \setminus \Lambda_1$ has a non-empty transverse intersection with $W^s(\Lambda_2) \setminus \Lambda_2$ and vice-versa. The $h-$closure of a hyperbolic basic set Λ is defined to be the closure of the union of the hyperbolic basic sets Λ_1 such that $\Lambda_1 \sim \Lambda$. This coincides with the $h-$closure of the hyperbolic periodic points which are contained in Λ.

A diffeomorphism f has *persistent homoclinic tangencies* if there are a hyperbolic basic set $\Lambda(f)$ for f with adapted neighborhood U and a neighborhood $\mathcal{N}$ of f in $\mathcal{D}^r(M)$ such that if $g \in \mathcal{N}$, then

$$\Lambda(g) = \bigcap_{n \in \mathbf{Z}} g^n(U)$$

has a homoclinic tangency.

It is known that, if $r \geq 2$, and f has a homoclinic tangency, then one can find g arbitrarily close to f in $\mathcal{D}^r(M)$ so that g has persistent homoclinic tangencies. This is proved in [26] when the homoclinic tangency is for a periodic orbit. In the general case, if f has a homoclinic tangency for a hyperbolic basic set, then standard methods (e.g. Lemma 8.4 in [27]) show that there are $g's$ arbitrarily C^r close to f which have homoclinic tangencies for periodic orbits. It follows from these considerations that there is a dense open subset $\mathcal{V}$ in $\mathcal{D}^r(M)$ such that if $f \in \mathcal{V}$ and f has a homoclinic tangency, then f indeed has persistent homoclinic tangencies. In addition to [26], various properties associated to persistent homoclinic tangencies are studied in [15], [10], [36], [27], and [33].

If Λ is an $f-$invariant set containing some hyperbolic periodic orbits, let $\chi(\Lambda)$ be the supremum of the characteristic exponents of those periodic orbits.

The next theorem, which has independent interest, shows that the presence of homoclinic tangencies gives a lower bound on the quantity $h_{sex}(f)$ which has the potential to be larger than $h_{top}(f)$. Later we will give examples of open sets of diffeomorphisms where this actually happens.

Theorem 6.1 *Fix $r \geq 2$, and $\mathcal{U}$ be the open subset of $\mathcal{D}^r(M)$ so that each $f \in \mathcal{U}$ has a hyperbolic basic set $\Lambda(f)$ which has persistent homoclinic tangencies. There is a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that if $f \in \mathcal{R}$, then*

$$f \text{ is not asymptotically } h\text{-expansive} \tag{48}$$

and, letting $\Lambda_1(f)$ be the homoclinic closure associated to $\Lambda(f)$, we have

$$h_{sex}(f) \geq \max\left(h_{top}(f), \frac{\chi(\Lambda_1(f)) \cdot r}{r-1}\right). \tag{49}$$

Remark. Examples of C^r non-asymptotically expansive diffeomorphisms in manifolds of dimension greater than 3 were constructed by Misiurewicz in [20]. Although the existence of such examples on surfaces has been known to experts for a long time, the examples here may be the first published version of such examples. Further, we show here an abundance of such examples.

We expect that the estimate (49) is actually an equality for typical non-hyperbolic C^r systems.

To start the proof of Theorem 6.1, let $\epsilon > 0$.

Since it is obvious that $h_{sex}(f) \geq h_{top}(f)$, Theorem 6.1 follows from the inequality

$$h_{sex}(f) \geq \frac{(\chi(\Lambda_1(f)) - \epsilon) \cdot r}{r-1}. \tag{50}$$

The first step in the proof of (50) is the following lemma.

Let (α_k) be an essential sequence of simplicial partitions as defined in Section 5. Let h_μ be the entropy of the invariant measure μ, and let $h_k = h_k(\mu) = h_\mu(\alpha_k, f)$ be the elements of the entropy structure $\mathcal{H} = (h_k)$.

Lemma 6.2 *Consider a family of ergodic measures $\mu_{0,i_1,i_2,\ldots,i_j} \in \mathcal{M}(\Lambda_1(f))$ indexed by all finite sequences of natural numbers $(i_1, i_2, \ldots, i_j) \in \mathbf{N}^j$, such that, for each $j \geq 0$,*

$$\lim_{i_{j+1}\to\infty} \mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}} = \mu_{0,i_1,i_2,\ldots,i_j}, \tag{51}$$

(for $j = 0$ we set $i_0 = 0$ so that the above includes $\lim \mu_{0,i_1} = \mu_0$).

Letting $\chi = \chi(\mu)$ denote the characteristic exponent of the ergodic measure μ and fixing a positive real number $\epsilon > 0$, suppose that

$$h(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}}) \geq \chi(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}}) - \epsilon \geq \frac{\chi(\mu_{0,i_1,i_2,\ldots,i_j}) - \epsilon}{r}, \tag{52}$$

and, for each k,

$$\lim_{i_{j+1}\to\infty} h_k(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}}) = 0. \tag{53}$$

Then

$$E\mathcal{H}(\mu_0) \geq (\chi(\mu_0) - \epsilon)\frac{r}{r-1}.$$

Proof. We will use the transfinite characterization of $E\mathcal{H}$ (Proposition 4.2), from which it follows that

$$E\mathcal{H} \geq h + \sup_{n\in\mathbf{N}} u_n.$$

First observe that, by (52), (53), and the definition of u_1,

$$\begin{aligned} u_1(\mu_{0,i_1,i_2,\ldots,i_j}) &\geq \inf_k \limsup_{i_{j+1}} \big(h(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}})) - h_k(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}})\big) \\ &\geq \frac{\chi(\mu_{0,i_1,i_2,\ldots,i_j}) - \epsilon}{r}. \end{aligned}$$

Suppose we have inductively proved that

$$u_n(\mu_{0,i_1,i_2,\ldots,i_j}) \geq \big(\chi(\mu_{0,i_1,i_2,\ldots,i_j}) - \epsilon\big) \sum_{s=1}^{n} \frac{1}{r^s}.$$

Then we can extend this to $n+1$ directly by the definition of u_{n+1} as follows:

$$\begin{aligned} u_{n+1}(\mu_{0,i_1,i_2,\ldots,i_j}) &\geq \inf_k \limsup_{i_{j+1}} \big[u_n(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}}) + \\ &\qquad + h(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}}) - h_k(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}})\big] \\ &\geq \limsup_{i_{j+1}} \left((\chi(\mu_{0,i_1,i_2,\ldots,i_j,i_{j+1}}) - \epsilon) \sum_{s=1}^{n} \frac{1}{r^s} + \frac{\chi(\mu_{0,i_1,i_2,\ldots,i_j}) - \epsilon}{r} \right) \\ &\geq (\chi(\mu_{0,i_1,i_2,\ldots,i_j}) - \epsilon) \left(\frac{1}{r} \sum_{s=1}^{n} \frac{1}{r^s} + \frac{1}{r} \right) \\ &= (\chi(\mu_{0,i_1,i_2,\ldots,i_j}) - \epsilon) \sum_{s=1}^{n+1} \frac{1}{r^s}. \end{aligned}$$

As a result,

$$E\mathcal{H}(\mu_0) \geq h(\mu_0) + \sup_n u_n(\mu_0) \geq (\chi(\mu_0) - \epsilon) \left(1 + \sum_{s=1}^{\infty} \frac{1}{r^s}\right) = (\chi(\mu_0) - \epsilon) \frac{r}{r-1}.$$

which proves Lemma 6.2.

Now, we go to the proof of (50) which gives Theorem 6.1. The proof is similar in spirit to that of Theorem 1.3. However, we don't work in the symplectic category and our perturbations must all be C^r. As is to be expected, this forces changes in many of the estimates. The resulting differences which appear are that some expressions involving $\chi(p)$ in (11) and (12) are replaced by $\frac{\chi(p)}{r}$ in (56) and (57), respectively.

Let us proceed. For each $f \in \mathcal{U}$, let $\Lambda(f)$ be a basic set which has persistent homoclinic tangencies.

Given a positive integer n, let $\tilde{H}_n(f)$ be the set of hyperbolic saddle points of least period n which are $h-$ related to $\Lambda(f)$, let $\tilde{H}(f) = \bigcup_n \tilde{H}_n(f)$, and let $\tilde{\tau}(f)$ denote the least positive integer n such that $\tilde{H}_n(f) \neq \emptyset$. Set $\tilde{\mathcal{R}}_{2,m}$ to be the set of diffeomorphisms $f \in \mathcal{U}$ such that $\tilde{\tau}(f) = m$.

For $n \geq m$, let us say that a diffeomorphism f satisfies property $\tilde{\mathcal{S}}_n$ if, for each $p \in \tilde{H}_n(f)$,

1. *there is a zero dimensional periodic hyperbolic basic set* $\Lambda(p,n)$ *for* f *such that*

$$\Lambda(p,n) \bigcap \partial\alpha_n = \emptyset, \tag{54}$$

2. $$\Lambda(p,n) \text{ is subordinate to } \alpha_n, \tag{55}$$

3. *there is an ergodic* $\mu \in \mathcal{M}(\Lambda(p,n))$ *such that*

$$\mid h_\mu(f) - \tfrac{\chi(p)}{r} \mid < \frac{\chi(p)}{nr}, \tag{56}$$

and

4. *for every ergodic* $\mu \in \mathcal{M}(\Lambda(p,n))$, *we have*

$$\rho(\mu, \mu_p) < \frac{1}{n} \text{ and } \mid \chi(\mu) - \tfrac{\chi(p)}{r} \mid < \frac{\chi(p)}{nr}, \tag{57}$$

Given positive integers $m \leq n$, let $\tilde{\mathcal{D}}_{m,n}$ denote the subset of $\tilde{\mathcal{R}}_{2,m}$ consisting of diffeomorphisms $f \in \mathcal{U}$ satisfying property $\tilde{\mathcal{S}}_n$.

Lemma 6.3 *For every positive integers* $m \leq n$ *the set* $\tilde{\mathcal{D}}_{m,n}$ *is dense and open in* $\tilde{\mathcal{R}}_{2,m}$.

The proof is similar to that of Lemma 5.1 except the we don't have to keep things symplectic, and we use results of Kaloshin [15] (and Gonchenko-Shilnikov-Turaev [10]) to get intervals of homoclinic tangencies. We will sketch the ideas, indicating the main changes to the previous arguments. All perturbations are assumed to be C^r small.

Step 1: For $p \in \tilde{H}_n(f)$, we first use Lemma 8.4 in [27] to perturb to get a homoclinic tangency for $O(p)$.

Step 2: We use Proposition 5 and Lemma 3 in [15] to get an interval of tangencies between $W^u(p)$ and $W^s(p)$.

Step 3: We take a further perturbation g to create N bumps as in figure 2, and also the set $\Lambda(p, N)$. However, to keep the perturbation C^r small, we replace (33) with

$$A(N) = \frac{\epsilon(a_2 - a_1)}{N^r}. \tag{58}$$

For convenience of notation (as in our earlier considerations near formula (27)), for positive real numbers a, b, we use the expression $a \sim b$ to mean

$$\frac{a}{b}$$

is bounded above and below by constants independent of k.

We see that (42) gets replaced by

$$\mid \lambda_u \mid^k \sim \frac{1}{A(N)} \sim N^r, \tag{59}$$

or

$$N \sim \mid \lambda_u \mid^{\frac{k}{r}}.$$

Also, for a unit vector $v \in \mathcal{C}_z^u$, we have

$$\mid Dg_z^k(v) \mid \sim \mid \lambda_u \mid^k \cdot s(v) \sim \mid \lambda_u \mid^k N^{1-r} \sim N \sim \mid \lambda_u \mid^{\frac{k}{r}}. \tag{60}$$

From, (59), we get

$$h_{top}(g, \Lambda(p, N)) = \frac{\log N}{k + T} \to \frac{1}{r} \log \mid \lambda_u \mid. \tag{61}$$

The arguments following (45) used (46) to show that $\chi(\mu) \sim \chi(p)$.

In a similar way we can now use (60) to show that $\chi(\mu) \sim \frac{\chi(p)}{r}$.

Proof of (50):

Let $f \in \bigcap_{n \geq m} \tilde{\mathcal{D}}_{m,n}$.

Let $\epsilon > 0$, and let p be a hyperbolic periodic point which is $h-$related to $\Lambda(f)$ so that

$$\chi(p) > \chi(\Lambda_1(f)) - \epsilon. \tag{62}$$

Let $\mu_0 = \mu(p)$ be the uniform measure on the orbit of p.

Using property $\tilde{\mathcal{S}}_n$ for larger and larger n, we can find a sequence μ_{0,i_1} of measures supported on periodic hyperbolic basic sets $\Lambda_{0,i_1} \subset \Lambda_1(f)$ such that

$$\mu_{0,i_1} \to \mu_0,\ \chi(\mu_{0,i_1}) \to \frac{\chi(p)}{r},\ h_{\mu_{0,i_1}} \to \frac{\chi(p)}{r}, \text{ and } h_k(\mu_{0,i_1}) \to 0$$

as $i_1 \to \infty$.

By (62) we may assume that

$$h_{\mu_{0,i_1}} \geq \chi(\mu_{0,i_1}) - \epsilon \geq \frac{\chi(\mu_0)}{r}.$$

This gives that $\mu \to h_\mu$ is not uppersemicontinuous. Thus, f cannot be asymptotically $h-$expansive. This is the first statement of Theorem 6.1.

Since μ_{0,i_1} is supported on the hyperbolic set Λ_{0,i_1}, we use Sigmund [38] to get a sequence of periodic points $p_{0,i_1,i_2} \in \tilde{H}(f)$ such that

$$\mu(p_{0,i_1,i_2}) \to \mu_{0,i_1} \text{ and } \chi(\mu(p_{0,i_1,i_2})) \to \chi(\mu_{0,i_1}) \text{ as } i_2 \to \infty.$$

Now, we use $\tilde{\mathcal{S}}_n$ repeatedly again replacing p with each p_{0,i_1,i_2} and get measures μ_{0,i_1,i_2} satisfying (51)–(53) for $j = 1$.

Continuing in this manner we get a family of measures indexed by sequences of natural numbers as in Lemma 6.2. Applying that Lemma, we get

$$\sup E\mathcal{H} \geq \frac{r(\chi(p) - \epsilon)}{r-1}.$$

Since ϵ was arbitrary, this proves Theorem 6.1.

Next, we wish to describe an open set $\mathcal{U}$ in $\mathcal{D}^r(M)$ so that, for each $f \in \mathcal{U}$, we have

$$h_{top}(f) < \frac{\chi(\Lambda_1(f))r}{r-1}. \tag{63}$$

This will complete the proof of Theorem 1.4.

We first describe the construction of $\mathcal{U}$ using C^r diffeomorphisms mapping a closed 2-disk D in $\mathbf{R}^2$ into itself. Then, we use standard techniques to embed this family of diffeomorphisms into an open set in $\mathcal{D}^r(M)$.

Let us begin with the time-one map f_1 of the vector field X in the plane $\mathbf{R}^2$ given by

$$X(x,y) = y\frac{\partial}{\partial x} + (x - x^2)\frac{\partial}{\partial y}. \tag{64}$$

This is a Hamiltonian system with one degree of freedom and Hamiltonian function $H(x,y) = \frac{y^2}{2} + \frac{x^3}{3} - \frac{x^2}{2}$.

The point $p = (0,0)$ is a saddle point for X and $p_1 = (1,0)$ is a center. Also, the right components of $W^u(p) \setminus \{p\}$ and $W^s(p) \setminus \{p\}$ coincide in a homoclinic loop for X. Denote these components by $W^u_+(p)$ and $W^s_+(p)$, respectively. Let $W^u_-(p)$ denote the left component of $W^u(p) \setminus \{p\}$. With standard modifications (e.g. as in [32]), we change the map f_1 (via an isotopy) to a C^r diffeomorphism f_2 with the following properties.

1. p is a fixed saddle point and p_1 is a fixed source of f_2, and f_2 has another hyperbolic fixed point p_2 which is a sink,

2. $W^u(p, f_2) \bigcap W^s(p, f_2) \setminus \{p\}$ consists of the orbit of a single homoclinic tangency q,

3. $0 < det(Df_2(p)) < 1$,

4. $Closure(W^u_+(p, f_2)) \setminus W^u_+(p, f_2) \subset W^u_-(p, f_2) \bigcup \{p_2, p\}$ (that is, $W^u_+(p, f_2)$ only accumulates on $W^u(p, f_2)$ in the left component $W^u_-(p, f_2)$),

5. there is a closed 2-disk D in $\mathbf{R}^2$ such that f_2 maps D into its interior, and $\{p, p_1, p_2, q\} \bigcup W^u(p) \subset D$, and

6. the collection of $\omega-$limit sets of all points in D consists of the orbit of q, the saddle fixed point p, the fixed source p_1, and the fixed sink p_2.

7. $\chi(p_1) < \chi(p)$ where $\chi(\cdot)$ denotes the largest characteristic exponent.

Since the only recurrent points of f_2 are fixed points, we have

$$h_{top}(f_2) = 0. \tag{65}$$

See Figure 3 for the maps f_1 and f_2 on D.

For a given diffeomorphism f, and a set E, define the number $R(f, E)$ by

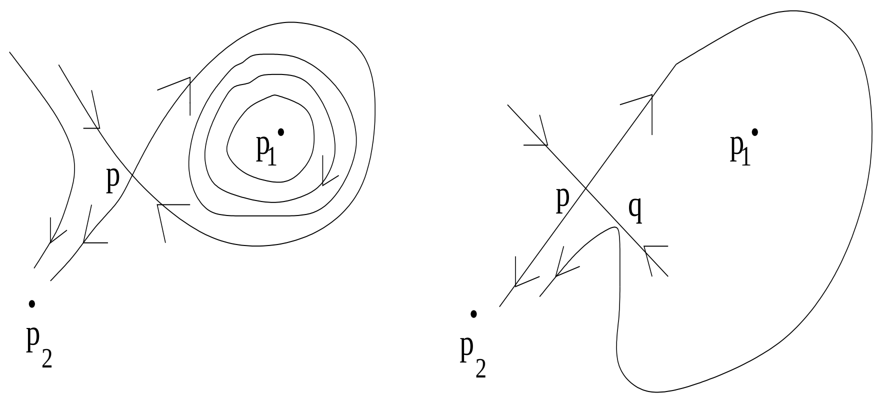

Figure 3: The maps f_1 and f_2.

$$R(f,E) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in E} | \, Df^n_x \, |.$$

Let $\lambda > 1$ be the expanding eigenvalue of $Df_2(p)$, so that $\chi(p) = \log \lambda$.

Now, the forward f_2-orbit of any $x \in D \setminus \{p_1\}$ is asymptotic to the fixed sink p_2 or the closure of the orbit of the homoclinic tangency q. In the latter case, the orbit eventually spends most of its time near the saddle point p. Thus we have

$$R(f_2, D) = \max(| \, Df_2(p_1) \, |, \chi(p)) = \chi(p). \tag{66}$$

Given a C^r curve γ in a Riemannian manifold M, we let $| \, \gamma \, |$ denote its arclength. We define the growth rate of γ for a diffeomorphism f to be

$$G(f,\gamma) = \limsup_{n\to\infty} \frac{1}{n} \log^+ | \, f^n \circ \gamma \, |$$

where $\log^+ x = \max(\log x, 0)$ for every real x.

Also the one-dimensional growth rate of a C^r map f is defined to be

$$G_1(f) = \sup_{\gamma} G(f,\gamma)$$

where the supremum is taken over all C^r curves γ in M.

In [29] it was proved that the topological entropy of a $C^{1+\alpha}$ diffeomorphism was bounded above by the maximal volume growth of smooth disks. Since there is no volume growth of two dimensional disks for diffeomorphisms on surfaces, we have that the topological entropy of a C^2 diffeomorphism f of a surface is bounded above by $G_1(f)$.

Applying this and results of Yomdin in [41], [40], we get

$$\limsup_{g \to f \text{ in } C^r} h_{top}(g) \leq h_{top}(f) + \frac{R(f)}{r}. \tag{67}$$

Actually, in [41], the upper bound given for $G_1(f)$ was $\frac{2R(f)}{r}$, but this was improved to $\frac{R(f)}{r}$ in [40].

Let $\epsilon > 0$.

Using (67), we take a small C^r neighborhood $\mathcal{U}_1$ of f_2 so that, for any $f \in \mathcal{U}_1$, we have

$$h_{top}(f) < h_{top}(f_2) + \frac{\log \lambda + \epsilon}{r} = \frac{\chi(p) + \epsilon}{r}. \tag{68}$$

Also, we choose an open set $\mathcal{U} \subset \mathcal{U}_1$ so that each $f \in \mathcal{U}$ has a hyperbolic basic set $\Lambda(f)$ with persistent homoclinic tangencies which is $h-$related to p. It follows from Theorem 6.1 that there is a residual subset $\mathcal{R} \subset \mathcal{U}$ such that if $f \in \mathcal{R}$, then

$$h_{sex}(f) \geq \frac{(\chi(p) - \epsilon) r}{r - 1}. \tag{69}$$

Thus, since

$$\frac{1}{r} < \frac{r}{r-1},$$

we have that $h_{sex}(f) > h_{top}(f) + c$ for some positive number $c = c(\mathcal{U}_1)$ for $f \in \mathcal{R}$ provided that ϵ is small enough. This proves Theorem 1.4 in the case of diffeomorphisms of a two dimensional disk D into its interior.

To get the result for $\mathcal{D}^r(M)$ with M an arbitrary surface, we proceed in the following standard way.

Consider an arbitrary surface M together with a C^∞ Morse function $\phi : M \to R$ on M. Let f_3 be the time-one map of the gradient vector field of ϕ in some Riemannian metric on M. Near a local minimum of ϕ, we can find a smooth two-disk D' such that $f_3(D') \subset interior(D')$. Using standard techniques we modify f_3 to a diffeomorphism $f : M \to M$ satisfying the following properties.

- f agrees with f_3 outside D',

- there is a smooth two disk $D'' \subset D'$ such that (f, D'') is C^r conjugate to (f_2, D),
- the collection of $\omega-$limit points of f consists of hyperbolic fixed points and a single orbit of homoclinic tangencies, and
- $R(f, D'') = \chi(p)$.

Now, we apply the same method we used for f_2 to the map f. We perturb into an open set $\mathcal{U}$ such that if $g \in \mathcal{U}$, then g has a hyperbolic set $\Lambda_1(g)$ which has persistent homoclinic tangencies, is $h-$related to the saddle point $p(g)$, and satisfies the analogs of (68) and (69). This completes the proof of Theorem 1.4.

7 Proof of Theorem 1.6

We recall some results from the paper of Manning and McCluskey [19]. We denote the Hausdorff dimension of a set Λ by $HD(\Lambda)$.

Let $\Lambda = \Lambda(f)$ be an infinite zero dimensional hyperbolic basic set for the C^1 diffeomorphism f. Let E^u_x denote the expanding subspace at $x \in \Lambda$, and let $| \cdot |$ be an adapted Riemannian norm. The function $\phi^u(x) = \phi^u(x, f) = -log| \, Df_x \mid E^u_x \, |$ is strictly negative and continuous. For each $t \in [0, 1]$ define the pressure

$$P(t\phi^u, f) = \sup_{\mu \in \mathcal{M}(\Lambda)} \left(h_\mu(f) + t \int \phi^u d\mu \right).$$

For $r \geq 1$, the function $t \to P(t\phi^u, f)$ is strictly decreasing, and satisfies

$$P(0, f) = h_{top}(f \mid \Lambda) > 0.$$

If $r \geq 2$, then, since Λ is not an attractor, Theorem 4.11 in [1] gives

$$P(\phi^u, f) < 0.$$

Thus, there is a unique $\delta^u \in [0, 1]$ such that $P(\delta^u \phi^u, f) = 0$.

Moreover, for each $x \in \Lambda$,

$$HD(W^u(x) \bigcap \Lambda) = \delta^u. \tag{70}$$

Define the *unstable Hausdorff dimension* of Λ to be the quantity δ^u in (70). Denote this quantity by $HD^u(\Lambda)$. Replacing f by f^{-1}, we obtain the *stable Hausdorff dimension* of Λ to be the unique number $\delta^s = HD^s(\Lambda)$ such that, for each $x \in \Lambda$,

$$HD(W^s(x) \bigcap \Lambda) = \delta^s. \tag{71}$$

Observe that if $\Lambda_1 \subseteq \Lambda_2$, then

$$HD^u(\Lambda_1) \leq HD^u(\Lambda_2) \text{ and } HD^s(\Lambda_1) \leq HD^s(\Lambda_2).$$

By Manning and McCluskey, the quantities $HD^s(\Lambda), HD^s(\Lambda)$ depend continuously on f in the C^1 topology. By work of Palis and Viana [34], we have

$$HD(\Lambda) = HD^u(\Lambda) + HD^s(\Lambda), \tag{72}$$

so,

$$\text{the map } f \to HD(\Lambda(f)) \text{ is continuous} \tag{73}$$

in the C^r topology for $r \geq 1$.

For $r \geq 2$, the quantities $HD^u(\Lambda), HD^s(\Lambda)$ are studied in detail in [33] where they are also called *limit capacities*.

Fix $r \geq 2$. Let $\mathcal{U}_1$ be the open subset of $\mathcal{D}^r(M^2)$ so that if $f \in \mathcal{U}_1$, then f has a hyperbolic basic set $\Lambda(f)$ with persistent homoclinic tangencies. Let $n > 0$ be a positive integer.

Claim: The set $\mathcal{U}_{1,n}$ of diffeomorphisms $f \in \mathcal{U}_1$ such that there is a hyperbolic basic set $\Lambda_1(f)$ h-related to $\Lambda(f)$ such that $HD(\Lambda_1(f)) > 2 - \frac{1}{n}$ is dense and open in $\mathcal{U}_1$.

Once the claim is proved, if follows that if $f \in \bigcap_n \mathcal{U}_{1,n}$, then the homoclinic closure of $\Lambda(f)$ has Hausdorff dimension two. This will prove Theorem 1.6.

Since it is immediate from (73) that $\mathcal{U}_{1,n}$ is open in $\mathcal{U}_1$, it suffices to prove that it is dense.

Consider $f \in \mathcal{U}_1$ with hyperbolic basic set $\Lambda(f)$ having persistent homoclinic tangencies.

Below, we use the notation $\Lambda_1 \sim \Lambda_2$ to mean that Λ_1 is homoclinically related to Λ_2.

Let p be a hyperbolic periodic point in $\Lambda(f)$ with $\chi(p) > 0$. Let $\epsilon > 0$. By Lemma 6.3, we can perturb f off a neighborhood of the orbit of p to g in $\mathcal{D}^r(M^2)$ so that g has an invariant hyperbolic basic set $\Lambda_2(g) \sim \Lambda(g)$ containing p such that there is an ergodic hyperbolic measure μ supported on $\Lambda_2(g)$ so that

$$\mid \chi(\mu) - \tfrac{\chi(p)}{r} \mid < \epsilon,$$

and

$$\mid h_\mu(g) - \tfrac{\chi(p)}{r} \mid < \epsilon.$$

Let

$$t = \frac{h_\mu(g)}{\chi(\mu)}.$$

Since $\chi(p, g) = \chi(p, f) > 0$ for all such g, if ϵ is close enough to 0, we can get t arbitrarily close to 1.

Thus, we can find a g C^r close to f so that there is a hyperbolic basic set $\Lambda^u(g)$ supporting a $g-$invariant ergodic measure μ such that there is a t arbitrarily close to 1 such that

$$h_\mu(g) - t\chi(\mu, g) = 0.$$

As is well-known, the Birkhoff ergodic theorem applied to the function $\phi^u(x, g)$ gives

$$-\chi(\mu) = \int \phi^u(x, g) d\mu(x).$$

We have shown the following:

Given $n > 0$, we can C^r perturb f to g so that there are a $g-$hyperbolic basic set $\Lambda^u(g) \sim \Lambda(g)$, a $\mu \in \mathcal{M}_e(\Lambda(g))$, and a $t \in (1 - \frac{1}{2n}, 1)$ so that

$$P_\mu(t\phi^u, g) = 0.$$

Since $P(t\phi^u, g)$ is the supremum of such measures, we have that

$$P(t\phi^u, g) \geq 0.$$

On the other hand we know that $t \to P(t\phi^u, g)$ is strictly decreasing and $P(\phi^u, g) < 0$. So, the number δ^u so that $P(\delta^u \phi^u, g) = 0$ is also in the interval $(1 - \frac{1}{2n}, 1)$.

Since $\delta^u = HD^u(\Lambda^u(g))$, we get

$$HD^u(\Lambda^u(g)) > 1 - \frac{1}{2n}. \tag{74}$$

But, g still has a homoclinic tangency. So, we can apply similar reasoning to g^{-1} to get a g_1 C^r near f having another g_1-hyperbolic basic set $\Lambda^s(g_1) \sim \Lambda(g)$, such that

$$HD^s(\Lambda^s(g_1)) > 1 - \frac{1}{2n}. \tag{75}$$

For g_1 close enough to g, we also have a $\Lambda^u(g_1)$ near $\Lambda^u(g)$ satisfying (74) and $\Lambda^u(g_1) \sim \Lambda(g_1)$.

Next, since

$$\Lambda^u(g_1) \sim \Lambda(g_1) \sim \Lambda^s(g_1),$$

we can use Lemma 8 in [26] to get a hyperbolic basic set $\Lambda_3(g_1) \sim \Lambda(g_1)$ containing both $\Lambda^u(g_1)$ and $\Lambda^s(g_1)$.

Hence, we have

$$\begin{aligned} HD(\Lambda_3(g_1)) &= HD^u(\Lambda_3(g_1)) + HD^s(\Lambda_3(g_1)) \\ &\geq HD^u(\Lambda^u(g_1)) + HD^s(\Lambda^s(g_1)) \\ &\geq 2 - \frac{1}{n} \end{aligned}$$

as required. This completes the proof of Theorem 1.6.

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