QUASI-ELLIPTIC PERIODIC POINTS IN CONSERVATIVE DYNAMICAL SYSTEMS.

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1. One of the most important and beautiful subjects in the theory of dynamical systems concerns the orbit structure near an elliptic periodic point of an area preserving diffeomorphism f of the two dimensional disk D^2. Recall that a periodic point p of such an f is a point for which $f^n(p)=p$ for some integer $n>0$. Assuming n is the least such integer, p is called *elliptic* if the derivative of f^n at p, T_pf^n, has non-real eigenvalues of norm one. If the eigenvalues of T_pf^n have norm different from one, p is called *hyperbolic*. It has been known for a long time that elliptic periodic orbits occur in many problems in mechanics, in particular, the restricted three body problems [3, 8]. When f is real analytic, Birkhoff established a normal form for f near an elliptic fixed point provided the eigenvalues of Tf are not roots of unity. If this normal form is not linear, he showed that the fixed point is a limit of infinitely many periodic points, and that among these accumulating periodic points both elliptic and hyperbolic types appear [28]. A theorem due to Kolmogorov, Arnold, and Moser asserts that many f-invariant circles enclose a general elliptic fixed point p, and that on each of these circles f behaves like a rotation through an angle θ with $\theta/2\pi$ strongly irrational [8, 9]. This result implies that general elliptic orbits are Liapounov stable.

By contrast, according to a theorem of Hartman and Grobman [12, 16], the local structure near a hyperbolic fixed point p is not complicated. The diffeomorphism f behaves topologically like its derivative. The points asymptotic to p under forward and backward iterates form smooth curves (the stable and unstable manifolds of p) meeting transversely at p, and the other nearby orbits lie on continuous curves which are easily described. Recently, E. Zehnder has shown that, generically, many hyperbolic periodic orbits near an elliptic periodic orbit have homoclinic points (non-periodic intersections of the stable and unstable manifolds of the same hyperbolic periodic orbit) [28].

Thus the rather intricate picture in Figure 1 (taken from [3]) generically occurs near any elliptic periodic point p. Each circle is invariant under a power

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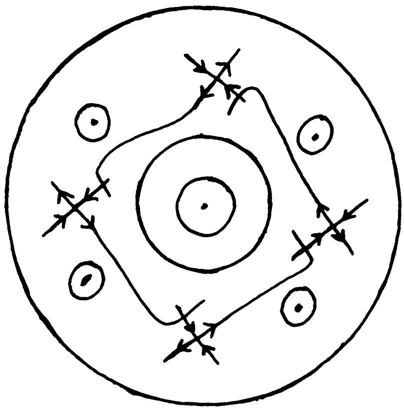

FIGURE 1.

of f, and the center of each circle is a limit of smaller circles invariant under higher powers of f and hyperbolic points with homoclinic points. A simplification due to Russmann in the proof of Kolmogorov-Arnold-Moser theorem enables one to show that all of this structure is found under the assumption that f is of class C^5 [23].

A natural question occurs: Where do elliptic periodic orbits appear? One consequence of our results here is the following. Let f be a C^1 area preserving diffeomorphism of D^2, and let y be a point of D^2 with U any neighborhood of y. Then, arbitrarily C^1-close to f, we may find a real analytic area preserving diffeomorphism g having an elliptic periodic point p in U. Of course, with a further approximation we may make p of general type. Thus with a small C^1 change in f all of the above structure appears near an arbitrary point y of D^2.

Let us now consider the general situation. A symplectic manifold is a C^∞ differentiable manifold M together with a closed non-degenerate differential two-form ω. A C^1 diffeomorphism $f: M \to M$ is *symplectic* if f preserves ω; i.e. $f^*\omega = \omega$. For applications as in the preceding paragraphs we may allow M to have a boundary ∂M, but in this case we assume that $f(\partial M) = \partial M$ and that the tangent bundle TM is the continuous bundle naturally induced by $T(M - \partial M)$. Examples of symplectic manifolds are orientable surfaces, even dimensional tori, compact complex algebraic manifolds, and the cotangent bundle T^*N of an arbitrary differentiable manifold N. For general information regarding symplectic diffeomorphisms we refer to [1], [3], and [20].

Let $\mathrm{Diff}^r_\omega(M)$ be the space of C^r symplectic diffeomorphisms of M with the C^r Whitney topology, $1 \leqslant r \leqslant \infty$ (see Section 2 for the definition). A subset

$\mathfrak{B} \subset \mathrm{Diff}^r(M)$ is *residual* if it contains a countable intersection of dense open sets. Residual sets are dense, and a countable intersection of residual sets is again residual. Properties which are true for residual sets are called *generic* properties.

A periodic point p of $f \in \mathrm{Diff}^r_\omega(M)$ of minimum period n is called *quasi-elliptic* if the derivative T_pf^n has a non-real eigenvalue of norm one, and all eigenvalues of norm one are not real. To be more exact, we say that p is k-elliptic, $1 \leqslant k \leqslant \frac{1}{2}\dim M$, if T_pf^n has $2k$ non-real eigenvalues of norm one, and its remaining eigenvalues have norm different from one. If all eigenvalues of T_pf^n have norm different from one, p is called *hyperbolic*.

A well-known class of symplectic diffeomorphisms without quasi-elliptic periodic points is the set of those which are Anosov. By definition, these possess continuous Tf-invariant sub-bundles E^u, E^s of TM with $TM = E^s \oplus E^u$, $Tf|E^u$ expanding, and $Tf|E^s$ contracting. More precisely, recall that a closed f-invariant subset $\Lambda \subset M$ is called *hyperbolic* if there are a continuous splitting of the tangent bundle over Λ, $T_\Lambda M = E^s \oplus E^u$, a Riemann norm $|\cdot|$ on TM, and constants $c > 0$, $0 < \lambda < 1$, such that

$$T_xf(E^s_x) = E^s_{f(x)}, \quad T_xf(E^u_x) = E^u_{f(x)} \qquad \text{for} \quad x \in \Lambda, \tag{a}$$

$$\left.\begin{aligned} &|T_xf^n(v)| \leqslant c\lambda^n|v| \qquad \text{for} \quad x \in \Lambda, \quad v \in E^s_x, \quad n \geqslant 0, \\ &|T_xf^n(v)| \geqslant c^{-1}\lambda^{-n}|v| \qquad \text{for} \quad x \in \Lambda, \quad v \in E^u_x, \quad n \geqslant 0. \end{aligned}\right\} \tag{b}$$

The conditions in (a) and (b) are independent of the Riemann norm if Λ is compact, but this is false if Λ is not compact (see [27]).

If M is a hyperbolic set for the diffeomorphism $f: M \to M$, then f is called Anosov. The Anosov diffeomorphisms in $\mathrm{Diff}^1_\omega(M)$ form an open set of diffeomorphisms containing no quasi-elliptic periodic points. Our first theorem says that, in the compact case, this is the maximal such open set.

THEOREM 1.1. *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a C^1 symplectic diffeomorphism with M compact, and assume f is not Anosov. Fix $x \in M$. Given neighborhoods U of x in M and V of f in $\mathrm{Diff}^1_\omega(M)$, there is a g in V having a 1-elliptic periodic point in U.*

This result has several interesting corollaries.

Recall that a diffeomorphism f is structurally stable if there is a neighborhood V of f in $\mathrm{Diff}^1(M)$ so that for $g \in V$, there is a homeomorphism $h: M \to M$ with $hf = gh$. In the early 1960's, Anosov proved that what are now called Anosov diffeomorphisms are structurally stable for compact M. Also, it has been known for some time that the presence of a 1-elliptic periodic point keeps a symplectic diffeomorphism from being structurally stable [20]. Thus, Theorem 1.1 gives

THEOREM 1.2. *A symplectic diffeomorphism of a compact manifold is structurally stable if and only if it is Anosov.*

It should be pointed out that there are many (non-symplectic) structurally stable diffeomorphisms which are not Anosov, and there are many Anosov diffeomorphisms which are not symplectic. The interest of Theorem 1.2 is that it lends further credence to the conjecture of Palis and Smale that Axiom A and strong transversality are necessary and sufficient conditions for structural stability. In this connection, see [18], [22].

Let us now apply a technique due to Pugh [15].

Let $n>0$ be a positive integer. It is easily proved with known techniques that the set of 1-elliptic periodic points of period n_1 with $n_1 \leqslant n$ for a symplectic diffeomorphism f varies continuously with f. Indeed, if p is such a point with $f^{n_1}(p)=p$, and $f^k(p) \neq p$, $0 \leqslant k < n$, then 1 is not an eigenvalue of $T_p f^{n_1}$. Hence, if $\mathrm{id} \times f^{n_1}: y \to (y, f^{n_1}(y))$, then $(\mathrm{id} \times f^{n_1})(M)$ is transverse to the diagonal Δ in $M \times M$ at (p,p), so $(\mathrm{id} \times f^{n_1})^{-1}(\Delta)$ varies continuously with f. Thus, the closure $Q(f)$ of all the 1-elliptic periodic points of f is a lower semi-continuous set valued function of f in $\mathrm{Diff}^1_\omega(M)$. Proposition 26 in [19] says that $Q(f)$ is continuous on a residual subset $\mathfrak{B}$ of $\mathrm{Diff}^1_\omega(M)$. Theorem (1.1) and the continuity of a 1-elliptic point of f imply that if $f \in \mathfrak{B}$ is not Anosov, then the 1-elliptic points of f are dense in M.

Thus,

THEOREM 1.3. *For compact M there is a residual subset $\mathfrak{B} \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathfrak{B}$, then either f is Anosov or the 1-elliptic periodic points of f are dense in M.*

For other generic properties of symplectic diffeomorphisms, the reader may consult [17], [19], [25], [26]. We will refer to these papers in more detail in later sections.

In Section 2, we will recall the definition of the Whitney topology, and we will present a more or less well-known perturbation lemma needed in the sequel. In Section 3 we extend Zehnder's homoclinic point theorem to arbitrary dimension in the C^r case, $0 < r \leqslant \infty$, by means of the Birkhoff normal form, and we show that generically every quasi-elliptic periodic point is a limit of hyperbolic periodic points. Section 4 contains the proof of Theorem 1.1. Of basic importance to the proof are the closing lemma of Pugh and Robinson [17], a weakened version of Takens's homoclinic point theorem [26], and the Birkhoff theorems used in Section 3. These results enable us to approximate a symplectic diffeomorphism $f \in \mathrm{Diff}^1_\omega(M)$, M compact, by g so that the homoclinic points of g are dense in M. Unless f happened to be Anosov, we then show that such a g can be C^1-perturbed to create a 1-elliptic point in any pre-assigned open set U.

In Section 5, the results of the preceding sections are extended to the non-compact case. The main novelty here is that Takens's homoclinic point theorem must be extended to this case. This requires new proofs since Takens's proofs rested strongly on the existence of a smooth invariant bounded measure μ [i.e., $\mu(M)<\infty$] which was positive on open sets. The result here may be used to give a proof of Theorem 1.1 independent of Takens's methods as well.

Finally, in Section 6 we indicate how the results of the preceding sections relate to Hamiltonian systems. One application is that a Hamiltonian system with at least two independent integrals and a bounded non-wandering orbit γ may be C^1-perturbed to a Hamiltonian system possessing infinitely many periodic, quasi-periodic, and homoclinic motions near γ.

In closing this section, let us list some C^1 generic properties of symplectic diffeomorphisms f of a compact symplectic manifold M.

(a) The hyperbolic periodic points of f are dense in M.
(b) Every hyperbolic periodic point of f has a homoclinic point.
(c) Either f is Anosov or the 1-elliptic periodic points are dense in M.

Property (a) is a consequence of the closing lemma of Pugh and Robinson and the Birkhoff fixed point theorem as in Section 3, while property (b) is due to Takens [26]. Property (c) is the content of Theorem 1.3. Extending methods of Robinson [21], D. Pixton has proved property (b) to be C^r-generic, $r \geqslant 1$, on the 2-sphere [13]. With this exception, the C^r-genericity of these properties, $r>1$, remains unknown and is one of the most challenging unsolved problems in the theory of dynamical systems.

2. The purpose of this section is to recall the definition of the C^r Whitney topology and to give a known perturbation lemma which will be needed in the sequel.

To define the Whitney topology on $\mathrm{Diff}^r_\omega(M)$, we will specify a basic neighborhood of $f \in \mathrm{Diff}^r_\omega(M)$. Pick two locally finite open coverings $\mathcal{U}=\{(U_i,\phi_i): i=1,2,\dots\}$, $\mathcal{V}=\{(V_i,\psi_i): i=1,2,\dots\}$ of M by C^r symplectic coordinate charts such that $\mathrm{Cl}\,U_i$ and $\mathrm{Cl}\,V_i$ are compact and $f(\mathrm{Cl}\,U_i)\subset V_i$ for all i. Thus $\phi_i: U_i \to \mathbf{R}^{2n}$, $\psi_i: V_i \to \mathbf{R}^{2n}$ are C^r diffeomorphisms with $\omega=\phi_i^*(\sum_{i=1}^n du_i \wedge dv_i)=\psi_i^*(\sum_{i=1}^n du_i\wedge dv_i)$, where $(u,v)=(u_1,\dots,u_n,v_1,\dots,v_n)$ are coordinates on $\mathbf{R}^{2n}$ and $\dim M = 2n$.

Let $\mathbf{R}_+$ denote the set of strictly positive real numbers, and let $\varepsilon: M \to \mathbf{R}_+$ be a continuous function.

Define $\mathcal{N}^r(f, \mathcal{U}, \mathcal{V}, \varepsilon)$ to be the set of diffeomorphisms g in $\mathrm{Diff}^r_\omega(M)$ such that

(a) $g(\mathrm{Cl}\, U_i) \subset V_i$ for all i,
(b) $|\partial^\alpha \psi_i g \phi_i^{-1}(x) - \partial^\alpha \psi_i f \phi_i^{-1}(x)| < \varepsilon(\phi_i^{-1} x)$ for $x \in \phi_i(U_i)$, $|\alpha| \leq r$, and all i.

Here $\alpha = (\alpha_1, \dots, \alpha_r)$ is a multi-index of non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_r$, and ∂^α denotes the corresponding partial derivative. The set $\mathcal{N}^r(f, \mathcal{U}, \mathcal{V}, \varepsilon)$ is a basic C^r Whitney neighborhood of f. That is, varying $\mathcal{U}, \mathcal{V}, \varepsilon$ with the above conditions gives a neighborhood base at f for the C^r Whitney topology. In all estimates, we will assume $\mathcal{U}, \mathcal{V}$ are fixed and we will shrink ε. Also, we will refer to the neighborhood $\mathcal{N}^r(f, \mathcal{U}, \mathcal{V}, \varepsilon)$ as $\mathcal{N}$ or $\mathcal{N}(f)$, and we will say that g is ε-C^r-close to f to mean $g \in \mathcal{N}^r(f, \mathcal{U}, \mathcal{V}, \varepsilon)$ with ε small. Of course, if M is compact we may replace ε by $\inf_{x \in M} \varepsilon(x)$ and use constant functions. Then $\mathcal{U}, \mathcal{V}$ may be chosen to be finite open coverings, and the topology is the uniform C^r topology. When M is not compact, $\mathrm{Diff}^r_\omega(M)$ fails to satisfy the first axiom of countability, but it does have the Baire property: countable intersections of dense open sets are dense.

The statement $\varepsilon_1 \leq \varepsilon_2$ for continuous real-valued functions means $\varepsilon_1(x) \leq \varepsilon_2(x)$ for all x.

Let d be a metric on M induced from some Riemann structure, and let $B_\delta(x)$ denote the set of $y \in M$ with $d(y,x) < \delta$.

Lemma 2.1 (Perturbation lemma). *Fix $f \in \mathrm{Diff}^r_\omega(M)$, $r \geq 1$, and choose $\mathcal{U}, \mathcal{V}$ as above. There are continuous functions $\varepsilon_0: M \to \mathbf{R}_+$ and $c: M \to \mathbf{R}_+$ (depending on f, U, V) such that for any $x \in M$, any $g \in \mathcal{N}^r(f, \mathcal{U}, \mathcal{V}, \varepsilon_0)$, and any continuous functions $0 < \delta \leq \varepsilon_0$, $0 < \varepsilon \leq \varepsilon_0$, the following facts hold.*

(1) *If $d(y,x) < c(x)\delta^r(x)\varepsilon(x)$, then there is a g_1 in $\mathcal{N}^r(g, \mathcal{U}, \mathcal{V}, \varepsilon)$ such that $g_1 g^{-1}(x) = y$, $g_1(z) = g(z)$ for $z \notin g^{-1}(B_{\delta(x)}(x))$, and $g_1^{-1}(z) = g^{-1}(z)$ for $z \notin B_{\delta(x)}(x)$.*
(2) *For any symplectic linear map $A: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ with $|T_{\phi_i(x)} \psi_i g \phi_i^{-1} - A| < c(x)\delta(x)^{r-1}\varepsilon(x)$ for all i with $x \in U_i$, there is a g_1 in $\mathcal{N}^r(g, \mathcal{U}, \mathcal{V}, \varepsilon)$ such that $T_{\phi_i(x)} \psi_i g \phi_i^{-1} = A$, $g_1(x) = g(x)$, and $g_1(z) = g(z)$ for $z \notin B_{\delta(x)}(x)$.*

Before proving the lemma let us recall the elementary facts about generating functions [3].

Let $(u,v) = (u_1, \dots, u_n, v_1, \dots, v_n)$ be coordinates on $\mathbf{R}^{2n}$, $\omega = \sum_{i=1}^n du_i \wedge dv_i$, and let $f(u,v) = (\xi(u,v), \eta(u,v))$ be a C^r symplectic diffeomorphism (relative to

ω) defined on a simply connected neighborhood V of the origin $(0,0)$ with $r>0$. Thus $\sum_{i=1}^{n} d\xi_i \wedge d\eta_i = \sum_{i=1}^{n} du_i \wedge dv_i$. Assume $f(0,0)=(\xi_0,\eta_0)$ and $\frac{\partial \eta}{\partial v}(u,v)$ is non-singular at each point of V. Thus $(u_1,\dots,u_n,\eta_1,\dots,\eta_n)$ define new C^r coordinates on a neighborhood of $(0,\eta_0)$, and $v=v(u,\eta)$ is given as a C^r function of u and η there. Since f is symplectic, the 1-form $\alpha=\sum_{i=1}^{n}(\xi_i\,d\eta_i + v_i\,du_i)$ is closed, so there is a C^{r+1} real valued function $S=S(u,\eta)$, unique up to a constant, defined for (u,η) near $(0,\eta_0)$ and satisfying $S_{\eta_i}=\xi_i$, $S_{u_i}=v_i$, and $S_{u_i\eta_j}$ non-singular at each point (u,η) in the domain of S. The function S is called a generating function for f relative to (u,v) and (ξ,η). Conversely, if $S(u,\eta)$ is a C^{r+1} function defined for (u,η) near $(0,\eta_0)$ such that the matrix $S_{u_i\eta_j}(u,\eta)$ is non-singular for each (u,η), then setting $S_{\eta_i}(u,\eta)=\xi_i(u,\eta)$ and $S_{u_i}(u,\eta)=v_i(u,\eta)$, we may solve for η in terms of u and v and obtain a symplectic diffeomorphism $(u,v)\to\big(\xi(u,\eta(u,v)),\eta(u,v)\big)$. Generating functions provide a convenient way of parametrizing C^r perturbations of f in that one merely chooses C^{r+1} perturbations of a generating function for f, and then passes to the associated diffeomorphisms.

We now proceed to prove part (1) of the perturbation lemma. The similar proof of part (2) will be omitted.

Assume $\mathfrak{U},\mathfrak{V}$ are locally finite open covers of M as above. We first consider perturbations of the identity map id on M.

Let $K_1,K_2,\dots$ be a sequence of compact subsets of M with $K_i\subset \operatorname{int} K_{i+1}$ and $M=\cup_{i\geq 1}K_i$. Since $\mathfrak{V}$ is locally finite, for each i there is an integer n_i such that at most n_i elements of $\mathfrak{V}$ meet K_i. Choose a positive real number $\alpha_i<1$ for each i so that if $z\in K_i$ and $(V,\psi)\in\mathfrak{V}$ with $z\in V$, then $B_{\alpha_i}(z)\subset V$. Take $\varepsilon_1=\varepsilon_1(z)$ a positive continuous function such that $\varepsilon_1(z)<\alpha_i/2$ for $z\in K_i$ for all i. Let $x\in K_i$, and fix $(V,\psi)\in\mathfrak{V}$, a symplectic coordinate chart with $x\in V$.

Assume $0<\varepsilon_2\leq\varepsilon_1$ and suppose $y\in M$ with $d(y,x)<\varepsilon_2(x)<\varepsilon_1(x)$. Let $(u,v)=(u_1,\dots,u_n,v_1,\dots,v_n)$ be coordinates on $\mathbf{R}^{2n}$ with $\omega=\psi^*(du_i\wedge dv_i)$. There is an orthogonal symplectic map $A:\mathbf{R}^{2n}\to\mathbf{R}^{2n}$ so that the affine map $A_1:(u,v)\to A\big((u,v)-\psi(x)\big)$ carries $\psi(y)$ to $(\mu,0,\dots,0)$ with $\mu>0$. Note also that $A_1\psi(x)=(0,0)$.

There is a constant $c_i>0$ such that

$$\mu=|\psi(y)-\psi(x)|\leq c_i d(y,x)<c_i\varepsilon_2(x).$$

Let $\lambda:\mathbf{R}\to[0,1]$ be a C^∞ real function so that $\lambda(z)=1$ for $z\leq\frac{1}{2}$ and $\lambda(z)=0$ for $z\geq 1$, and let $\beta=\sup_{z\in\mathbf{R}}\{1,|\lambda'(z)|,\dots,|\lambda^{r+1}(z)|\}$. The function $S_1(u,v)=(u_1+\mu)v_1+\sum_{i>1}u_iv_i$ is the generating function of the translation $(u_1,\dots,u_n,v_1,\dots,v_n)$

$\to(u_1+\mu, u_2, \ldots, u_n, v_1, \ldots, v_n)$, while $S_0(u,v)=\Sigma_{i>1}u_iv_i$ is the generating function of the identity.

For $0 \leqslant \delta_1 \leqslant \varepsilon_1$, set

$$S(u,v)=\lambda\left(\frac{|(u,v)|}{\delta_1(x)}\right)S_1(u,v)+\left[1-\lambda\left(\frac{|(u,v)|}{\delta_1(x)}\right)\right]S_0(u,v)$$
$$\text{for} \quad |(u,v)| \leqslant \delta_1(x)$$
$$= S_0(u,v) \qquad \text{for} \qquad |(u,v)| \leqslant \delta_1(x).$$

Then S is C^∞, and a direct computation shows that $|S-S_0|_{r+1} \leqslant \beta\bar{c}\mu\delta_1(x)^{-r}$, where $\bar{c}$ is a constant depending only on n and r. Here, $|S-S_0|_{r+1}$ denotes the C^{r+1} norm, i.e., the maximum of the norms of the kth order partial derivatives with $0 \leqslant k \leqslant r+1$. Thus, $|S-S_0|_{r+1} \leqslant \beta\bar{c}c_i\varepsilon_2(x)\delta_1(x)^{-r}$, and if we let $h(u,v)=\left(S_{v_1}(u,v),\ldots,S_{v_n}(u,v),S_{u_1}(u,v),\ldots,S_{u_n}(u,v)\right)$, then $|h-\mathrm{id}|_r \leqslant \beta\bar{c}c_i\varepsilon_2(x)\delta_1(x)^{-r}$. Also, $A_1^{-1}hA_1$ takes $\psi(x)$ to $\psi(y)$ and $|A_1^{-1}hA_1-\mathrm{id}|_r=|h-\mathrm{id}|_r$, since A_1 is affine with an orthogonal derivative. Let $\mathcal{V}_B'$ be a cover of M by open coordinate charts (V_j',ψ_j') so that $f(\mathrm{Cl}\,U_j)\subset V_j' \subset \mathrm{Cl}\,V_j' \subset V_j$ for all j and $\psi_j'=\psi_j|V_j$. Also, choose α_i and the charts V_j' so that if $z\in K_i\cap V_j'$ for some i,j, then $B_{\alpha_i}(z)\subset V_j'$.

Let $h_i: M\to M$ be defined by $h_1(z)=\psi^{-1}h\psi(z)$ for $z\in V$ and $h_1(z)=z$ for $z\notin V$. Then h_1 will be in $\mathfrak{N}^r(\mathrm{id},\mathcal{V}',\mathcal{V},e_{1i}c_i\varepsilon_2\delta_1^{-r})$ where e_{1i} is a constant depending on β, n, r, and the n_i coordinate charts in $\mathcal{V}$ meeting K_i. Also, $h_1(z)=z$ for $d(x,z)\geqslant e_{2i}\delta_1(x)$, with e_{2i} depending on i and the Riemann metric.

To summarize, given $x\in K_i$, $y\in M$ with $d(x,y)\leqslant \varepsilon_2(x)$, we have h_1 in $\mathfrak{N}^r(\mathrm{id},\mathcal{V}',\mathcal{V},e_{1i}c_i\varepsilon_2\delta_1^{-r})$, so that $h_1(x)=y$ and $h_1(z)=z$ for

$$d(x,z)\geqslant e_{2i}\delta_1(x).$$

Replacing $\varepsilon_2(x)$ by $\varepsilon_2(x)\delta_1(x)^r$, we see that the corresponding h_1 will be in $\mathfrak{N}^r(\mathrm{id},\mathcal{V}',\mathcal{V},e_{1i}c_i\varepsilon_2)$.

Let e_1, e_2 be continuous functions on M such that for $z\in K_i$, $e_1(z)\geqslant e_{1i}c_i$ and $e_2(z)\geqslant e_{2i}$. Take $\delta(x)=e_2(x)\delta_1(x)$ and $\varepsilon(x)=e_1(x)\varepsilon_2(x)$. Then

(*) *Given*

$$d(y,x)<\frac{\varepsilon(x)}{e_1(x)e_2(x)^r}\delta(x)^r,$$

the h_1 obtained above is in $\mathfrak{N}^r(\mathrm{id},\mathcal{V}',\mathcal{V},\varepsilon(x))$ with $h_1(x)=y$ and $h_1(z)=z$ for $d(x,z)\geqslant\delta(x)$.

As a notational convenience, for $z \in K_i$ $g_1, g \in \mathfrak{N}^r(f, \mathfrak{U}, \mathfrak{V}, \varepsilon_1)$, $h \in \mathfrak{N}^r(\mathrm{id}, \mathfrak{V}', \mathfrak{V}, \varepsilon_1)$, define

$$|(g_1 - g)(z)|_r = \sup\left\{ \left| \left[\partial^\alpha \psi_i g_1 \phi_i^{-1} - \partial^\alpha \psi_i g \phi_i^{-1} \right] \left[\phi_i(z) \right] \right| : |\alpha| \leqslant r, z \in U_i \right\}$$

and

$$|(h - \mathrm{id})(z)|_r = \sup\left\{ \left| \partial^\alpha \left(\psi_i h_i^{-1} - \mathrm{id} \right) \left(\psi_i(z) \right) \right| : |\alpha| \leqslant r, z \in V_i' \right\}.$$

Assume ε_1 small enough so that $g \in \mathfrak{N}^r(f, \mathfrak{U}, \mathfrak{V}, \varepsilon_1)$ and $h \in \mathfrak{N}^r(\mathrm{id}, \mathfrak{V}', \mathfrak{V}, \varepsilon_1)$ imply that g and h are both diffeomorphisms and $h \circ g(\mathrm{Cl}\, U_i) \subset V_i$ for all i.

Assume $0 < \varepsilon_2 < \varepsilon_1$ small enough so that if $g \in \mathfrak{N}^r(f, \mathfrak{U}, \mathfrak{V}, \varepsilon_2)$ and $h \in \mathfrak{N}^r(\mathrm{id}, \mathfrak{V}', \mathfrak{V}, \varepsilon_2)$, then $h \circ g \in \mathfrak{N}^r(f, \mathfrak{U}, \mathfrak{V}, \varepsilon_1)$. There is a continuous function $c_3 : M \to \mathbf{R}_+$ such that for $z \in M$, $g \in \mathfrak{N}^r(f, \mathfrak{U}, \mathfrak{V}, \varepsilon_2)$, and $h \in \mathfrak{N}^r(\mathrm{id}, \mathfrak{V}', \mathfrak{V}, \varepsilon_2)$, we have

$$|(h \circ g - g)(z)|_r \leqslant c_3(z) |(h - \mathrm{id})(z)|_r.$$

We may assume that $c_3(z) \geqslant 1$ for all z. Finally, let $c(x) = \left(c_3(x) e_1(x) e_2(x)^r \right)^{-1}$, and set $\varepsilon_0(x) = \min\left(1, \varepsilon_2(x)\right)$ for all $x \in M$. Now suppose $x, y \in M$, $0 < \delta \leqslant \varepsilon_0$, $0 < \varepsilon \leqslant \varepsilon_0$, $g \in \mathfrak{N}^r(f, \mathfrak{U}, \mathfrak{V}, \varepsilon_0)$, and $d(y, x) < c(x) \delta(x)^r \varepsilon(x)$. By (*), there is an h_1 in $\mathfrak{N}^r(\mathrm{id}, \mathfrak{V}', \mathfrak{V}, \varepsilon / c_3)$ with $h_1(x) = y$ and $h_1(z) = z$ for $d(x, z) \geqslant \delta(x)$. The proof of part (1) of the lemma now follows with $g_1 = h_1 \circ g$.

3. In this section we consider homoclinic points of symplectic diffeomorphisms. Let us recall that if p is a hyperbolic periodic point of minimum period n for the diffeomorphism f on M, one defines the stable manifold of p or (p, f) by $W^s(p) = W^s(p, f) = \{ y \in M : d(f^{nk} y, f^{nk} p) \to 0 \text{ as } k \to \infty \}$ where d is a topological distance function on M. Similarly, the unstable manifold of p is $W^u(p) = W^u(p, f) = W^s(p, f^{-1})$. It is known that these sets are injectively immersed copies of Euclidean spaces of complementary dimensions meeting transversely at p [6]. The stable manifold of the orbit $o(p)$ is the set $W^s(o(p)) = \bigcup_{0 \leqslant j < n} f^j(W^s(p))$, and $W^u(o(p))$ is defined similarly. A point x in $W^u(o(p)) \cap W^s(o(p)) - o(p)$ is called a homoclinic point of p. If x is a transverse intersection of $W^s(o(p))$ and $W^u(o(p))$, it is called a transversal homoclinic point. We will say that x is a (transversal) homoclinic point of f if it is a (transversal) homoclinic point of some hyperbolic periodic point of f. It is known that C^r-generically $(r > 0)$, all homoclinic points are transversal [19].

Homoclinic points were originally discovered by Poincaré in the restricted three body problem.

In [24], Smale showed that every transversal homoclinic point is a limit of infinitely many periodic points. In fact, on a closed set containing such a homoclinic orbit, a power of f is topologically conjugate to a shift automorphism on finitely many symbols. For physical applications of homoclinic points see [2] and [9], which contain further references.

In [26], F. Takens proved that C^1-generically any hyperbolic periodic point of a symplectic diffeomorphism on a compact symplectic manifold has homoclinic points. Also, E. Zehnder has shown the generic existence of homoclinic points near any elliptic periodic point for analytic area preserving diffeomorphisms of the plane [28]. Recently, R. C. Robinson proved that if the unstable manifold of a hyperbolic fixed point of a C^r diffeomorphism f on S^2 accumulates on its stable manifold, then f may be C^r-perturbed to g having a homoclinic point [21]. Subsequently, D. Pixton generalized the result to hyperbolic periodic points of arbitrary period [13]. If f is area preserving, the perturbations g may be taken area preserving as well. With a further argument, Pixton was able to establish the C^r version of Takens's theorem, $r>1$, on S^2 (in fact, on any region in S^2 on which the Jordan curve theorem holds). Since the Birkhoff fixed point theorem implies that generically an elliptic periodic point on S^2 is a limit of hyperbolic periodic points, Pixton's result implies the C^r version of Zehnder's.

Our first task is to extend Zehnder's theorem to arbitrary dimension.

Proposition 3.1. *There is a residual set $\mathfrak{B} \subset \mathrm{Diff}^r_\omega(M)$, $1 \leqslant r \leqslant \infty$, such that if $f \in \mathfrak{B}$, then each quasi-elliptic periodic point of f is a limit of transversal homoclinic points of f.*

Proof. For $f \in \mathrm{Diff}^r_\omega(M)$, let $Q(f)$ be the closure of the set of quasi-elliptic periodic points of f which are limits of transversal homoclinic points of f. It is easily seen that $Q(f)$ is a lower semi-continuous set valued function from $\mathrm{Diff}^r_\omega(M)$ into the set of closed subsets of M with a bounded metric induced from Hausdorff metrics on a sequence $H_1 \subset H_2 \subset \cdots$ of compact subsets of M with $\bigcup_{i=1}^{\infty} H_i = M$. Thus, $Q(f)$ is continuous on a residual set $\mathfrak{B} \subset \mathrm{Diff}^r(M)$, as may be proved with an obvious extension of the methods in Section XVIII of [19]. We prove that for $f \in \mathfrak{B}$, each quasi-elliptic periodic point of f is in $Q(f)$. For this, it suffices to show that if p is a quasi-elliptic periodic point of $f \in \mathrm{Diff}^r_\omega(M)$, $\mathfrak{N}$ is any neighborhood of f in $\mathrm{Diff}^r_\omega(M)$, and U is any neighborhood of p in M, then there is a $g \in \mathfrak{N}$ such that p is quasi-elliptic for g, and g has a homoclinic point in U.

Assume p is a quasi-elliptic periodic point of f with $f^\tau(p)=p$, and $f^k(p) \neq p$ for $0<k<\tau$. Let $T_pM = E^c \oplus E^u \oplus E^s$ be the invariant direct sum decomposition such that $T_pf^\tau | E^c$ has eigenvalues of norm one, $T_pf^\tau | E^u$ has eigenvalues of norm bigger than one, and $T_p f^\tau | E^s$ has eigenvalues of norm less than one; and write $\dim E^c = c$, $\dim E^u = u$, $\dim E^s = s$. Note that c is even and $s = u$, since any eigenvalue of $T_p f^\tau$ occurs together with its inverse [20]. The principal eigenvalues of $T_p f^\tau$ are those with norm bigger than one or norm equal to one and positive imaginary part. Perturbing f slightly via generating functions, we may assume f is C^∞ and the principal eigenvalues of f are multiplicatively independent over the integers. Set $k = c/2$. For any fixed $r > 0$, there are a neighborhood U_1 of p in M and C^{r+1} symplectic coordinates $\phi: U_1 \to \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}^u \times \mathbf{R}^s$ with

$$\phi(z) = \big(x_1(z), \ldots, x_k(z), y_1(z), \ldots, y_k(z), x_{k+1}(z), \ldots, x_{k+u}(z), y_{k+1}(z), \ldots, y_{k+u}(z)\big).$$

Furthermore,

(1) $\phi(p) = (0,0,0,0)$, $\omega = \phi^*\left(\sum_{j=1}^{k+u} dx_j \wedge dy_j\right)$, and

(2) if

$$v_1 = (x_1, \ldots, x_k, y_1, \ldots, y_k), \qquad v_2 = (x_{k+1}, \ldots, x_{k+u}), \qquad v_3 = (y_{k+1}, \ldots, y_{k+u}),$$

and

$$\phi f \phi^{-1}(v_1, v_2, v_3) = \big(f_1(v_1, v_2, v_3), f_2(v_1, v_2, v_3), f_3(v_1, v_2, v_3)\big),$$

then $f_2(v_1, 0, 0) = 0$, $f_3(v_1, 0, 0) = 0$, and $f_1(v_1, 0, 0)$ preserves $\sum_{j=1}^k dx_j \wedge dy_j$.

These facts follow from the center and stable manifold theory in [7] together with the generalizations of Darboux's theorem in [20].

By the Birkhoff normal form theorem [20], we may choose ϕ so that taking $z_i = x_i + \sqrt{-1}\, y_i$, $i = 1, \ldots, k$, to be complex coordinates on $\mathbf{R}^k \times \mathbf{R}^k$, we have $f_1(z_1, \ldots, z_k, 0, \ldots, 0) = (\xi_1, \ldots, \xi_k, 0, \ldots, 0) + h(z_1, \ldots, z_k, 0, \ldots, 0)$ where $\xi_i = z_i e^{\sqrt{-1}\, L}$ with

$$L = L(z_1, \ldots, z_k) = \sum_{\substack{i_1, \ldots, i_k \geq 0 \\ |i| \leq r}} a_{i_1 \ldots i_k} |z_1|^{2i_1} \cdots |z_k|^{2i_k}$$

and the first r partial derivatives of h relative to $z_1, \ldots, z_k$ vanish at $(0, \ldots, 0)$. Thus, L is a polynomial mapping in the variables $|z_1|^2, \ldots, |z_k|^2$ with the map $g_1(z_1, \ldots, z_k, 0, \ldots, 0) = (z_1 e^{\sqrt{-1}\, L}, \ldots, z_k e^{\sqrt{-1}\, L})$ giving the r-jet of $f_1(z_1, \ldots, z_k, 0, \ldots, 0)$ at $z_1 = \ldots = z_k = 0$. Hence $g_1 | \mathbf{R}^k \times \mathbf{R}^k \times (0,0)$ is C^r close to

$f_1|\mathbf{R}^k \times \mathbf{R}^k(0,0)$ in a small neighborhood V of $(0,0,0,0)$ with $V \subset \operatorname{int}\phi(U_1)$. Using generating functions as in the proof of Lemma 2.1, we may find a symplectic diffeomorphism g C^r-close to f_1 such that $\phi g \phi^{-1}|V \times \{0\} \times \{0\} = g_1|V$ and $\phi g \phi^{-1}(z) = f_1(z)$ for z outside a small neighborhood of V. Then g preserves k two dimensional ω-symplectic submanifolds of M near p, say $T_1, \dots, T_k$. By means of the Birkhoff fixed point theorem, g may be chosen so that we get a hyperbolic periodic point p_1 (of high period) for $g^\tau|T_1$ near p. By Zehnder's theorem (or Pixton's theorem) we may assume that p_1 has a homoclinic point for $g^\tau|T_1$. Then p_1 is a periodic point of g near p whose hyperbolic eigenspaces have their dimensions strictly bigger than those of p. If p_1 is hyperbolic for g, we are done. If not, proceed similarly near p_1 by perturbing g to g_2 having a periodic point p_2 near p_1 whose hyperbolic eigenspaces have larger dimension and whose strong stable and unstable manifolds meet off the orbit of p_2. Continuing this way by induction on the dimension of the hyperbolic eigenspaces yields Proposition 3.1.

Now recall that the closing lemma of Pugh and Robinson [17] implies that C^1-generically for compact M, the periodic points of f are dense in M. Robinson [19] proves that generically every periodic point is either hyperbolic or quasi-elliptic. Thus Proposition 3.1 yields

COROLLARY 3.2. *If M is a compact symplectic manifold, there is a residual subset $\mathfrak{B} \subset \mathrm{Diff}^1(M)$ such that for $f \in \mathfrak{B}$, the hyperbolic periodic points of f are dense in M.*

4. This section contains the proof of Theorem 1.1.

We say that two submanifolds N_1, N_2 of M are tangent at a point y in M if $T_yN_1 \cap T_yN_2$ contains a non-zero vector and $T_yN_1 + T_yN_2 \neq T_yM$.

LEMMA 4.1. *Suppose M is a symplectic manifold and $f: M \to M$ is a C^r symplectic diffeomorphism, $r \geq 1$, having a hyperbolic periodic point p such that $W^u(p,f)$ is tangent to $W^s(p,f)$ at a point y. Given neighborhoods U of y in M and $\mathfrak{N}$ of f in $\mathrm{Diff}^r_\omega(M)$, there is a g in $\mathfrak{N}$ having a 1-elliptic periodic point in U.*

Remark. This lemma is a variant of Proposition 1 in [11] for nonsymplectic diffeomorphisms. However, the proof given here is somewhat simpler in that we replace the explicit computations of [11] which depended on the Sternberg linearization theorem by a cruder topological argument.

Proof. We assume p is a fixed point, since the argument is similar in the general case. We will find a small neighborhood $U_1 \subset U$ of y and a curve $\{g_t\}$,

$-\varepsilon \leqslant t \leqslant \varepsilon$, of diffeomorphisms in $\mathfrak{N}$ such that

(1) $g_t(x)=f(x)$ for $x \notin f^{-1}(U_1)$
(2) there are an m-disk D^m in M and an integer $N>0$ such that
   (a) $g_\varepsilon^N | D^m$ has a fixed point in $\operatorname{int} U_1$,
   (b) $g_{-\varepsilon}^N(D^m) \cap D^m \cap U_1 = \varnothing$
   (c) $g_t^N | bdD^m$ has no fixed points for $-\varepsilon \leqslant t \leqslant \varepsilon$.

Then along the arc g_t^N there is a first number t_0 such that $g_{t_0}^N$ has a fixed point, say x_0, in $\operatorname{int} D^m \cap U_1$. It follows that $T_{x_0} g_{t_0}^N$ has 1 as an eigenvalue [otherwise $\mathrm{id} \times g_{t_0}^N : D^m \to M \times M$ would be transverse to the diagonal in $M \times M$ at (x_0, x_0), contradicting the choice of t_0].

Now by Theorem (3.3.II) in [20], a generic curve of symplectic diffeomorphisms has the property that if a new periodic point p is created, it has 1 as an eigenvalue of multiplicity two, the other eigenvalues of absolute value one are non-real, and there are two periodic points near p of the same period, one of which is 1-elliptic.

Thus the above curve g_t may be perturbed to $\tilde{g}_t$ so that some $\tilde{g}_{t_1}$ with t_1 near t_0 has a 1-elliptic periodic point near x_0.

We now proceed to construct the curve $\{g_t\}$. For this purpose our first task is to make y into an isolated removable intersection of $W^s(p,g)$ and $W^u(p,g)$ for some g in $\mathfrak{N}$ with $g(x)=f(x)$, $x \notin f^{-1}U$. More precisely, we wish to find $g \in \mathfrak{N}$ with $g(x)=f(x)$, $x \notin f^{-1}U$, such that

(3) there are disks $D^u \subset W^u(p,g) \cap U$, $D^s \subset W^s(p,g) \cap U$ with $\{y\} = D^s \cap D^u = \operatorname{int}_{W^s} D^s \cap \operatorname{int}_{W^u} D^u$ ($\operatorname{int}_{W^s} D^s$ is the interior of D^s in W^s, etc.),
(4) in local coordinates on U there is a unit vector v tangent to y with
   (a) $D^u + tv \cap D^s = \varnothing$, $t>0$,
   (b) $D^u + tv \cap D^s \neq \varnothing$ and transverse for $t<0$.

Having found g, we may embed it in a curve $\{g_t\}$ with $g_0 = g$ so that for some small neighborhood $U_1 \subset U$ of y, and all t, $D^u + tv \cap U_1 \subset W^u(p, g_t)$ and $D^s \subset W^s(p, g_t)$.

This being done, we choose the D^m required in (1) and (2) as follows (see the analogous construction in [11]).

Let $\bar{D}^\sigma$ denote the unit ball in $\mathbf{R}^\sigma$ for $\sigma = s, u$. Pick coordinates $\psi : U_2 \to \bar{D}^s \times \bar{D}^u$ on a neighborhood U_2 of p with $U_2 \cap U = \varnothing$ such that $\psi(p) = (0,0)$, $\psi^{-1}(\bar{D}^s \times \{0\}) \subset W^s(p,g)$, and $\psi^{-1}(\{0\} \times \bar{D}^u) \subset W^u(p,g)$. This may be done so that $y \in \psi^{-1}(\bar{D}^s \times \{0\})$. Let $\bar{\pi}^\sigma : \bar{D}^s \times \bar{D}^u \to \bar{D}^\sigma$ be the projection, $\sigma = s,u$.

For large n, $\psi f^{-n}(y)$ is near $(0,0)$ in $\{0\} \times \bar{D}^u$. Choose $n_1 > 0$ so that $\psi f^{-1}(y) \in \{0\} \times \bar{D}^u$ for $n \geqslant n_1$.

Letting D_{n_1} be a small u-disk about $\psi f^{-n_1}(y)$ in $\{0\}\times \bar{D}^u$, we may then take D^m to be the component of $f^{-n}\psi^{-1}(\bar{\pi}^u)^{-1}(D_{n_1})$ containing $f^{-n}(y)$ with n very large. Parts (1), (2)(b), and (2)(c) follow from the construction of $\{g_t\}$. Part (2)(a) is proved using Lemma (2.10) in [10], where the necessary estimates may be obtained as in the proof at the bottom of p. 148 of [10].

It remains to find $g \in \mathfrak{N}$ as in (3) and (4).

Choose symplectic coordinates $(\bar{x},\bar{y})=(x_1,\dots,x_n,y_1,\dots,y_n)$ on a neighborhood U_2 of y with $W^s(y,f)\cap U_2=\{(\bar{x},\bar{y}):\bar{x}=0\}$, and choose symplectic coordinates $(u,v)=(u_1,\dots,u_n,v_1,\dots,v_n)$ on a neighborhood U_3 of $f^{-1}y$ with $W^u(f^{-1}y,f)\cap U_3=\{(u,v):v=0\}$. Let $f_i=x_i\circ f$, $g_i=y_i\circ f$. Observe that the coordinates $(u,v),(\bar{x},\bar{y})$ may be chosen so that

$$(5)\quad \det\frac{\partial g_i}{\partial v_j}(u,v)\neq 0 \text{ for all } (u,v) \text{ small.}$$

Indeed, having first chosen (u,v) so that $\{v=0\}=W^u(f^{-1}y,f)\cap U_3$, we may replace u by $u+\varepsilon v$, where ε is a small positive number, and get (5).

Now (5) guarantees that $(u_i),(g_i)$ determine coordinates on a neighborhood U_3^1 of $f^{-1}y$. So there is a real generating function $S_1(u_i,g_i)$ with $S_{1u_i}=v_i$, $S_{1g_i}=f_i$, $i=1,\dots,n$, and $\det S_{1u_ig_j}\neq 0$. We realize symplectic perturbations of f via perturbations of S_1 as usual. Let $S_2(u,g)$ be near S_1, so $S_{2u_ig_j}$ is non-singular. Define $v_i=S_{2u_i}$, $\tilde{f}_i=S_{2g_i}$. Using (u_i,v_i) as coordinates near $f^{-1}y$, define $\tilde{f}(u,v)=\psi_1^{-1}(\tilde{f}_i,g_i)$, where $\psi_1(z)=(\bar{x}(z),\bar{y}(z))$ are the coordinates for z near y. We want perturbations such that $\tilde{f}(z)=f(z)$ for z off a small neighborhood $f^{-1}y$. For this, we simply choose functions S_2 with $S_{2g_i}(u,g)=S_{1g_i}(u,g)$ for (u,v) outside a small neighborhood of $f^{-1}y$. To prove (3), we first choose S_2 near S_1 as above so that

$$S_{2g_1u_1u_1}(0,0)=\left.\frac{\partial^2\tilde{f}_1}{\partial u_1^2}\right|_{u=0,v=0}\neq 0,$$

$$S_{2g_1u_1}(0,0)=\frac{\partial\tilde{f}_1}{\partial u_1}(0,0)=0,$$

and

$$S_{2g_iu_j}(0,0)=\frac{\partial\tilde{f}_i}{\partial u_j}(0,0),\qquad 2\leqslant i,j\leqslant n,$$

is a non-singular $(n-1)\times(n-1)$ matrix. Clearly, we may arrange this with S_2 C^{r+1} near S_1 and equal to S_1 off a small neighborhood of $u=0$, $g=0$. Now assume $S_{2g_1u_1u_1}(0,0)>0$, the other case being similar (i.e., we need to replace t by $-t$ in the following). Let η_t be a curve of symplectic diffeomorphisms,

$-\varepsilon<t<\varepsilon$, with $\eta_t(x)=x$ off a neighborhood of y and $\eta_t(x_1,\ldots,x_n,y_1,\ldots,y_n)=(x_1+t,x_2,\ldots,x_n,y_1,\ldots,y_n)$ for $\bar{x},\bar{y}$ near $(0,0)$. Then for ε small, $\tilde{f}_t=\eta_t\circ\tilde{f}$ satisfies (3) and (4). That is, there are disks $D_t^u\subset W^u(\tilde{f}_t)$, $D_t^s\subset W^s(\tilde{f}_t)$ such that D_t^u and D_t^s meet transversely at two points near y for $t<0$, are tangent at y for $t=0$, and do not meet near y for $t>0$. This completes the proof of Lemma 4.1.

Proceeding to the proof of Theorem 1.1, assume (M,ω) is a compact symplectic manifold and $f:M\to M$ is a C^1 symplectic diffeomorphism which is not Anosov. Fix $x\in M$, and let U be a neighborhood of x in M. Let $\mathcal{H}\subset \mathrm{Diff}^1_\omega(M)$ be the set of diffeomorphisms g for which the transversal homoclinic points of the hyperbolic periodic points of g are dense in M.

By Proposition 3.1 and Proposition 5.1 or Takens's Corollary 4 [26], $\mathcal{H}$ is dense in $\mathrm{Diff}^1_\omega(M)$. Letting $\mathcal{N}$ be any neighborhood of f in $\mathrm{Diff}^1_\omega(M)$, we will show that

(6) there is a g in $\mathcal{N}$ having a hyperbolic periodic point p such that $W^u(p,g)$ and $W^s(p,g)$ are tangent at some point in U.

Then Lemma 4.1 applies to give the proof of Theorem 1.1. For $g\in \mathrm{Diff}^1_\omega(M)$, let $H_U(g)$ be the set of transversal homoclinic points of g whose orbits meet U. Then $H_U(g)$ is a g-invariant set, and if $g\in\mathcal{H}$, we have $\overline{H_U(g)}\supset U$. If $L:E\to F$ is a linear map between normed linear spaces, we let $m(L)=\inf_{|v|=1}|Lv|$ be the minimum norm of L. Here and in the sequel, all norms and minimum norms are those induced from the fixed Riemann metric on M. For a transversal homoclinic point p of $g\in\mathcal{H}$, define $u(p,g)$ to be the smallest positive integer n such that $m\big(T_pg^n|T_pW^u(p,g)\big)\geqslant 2$.

Proposition 4.2. *Suppose f is not Anosov. Then there are sequences $g_i\in\mathcal{H}$, $p_i\in H_U(g_i)$ with $g_i\to f$ in* $\mathrm{Diff}^1_\omega(M)$ *and* $\max\{u(p_i,g_i),u(p_i,g_i^{-1})\}\to\infty$ *as $i\to\infty$.*

Proof. Assume there exist a neighborhood $\mathcal{N}_1$ of f in $\mathrm{Diff}^1_\omega(M)$ and a positive integer $\nu>0$ such that

$$\sup_{\substack{p\in H_U(g)\\ g\in\mathcal{N}_1\cap\mathcal{H}}}\left\{u(p,g),u(p,g^{-1})\right\}\leqslant\nu.$$

We will show that f is Anosov.

First, let us show that

$$\mathrm{Cl}\Big(\bigcup_{-\infty<n<\infty} f^n(U)\Big)\underset{\mathrm{def}}{=}\Lambda$$

is a hyperbolic set.

Let $K_1=\inf\{m(T_zf^j):0\leqslant j\leqslant N,\ z\in M\}$. There is a neighborhood $\mathfrak{N}_2\subset\mathfrak{N}_1$ of f such that $\inf\{m(T_zg^j):0\leqslant j\leqslant \nu,\ z\in M, g\in\mathfrak{N}_2\}>K_1/2$. Choose an integer $h>0$ so that $2^hK_1/2>2$ and $2^h>2$. We assert

(7) $m(T_pg^{h\nu})>2$ for all $p\in H_U(g)$, $g\in\mathfrak{N}_2$.

Indeed, for each $p\in H_U(g)$, there is an integer $0<n(p)\leqslant\nu$ such that $m\big(T_pg^{n(p)}|E_p^u(g)\big)\geqslant 2$, since $u(p,g)\leqslant\nu$. Thus, for $p\in H_u(g)$, there is a sequence of integers $0<m_1<m_2<\cdots<m_j<h\nu$ with $m_i-m_{i-1}\leqslant\nu$, $h\nu-m_j\leqslant\nu$, and $m\big(T_{p_{i-1}}g^{m_i-m_{i-1}}|E^u_{p_{i-1}}(g)\big)\geqslant 2$, where $p=p_0$, $p_i=g^{m_i}(p_0)$, $0<i\leqslant j$. It is clear that $j\geqslant h$, so

$$\begin{aligned} m\big(T_pg^{h\nu}|E_p^u(g)\big) &\geqslant m\Big(T_{p_j}g^{h\nu-m_j}T_{p_{j-1}}g^{m_j-m_{j-1}}\cdots T_pg^{m_1}|E_p^u(g)\Big)\\ &\geqslant m\Big(T_{p_j}g^{h\nu-m_j}|E^u_{p_j}(g)\Big)\cdot 2^j\\ &\geqslant \min\left(\frac{K_1}{2},1\right)\cdot 2^h>2. \end{aligned}$$

Thus, if $n_1=h\nu$, we have $m(T_pg^{n_1}|E_p^u(g))>2$ for all $p\in H_U(g)$, $g\in\mathfrak{N}_2$, so (7) holds. Similarly, there is an integer $n_2>0$ such that $m\big(T_pg^{-n_2}|E_p^s(g)\big)>2$ for $p\in H_U(g)$, $g\in\mathfrak{N}_2$.

Now let $y\in\Lambda$ and consider two sequences $E^u_{p_i}(g_i)$, $E^u_{p_i}(g_i')$ with $p_i\to y$, $p_i'\to y$, $g_i\to f$, $g_i'\to f$ as $i\to\infty$ and $\{g_i,g_j'\}\subset\mathcal{H}$. We claim that there is a subspace $E^u_y\subset T_yM$ such that $E^u_{p_i}(g_i)\to E^u_y$ and $E^u_{p_i'}(g_i')\to E^u_y$ as $i\to\infty$ in the Grassmann sense. First, choosing E^u_y to be the limit of a subsequence of $\{E^u_{p_i}(g_i)\}$ and replacing $\{E^u_{p_i}(g_i)\}$ by that subsequence, we may assume that $E^u_{p_i}(g_i)\to E^u_y$. By way of contradiction, suppose that $E^u_{p_j'}(g_j')\not\to E^u_y$. Then there is a unit vector $v\in T_yM$ such that $v\notin E^u_y$ and v is in a subsequential limit of $\{E^u_{p'_j}(g_j')\}$. There is an integer $n_3>\max\{n_1,n_2\}$ such that for $n\geqslant n_3$, $\big|T_{g_n}^{-n_3}(v)\big|>2$, since $v\notin E^u_y$ and $E^u_{p_i}(g_i)\to E^u_y$. On the other hand, for infinitely many j's, v is near $E^u_{p_j'}(g_j')$, so $\big|T_{g_j'}^{-n_3}(v)\big|<\frac{1}{2}$. But $g_j'\to f$ and $g_j\to f$, so we reach the absurdity that $\frac{1}{2}>\big|T_yf^{-n_3}(v)\big|>2$. Thus, all $E^u_{p_i}(g_i)$'s converge to the same E^u_y. Moreover, $m(T_yf^{n_3}|E^u_y)>2$.

Similarly, the $E^s_{p_i}(g_i)$'s converge to a subspace $E^s_y\subset T_yM$, and $m\big(T_yf^{-n_3}|E^s_y\big)>2$. Since $T_{p_i}M=E^u_{p_i}\oplus E^s_{p_i}$, one sees easily that $T_yM=E^u_y\oplus E^s_y$, and reasoning analogous to the above gives that $\{E^s_y\}$ and $\{E^u_y\}$ are continuous sub-bundles of TM. Compactness of Λ and continuity of Tf enable one to choose $n_3>0$ so that

$$m\big(T_yf^{n_3}|E^u_y\big)>2 \quad\text{and}\quad m\big(T_yf^{-n_3}|E^s_y\big)>2$$

for all $y \in \Lambda$. Let $c_1 = \inf\{m(T_y f^j | E_y^u) : y \in \Lambda,\ 0 \leqslant j \leqslant n_3\}$ and choose $2 > \beta > 1$ with $\beta^{n_3} < 2$ and $k(\log 2 - n_3 \log \beta) - n_3 \log \beta > 0$ for every integer $k > 0$. Fix $n \geqslant 0$. Then $m(T_y f^n | E_y^u) \geqslant c_1 \beta^n > (c_1/2)\beta^n$ if $n \leqslant n_3$. If $n > n_3$, write $n = kn_3 + r$ with $k > 0$ and $0 \leqslant r < n_3$. Then

$$\begin{aligned} m\left(T_y f^n | E_y^u\right) &\geqslant m\left(T_y f^r\right) m\left(T_y f^{kn_3}\right) \\ &\geqslant c_1 2^k \geqslant c_1 \beta^{(k+1)n_3} > c_1 \beta^{kn_3 + r} \\ &= c_1 \beta^n > \frac{c_1}{2}\beta^n. \end{aligned}$$

Similarly, there are constants $c_2 > 0$, $\beta_1 > 1$ with $|T_y f^n | E_y^s| < c_2 \beta_1^{-n}$ for $n \geqslant 0$, so Λ is a hyperbolic set for f.

Now we show that Λ has a local product structure [5]; i.e., for $\varepsilon > 0$ small, and $x, y \in \Lambda$, we have $W_\varepsilon^u(x) \cap W_\varepsilon^s(y) \subset \Lambda$. Assume ε chosen so that if $d(x,y) < 2\varepsilon$, then $W_\varepsilon^u(x) \cap W_\varepsilon^s(y) \neq \varnothing$ and $W_\varepsilon^u(x)$ is transverse to $W^s(x)$. As usual, let d be the metric on Λ. Choose $0 < \delta < \varepsilon$ so that if $W_\delta^u(x) \cap W_\delta^s(y) \neq \varnothing$, then $d(x,y) < 2\varepsilon$. Let $x, y \in \Lambda$. We claim $W_\delta^u(x) \cap W_\delta^s(y) \subset \Lambda$. Of course, we may assume $W_\delta^u(x) \cap W_\delta^s(y) \neq \varnothing$, so $d(x,y) < 2\varepsilon$. Let $z \in W_\delta^u(x) \cap W_\delta^s(y)$. From the definition of Λ there are transversal homoclinic points x_1, y_1 of f such that x_1 is near x, y_1 is near y, $o(x_1) \cap \operatorname{int} U \neq \varnothing$, and $o(y_1) \cap \operatorname{int} U \neq \varnothing$. Applying Smale's homoclinic point theorem (e.g., as in the appendix of [10]), we see there are hyperbolic periodic points p_1, q_1 of f with p_1 near x, q_1 near y, such that

(a) $W_\delta^u(p_1) \cap W_\delta^s(q_1)$ has a point near z,
(b) $W_\delta^s(p_1) \cap W_\delta^u(q_1) \neq \varnothing$,
(c) $o(p_1) \cap \operatorname{int} U \neq \varnothing$, $o(q_1) \cap \operatorname{int} U \neq \varnothing$.

Now p_1 and q_1 are homoclinically related [10], so z will be near the closure of the transversal homoclinic points of p_1 (and q_1). But p_1 is in this latter set as well, so there are transversal homoclinic points meeting $\operatorname{int} U$ whose orbits come near z. This proves that $z \in \Lambda$.

To complete the proof of Proposition 4.2, we show $\Lambda = M$. For this, we need only show that Λ is an open subset of Λ, since it is closed by definition and M is connected. In view of the local product structure, it suffices to show that for each $y \in \Lambda$, $W_\delta^u(y) \cup W_\delta^s(y) \subset \Lambda$, where δ is as above. If $x \in H_U(f)$, there is a hyperbolic periodic point p whose orbit meets $\operatorname{int} U$ such that $W_\delta^u(x) \subset W^u(o(p))$ and $W_\delta^s(x) \subset W^s(o(p))$.

Since the orbit of p meets the interior of U, it follows that $W^s(o(p)) \cup W^u(o(p))$ is contained in the orbit of $\operatorname{int} U$ which is in Λ. So $W^s(o(p)) \cup W^u(o(p)) \subset \Lambda$, and hence $W_\delta^u(x) \cup W_\delta^s(x) \subset \Lambda$. But there is a sequence x_i in $H_U(f)$ with $x_i \to y$ as $i \to \infty$. Thus, $W^u_\delta(x_i) \to W^u_\delta(y)$ and $W^s_\delta(x_i) \to W^s_\delta(y)$, which gives $W^s_\delta(y) \cup W^u_\delta(y) \subset \Lambda$.

We now prove (6).

By Proposition 4.2, we know there are sequences $g_i \to f$ in $\mathrm{Diff}^1_\omega(M) \cap \mathcal{H}$ and p_i in $H_U(g_i)$ with $\max\{u(p_i, g_i), u(p_i, g_i^{-1})\} \to \infty$ as $i \to \infty$. Assume $u(p_i, g_i) \to \infty$, as the other case is similar. Let $c_1 > 0$ be a fixed constant such that $|\omega(v_1, v_2)| \leqslant c_1 |v_1||v_2|$ for all $v_1, v_2 \in TM$. Letting $n_i = u(p_i, g_i) - 1$, we have, for each i, a unit vector $v_i \in T_{p_i} W^u(g_i)$ with $\left|T_{p_i} g_i^{n_i}(v_i)\right| < 2$. Let $C_2 = \min\left(\frac{1}{2}, \inf_{x \in M} c(x)\right)$ where c is as in the perturbation lemma (2.1). Then for $0 < \varepsilon < 1$, we may make an $\varepsilon - C^1$-small change from f to f_1 so that

(a) $f(y) = f_1(y)$ for y outside a small neighborhood of $\{p_i, f(p_i), \ldots, f^{n_i}(p_i)\}$,
(b) $f_1^j(p_i) = f(p_i)$ for $0 \leqslant j \leqslant n_i$,
(c) $p_i \in H_U(f_1)$,
(d) $v_i \in T_{p_i} W^u(p_i, f_1)$,
(e) $\left|T_{p_i} f_1^{n_i}(v_i)\right| < 2(1 - c_2\varepsilon)^{n_i}$.

Observe that $\omega | W^u(p_i, f_1) \equiv 0 \equiv \omega | W^s(p_i, f_1)$. Indeed, if $a_1, a_2 \in T_z W^u(p_i, f_1)$ then $T_z f^{-n}(a_1)$ and $T_z f^{-n}(a_2)$ approach zero as $n \to \infty$. But $\omega_z(a_1, a_2) = \omega_{f^{-n}(z)}(T_z f^{-n} a_1, T_z f^{-n} a_2)$, so $\omega_z(a_1, a_2) = 0$. A similar argument implies that $\omega | W^s(p_i, f) \equiv 0$.

Now $|v_i| = 1$ and $v_i \in T_{p_i} W^u(p_i, f_1)$, so there is a vector $v_i' \in T_{p_i} W^s(p_i, f_1)$ such that $\omega(v_i, v_i') = 1$. Hence,

$$\begin{aligned} 1 = \omega(v_i, v_i') &= \omega\left(T_{p_i} f_1^{n_i} v_i, T_{p_i} f_1^{n_i} v_i'\right) \\ &\leqslant c_1 \left|T_{p_i} f_1^{n_i} v_i\right| \left|T_{p_i} f_1^{n_i} v_i'\right| \\ &< 2c_1(1 - c_2\varepsilon)^{n_i} \left|T_{p_i} f_1^{n_i} v_i'\right|. \end{aligned}$$

So $\left|T_{p_i} f_1^{n_i} v_i'\right| > (2c_1)^{-1}(1 - c_2\varepsilon)^{-n_i}$. With an $\varepsilon - C^1$-small change in f_1 to f_2 on a small neighborhood of $f^{-1}(p_i)$, we may make

$$T_{f_1^{-1}(p_i)} f_2\left(T_{p_i} f_1^{-1}(v_i)\right) = (\alpha v_i, \beta v_i') \in T_{p_i} W^u(p_i, f_2)$$

with $\beta/\alpha > c_2\varepsilon$. Then $T_{p_i} f_2^{n_i}(\alpha v_i, \beta v_i')$ has the form (ξ_1, ξ_2) with

$$\xi_1 \in T\left[f_2^{n_i}(p_i)\right] W^u\left(f_2^{n_i}(p_i), f_2\right), \xi_2 \in T\left[f_2^{n_i}(p_i)\right] W^s\left(f_2^{n_i}(p_i), f_2\right),$$

(where for typographical reasons we have temporarily written $T[f]$ for T_f), and

$$\frac{|\xi_2|}{|\xi_1|} > c_2\varepsilon(c_1 4)^{-1}(1 - c_2\varepsilon)^{-2n_i}.$$

Now n_i may be chosen arbitrarily large independent of ε, so that we may then further change f_2 to f_3 on a small neighborhood of $f_2^{n_i}(p_i)$ to obtain the desired tangency. Notice that all of these changes leave the f-orbit of p_i unchanged and leave f unaltered on a neighborhood of the hyperbolic periodic orbit $o(q)$ which has p_i as a homoclinic point. Thus the tangency actually is a homoclinic tangency for f_3. Moreover, $o(q)$ meets U, so f_3 has a homoclinic tangency (in the orbit of p_i) in U. This completes the proof of (6), and hence of Theorem 1.1.

5. Here we extend the results of the preceding sections to the non-compact case. Our first task is to generalize Takens's homoclinic point theorem [26]. Recall this theorem says that if M is compact, then there is a residual subset $\mathfrak{B} \subset \mathrm{Diff}^1_\omega(M)$ such that every hyperbolic periodic point of $f \in \mathfrak{B}$ has homoclinic points.

In the first place, one cannot expect this result to carry over directly to the non-compact case. For example, the linear symplectic map $f(x,y)=(\lambda x, \lambda^{-1}y)$, $0<\lambda<1$, has $(0,0)$ as a hyperbolic fixed point, and any diffeomorphism C^1 Whitney close to f is topologically equivalent to f. In particular, no small C^1 perturbation has homoclinic points. The reason for this is that f has no non-trivial non-wandering points. In the compact case, every point is non-wandering, and this is a significant factor contributing to homoclinic phenomena.

The relevant concept here is actually something slightly different from non-wandering. Roughly speaking, one need only require that x become non-trivially non-wandering after perturbations. More precisely, we say that a point x in M is *adherently periodic* for f if there are a compact subset $K \subset M$ and sequences (p_i) in K, (g_i) in $\mathrm{Diff}^1_\omega(M)$ such that

(a) p_i is a periodic point of g_i with $o(p_i) \subset K$,
(b) $p_i \to x$ and $g_i \to f$ as $i \to \infty$,
(c) $\mathrm{per}(p_i) \to \infty$ as $i \to \infty$,
(d) $g_i(z)=f(z)$ for $z \in M-U_K$ with U_K some compact neighborhood of K.

Here $\mathrm{per}(p_i)$ is the smallest positive integer τ with $g^\tau{}_i(p_i)=p_i$.

Observe that condition (c) makes it possible for a point to be periodic without being adherently periodic, as occurs in the above linear example. But it follows from Proposition 3.1 that every non-hyperbolic periodic point is adherently periodic. Also, the closing lemma of Pugh and Robinson implies that non-periodic points x which are non-wandering *relative to* K are adherently periodic. By this latter concept for x we mean there are sequences $y_i \in K$, $n_i \to \infty$ such that $y_i \to x$, $f^{n_i}(y_i) \to x$ and $f^j(y_i) \in K$ for $0 \leqslant j \leqslant n_i$. One can give examples of adherently periodic points for f which are not non-wandering for f using appropriate versions of cycles (defined as in [11]) on non-compact manifolds. In the non-symplectic case, such examples can be given on any manifold. It is an interesting problem to characterize the adherently periodic points of f without resorting to diffeomorphisms near f.

Condition (d) above is given because it enables us to conclude that (g_i) is eventually in any C^1 Whitney neighborhood of f. Without this we could not get by with (g_i) as a sequence, but would have to take it as a net indexed by a neighborhood base of f.

The main result of this section is

THEOREM 5.1. *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a C^1 symplectic diffeomorphism, and let x be an adherently periodic point for f. Then, given neighborhoods U of x in M and $\mathfrak{N}$ of f in $\mathrm{Diff}^1_\omega(M)$, there is a g in $\mathfrak{N}$ having a transversal homoclinic point in U.*

Proof. Choose K, U_K, p_i, g_i as in the definition of "adherently periodic." Thus $g_i \to f$, $p_i \to x$ with p_i a periodic point of g_i in K and $g_i = f$ off U_k. By [19], the p_i's may be assumed to be quasi-elliptic or hyperbolic. If $\{p_i\}$ has a subsequence of quasi-elliptic points, we are done, by Proposition 3.1. Thus, we may assume all the p_i's are hyperbolic (after eliminating finitely many of them). Also, letting $\tau_i = \mathrm{per}(p_i)$, we have that $\tau_i \to \infty$ as $i \to \infty$.

Let $T_qM = E^s_q \oplus E^u_q$ be the hyperbolic splitting of T_qM for $q \in o(p_i)$. Proceeding as in Section 4, for $q \in o(p_i)$ define $u(q, g_i) = \inf\{n > 0 : m(T_q g_i | E^u_q) \geqslant 2\}$. Set $\xi(q, g_i) = \max\{u(q, g_i), u(q, g_i^{-1})\}$. First suppose

$$\sup\{\xi(q, g_i);\ q \in o(p_i),\ i = 1, 2, \dots\} \leqslant n_1.$$

Since $\tau_i \to \infty$, and all of the p_i's lie in K, we may assume that for i larger there are points q_{i_1}, q_{i_2} in $o(p_i, g_i)$ which are close to each other. Then (1) and stable manifold theory as in [6] will imply that q_{i_1} and q_{i_2} are homoclinically related for large i. Therefore each of these p_i's has homoclinic points. Thus, (1) implies that g_i has homoclinic points near x for i large. So we may assume that (1) fails. That is, assume there are subsequences of $(p_i), (g_i)$ [denoted again as $(p_i), (g_i)$] with p_i a hyperbolic periodic point of g_i in K, and there are points $q_i \in o(p_i)$ such that $\xi(q_i, g_i) \to \infty$ as $i \to \infty$. We may assume $u(q_i, g_i) \to \infty$, the other case being similar.

One may verify that the techniques of J. Franks [4] and V. Pliss [14][1] may be modified for symplectic diffeomorphisms to give that either

(2) for some constants $c_1>0$, $\lambda>1$, and all large i, $m(T_q g_i^{\tau_i}|E_q^u) \geqslant c_1\lambda^{\tau_i}$ and $|T_q g_i^{\tau_i}|E_q^s| \leqslant c_1^{-1}\lambda^{-\tau_i}$, where $q\in o(p_i)$, or

(3) some small C^1 perturbation g of g_i, for arbitrarily large i, has a periodic point with eigenvalue one near x.

Indeed, the appropriate transcriptions of the methods in [4] and [14] to our case would involve linear maps near the identity map on E_q^u or E_q^s. Such maps may be realized in symplectic automorphisms near the identity on T_qM. To see this, let $\{v_1,\dots,v_n,v_1',\dots,v_n'\}$ be a symplectic basis for T_qM with $v_i\in E_q^u$, $v_i'\in E_q^s$, and let $A: E_q^u\to E_q^u$ be a linear isomorphism with matrix A_1 relative to $\{v_1,\dots,v_n\}$. The linear map B whose matrix relative to $\{v_1,\dots,v_n,v_1',\dots,v_n'\}$ is

$$\begin{pmatrix} A_1 & 0 \\ 0 & (A_1^t)^{-1} \end{pmatrix}$$

is symplectic. Also, the same construction applies if one replaces E_q^u by E_q^s. This remark, together with Lemma 2.1 and the techniques of [4], [14], implies that (2) or (3) is true.

If (3) holds, the perturbation g may be gotten along a smooth curve of symplectic diffeomorphisms, and hence Theorem (3.3.IV) of [20] implies that a further perturbation yields a quasi-elliptic point near x for some $\bar{g}$ that is C^1 near f. Then Theorem 5.1 follows from Proposition 3.1.

Thus we may assume (2) holds.

Let $\varepsilon_0=\varepsilon_0(x)$ and $c=c(x)$ be as in the perturbation lemma (Lemma 2.1), and let $\mathfrak{N}(f,\mathfrak{U},\mathfrak{V},\varepsilon)$ be a neighborhood of f in $\mathrm{Diff}^1_\omega(M)$ with $\varepsilon<\varepsilon_0$. Let $\varepsilon_1(x)=\frac{1}{2}\varepsilon(x)$ for $x\in M$. Since $g_i=f$ off U_K, we may choose a positive integer $\nu>0$ so that $i\geqslant\nu$ implies that $\mathfrak{N}(g_i,\mathfrak{U},\mathfrak{V},\varepsilon_1)\subset\mathfrak{N}(f,\mathfrak{U},\mathfrak{V},\varepsilon)$. Let $c_2=\inf_{x\in U_K}c(x)$. We will show there are an integer $i\geqslant\nu$ and a $g\in\mathfrak{N}(g_i,\mathfrak{U},\mathfrak{V},\varepsilon_1)$ such that g has a homoclinic point in U. This will prove the theorem, since Robinson shows that all homoclinic points may be made transverse with arbitrarily small approximations [19].

From now on we only consider $i\geqslant i_0$. For $q\in o(p_i)$ define

$$\alpha(q)=\inf\left\{\arccos\frac{v_1,v_2}{|v_1||v_2|}: v_1\in E_q^u-\{0\},\ v_2\in E_q^s-\{0\}\right\}.$$

[1] I am indebted to D. Pixton for making Pliss's results available to me. With these results, we were able to simplify the proof of Theorem 5.1. Originally, we had a proof using a weaker version of Pliss's results which was established via an extension of the closing lemma methods in [17].

Thus $\alpha(q)$ is the smallest angle between E^u_q and E^s_q. Let

$$\alpha = \liminf_{i\to\infty}\left\{\inf_{q\in o(p_i)}\alpha(q)\right\}.$$

Consider first the case

(4) $\alpha = 0$.

Let $\varepsilon_2 = \inf_{x\in U_k}\varepsilon_1(x) > 0$.

By (4), we may choose a subsequence which we also call p_i and points $q_i\in o(p_i)$ so that $\alpha(q_i)\to 0$. Since $W^u(q)$ $[W^s(q)]$ is tangent to E^u_q $[E^s_q]$ for all q, it follows that there are points $q_j\in o(p_j)$, $x_j\in W^u(q_j)$, and $y_j\in W^s(q_j)$ such that

(5) $d(x_j,y_j)\cdot d(x_j,q_j)^{-1}\to 0$ as $j\to\infty$,
(6) $d(x_j,o(q_j)-\{q_j\}) > 2d(x_j,q_j)$ and $d(y_j,o(q_j)-\{q_j\}) > 2d(x_j,q_j)$ for all j,
(7) $d(x_j,q_j)\to 0$ as $j\to\infty$.

Let $M_1 = \max\{\sup_{x\in U_K}|T_xf^{-1}|,1\}$ and $M_2 = \inf_{x\in U_K}m(T_xf^{-1})$.

Choose $\nu_1 > 0$ so that for $i \geqslant \nu_1$, we have $\sup_{x\in U_K}|T_xg_i^{-1}| < 2M_1$ and $\inf_{x\in U_K}m(T_xg_i^{-1}) > M_2/2$. By (2), we may also assume

(8) for $i > \nu_1$, $c_1\lambda^{\tau_i} > \max(2M_2M_1,2)$ and $c_1^{-1}\lambda^{-\tau_i} < \min\left(\frac{1}{2},(2M_2M_1)^{-1}\right)$.

For $\delta_1 > 0$ small enough and $\nu_2 > \nu_1$ large enough, we have that $0<\delta\leqslant\delta_1$, $u\in U_K$, $i\geqslant\nu_2$ imply $g_1^{-1}(B_\delta(u))\subset B_{3M_1\delta}(g^{-1}{}_i(u))$. Set $\delta_j = (18M_1^2)^{-1}d(x_j,q_j)$. Choose j large, using (5), so that $d(x_j,y_j) < c_2\delta_j\cdot\varepsilon_2 \leqslant c_2\delta_j\cdot\varepsilon_1(x_j)$.

By the perturbation lemma [Lemma 2.1(1)], there is a $g\in\mathfrak{N}(g_j,\mathfrak{U},\mathfrak{V},\varepsilon_1)$ such that

(9) $gg_j^{-1}(x_j) = y_j$,
(10) $g(z) = g_j(z)$ for $z\notin g_j^{-1}(B_{\delta_j}(x_j))$,
(11) $g^{-1}(z) = g_j^{-1}(z)$ for $z\notin B_{\delta_j}(x_j)$.

Now, for large j, $g_j^{-1}(B_{\delta_j}(x_j))\subset B_{3M_1\delta_j}(g_j^{-1}(x_j))$, so by (6), $g_j^{-1}(B_{\delta_j}(x_j))\cap B_{\delta_j}(x_j) = \varnothing$. Also,

(12)
$$\begin{aligned} d\left(g_j^{-1}B_{\delta_j}(x_j),g_j^{-1}(q_j)\right) &> \frac{M_2}{2}\left(1-\frac{1}{18M_1^2}\right)d(x_j,q_j)\\ &> \frac{M_2}{3}d(x_j,q_j)\\ &> \frac{M_2}{3}\cdot 3M_1 d\left(gu-{}_j^1x_j,g_j^{-1}q_j\right)\\ &= M_1M_2 d\left(g_j^{-1}x_j,g_j^{-1}q_j\right)\end{aligned}$$

By (8) and (11), $g_j^{-\tau_j}$ contracts $W^u(g_j^{-1}q_j, g_j)$ by at least $(2M_2M_1)^{-1}$ near $g_j^{-1}q_j$. So (12) implies that $g_j^{-l}(x_j) \notin B_{\delta_j}(x_j)$ for all $l \geqslant 1$. Thus, (10) gives that $g_j^{-1}(x_j) \in W^u(g^{-1}(q_j), g)$, and $y_j \in W^u(q_j, g)$. Similarly, $g_j^l(y_j) \notin g_j^{-1}B_{\delta_j}(x_j)$ for all $l \geqslant 0$, so (11) puts y_j in $W^s(q_j, g)$. From (9), we have that y_j is a homoclinic point for g. Also, $o(q_j, g) \subset \mathrm{Cl}\, o(y_j, g)$, so g has a homoclinic point in U.

Suppose now that $\alpha > 0$. Since $u(q_i, g_i) \to \infty$ as $i \to \infty$, we may choose a vector $v_i \in E^u_{q_i}$ such that $|v_i| = 1$ and $|T_{q_i} g_i^{n_i}(v_i)| < 2$ with n_i and i arbitrarily large. In what follows $c_3, c_4, \ldots$ will denote constants independent of i defined by the first equation in which they appear. We may find $c_3 > 0$ such that $|\omega_y(u, v)| \leqslant c_3 |u|\,|v|$ for all $y \in U_K$. Take i large enough so that $g_i \in \mathfrak{N}(f, \mathfrak{U}, \mathfrak{V}, \varepsilon_1)$. By Lemma 2.1, for $\varepsilon_3 = c_2\varepsilon_2 < 1$ we may find $h_1 \varepsilon_3 - C^1$ near g_i such that $o(q_i, h_1) = o(q_i, g_i)$, $|T_{q_i} h_1^{n_i}(v_i)| < 2(1 - c_2\varepsilon)^{n_i}$, and q_i is a hyperbolic periodic point for h_1 with $v_i \in E^u_{q_i}(h_1)$, $|v_i| = 1$. Choose $v_i' \in E^s_{q_i}(h_1)$ with $\omega(v_i, v_i') = 1$. Thus,

$$
\begin{aligned}
1 = \omega(v_i, v_i') &= \omega\big(T_{q_i} h_1^{n_i} v_i, T_{q_i} h_1^{n_i}(v_i')\big) \\
&\leqslant c_3 \big|T_{q_i} h_1^{n_i}(v_i)\big|\,\big|T_{q_i} h_1^{n_i}(v_i')\big| \\
&< 2c_3(1 - c_2\varepsilon_3)^{n_i} \big|T_{q_i} h_1^{n_i}(v_i')\big|
\end{aligned}
$$

or $|T_{q_i} h_1^{n_i}(v_i')| > (2c_3)^{-1}(1 - c_2\varepsilon_3)^{-n_i}$. Since $\alpha > 0$, there is a constant $c_4 > 0$ such that $|v_i'| \leqslant c_4$. Hence,

$$
\begin{aligned}
\big|T_{q_i} h_1^{n_i}(v_i')\big| &> (2c_3)^{-1} c_4^{-1} (1 - c_2\varepsilon_3)^{-n_i} c_4 \\
&> c_5 (1 - c_2\varepsilon_3)^{-n_i} |v_i'|.
\end{aligned}
$$

Choose local coordinates U_q around each $q \in o(q_i)$ so that $q \neq q'$ implies that $U_q \cap U_{q'} = \varnothing$, and disks about q in E^u_q, E^s_q, may be identified with disks about q in $W^u(q), W^s(q)$, respectively. Write (x_1, x_2) for a point in U_q with $x_1 \in W^u(q)$, $x_2 \in W^s(q)$. Assume v_i, v_i' small enough so that $T_{q_i} h_1^j(v_i)$ and $T_{q_i} h_1^j(v_i')$ are in $U_{h_1^j(q_i)}$ for $0 \leqslant j \leqslant n_i$. Using (8), Lemma 2.1(1), and the fact that $\alpha > 0$, we may find h_2 $\varepsilon_3 - C^1$ near h_1 such that (with the above identifications)

(13) $h_2(h_1^{-1} v_i, o) = (\beta_1 v_i, \beta_2 v_i')$ with $|\beta_2| / |\beta_1| > c_6 \varepsilon_3$,
(14) $(\beta_1 v_i, \beta_2 v_i') \in W^u(q_i, h_2)$,
(15) $h_2(z) = h_1(z)$ for $z \notin U_{h_1^{-1}(q_2)}$.

Then, if β_1 and β_2 are very small, we have $h_2^{n_i}(\beta_1 v_i, \beta_2 v_i') = (y_i, y_i')$ with $|y_i'| / |y_i| > c_7 (1 - c_2\varepsilon_3)^{-2n_i} \varepsilon_3$.

Thus, $(y_i, y_i') \in W^u(h_2^{n_i}(q_i), h_2)$ is very near $W^s(h_2^{n_i}(q_i), h_2)$ for n_i large. Again using (8) and Lemma 2.1, we may find h_3 such that $h_3(y_i, y_i')$ is a homoclinic point for $W^u(h_3^{n_i+1}(q_i), h_3)$. As before, h_3 has a homoclinic point on the orbit of $h_3(y_i, y_i')$ in U, and Theorem 5.1 is proved.

Remark. In view of the Smale homoclinic point theorem [10, Appendix], if x and f are as in Theorem 5.1, the g near f will have infinitely many periodic orbits near x. This result ought to be true even if f is not symplectic (i.e., one ought to be able to introduce infinitely many periodic points near an adherently periodic orbit). However, we have not been able to prove this in the non-symplectic case, even though the closing lemma implies it is true *generically*.

For $f \in \mathrm{Diff}^1_\omega(M)$, let $\mathcal{C}(f)$ denote the set of adherently periodic points of f. This is a closed f-invariant subset of M, and every orbit in it is bounded. From Theorem 5.1 and Proposition 4.1, we obtain the following analogs of Theorems 1.1 and 1.3 in the non-compact setting.

THEOREM 5.2. *Let $f \in \mathrm{Diff}^1_\omega(M)$, and suppose $\mathcal{C}(f)$ is not hyperbolic. Let $x \in \mathcal{C}(f)$. Given neighborhoods U of x in M and V of f in $\mathrm{Diff}^1_\omega(M)$, there is a g in V having a 1-elliptic periodic point in U.*

THEOREM 5.3. *There is a residual subset $\mathcal{B} \subset \mathrm{Diff}^1_\omega(M)$ such that if $f \in \mathcal{B}$ and $\mathcal{C}(f)$ is not hyperbolic, then $\mathcal{C}(f)$ is contained in the closure of the 1-elliptic points of f.*

6. In this section we observe that the theorems given earlier are valid for Hamiltonian systems with standard modifications (see [19], [26]). We also obtain a strengthened version of Theorem 5.1 if there are at least two independent integrals.

Let us recall the elementary properties of Hamiltonian systems.

Let (M,ω) be a symplectic manifold. Given a C^{r+1} real valued function $H: M \to \mathbf{R}$, one defines the C^r Hamiltonian vector field X_H associated to H by the formula

$$\omega_x(X_H(x), Y) = dH_x(Y) \qquad \text{for all} \quad x \in M, \quad Y \in T_xM.$$

That is, $dH = i_{X_H}\omega$, where i_{X_H} is the interior product. The set of C^r Hamiltonian vector fields is denoted $\chi^r_{\mathcal{H}}(M)$, and the Whitney C^r topology on this set corresponds to the Whitney C^{r+1} topology on the space $C^{r+1}(M,\mathbf{R})$ of C^{r+1} real functions on M. In symplectic coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ the differential equations for X_H become Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = \frac{-\partial H}{\partial X_i}.$$

All the differential equations of classical mechanics may be expressed as Hamiltonian vector fields, and in fact, so may any equations derivable from a variational principle (see [8]).

The flow ϕ_H^t associated to X_H which is defined by

$$\left.\frac{d}{dt}\right|_{t=s} \phi_H^t(x) = X_H\left(\phi_H^s(x)\right).$$

is called a Hamiltonian flow with Hamiltonian H. Since $d\omega = 0$, the Lie derivative $L_{X_H}\omega = i_{X_H}d\omega + di_{X_H}\omega = 0 + d^2H = 0$, so ϕ_H^t is a symplectic diffeomorphism for each t. Also, H is constant on orbits of ϕ_H^t, since

$$X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0.$$

Let x be a periodic point for X_H [i.e., $\phi_H^t(x) = 0$ for some $t > 0$] with $X_H(x) \neq 0$. Let Σ be a small piece of hypersurface in M transverse to $X_H(x)$ at x, and let $\Sigma_x = \Sigma \cap H^{-1}(H(x))$. If Σ is small enough, then Σ_x is a $2n-2$ dimensional submanifold of M and the pullback $i^*\omega$ is non-degenerate, where $i: \Sigma_x \to M$ is the inclusion. Let $y \in \Sigma_x$ be near x and such that $\phi_H^{t(y)} \in \Sigma_x$ for some $t(y) > 0$. For z near y in Σ_x, there is a time $t(z) > 0$ such that $\phi_H^{t(z)}(z) \in \Sigma_x$, and the implicit function theorem gives that $t(z)$ is a differentiable function of z.

Also, the map $z \to \phi_H^t(z)$ is a local diffeomorphism, called the Poincaré map, defined on an open neighborhood of x in Σ_x which preserves $i^*\omega$. Moreover, the technique in [25] shows how to realize certain C^r perturbations of $\phi_H^{t(\cdot)}(\cdot)$ as Poincaré maps of C^{r+1} perturbations H_1 of H. Also see [18] for C^1 perturbations of the Poincaré map. It is easily checked that these perturbation techniques, together with methods described earlier in this paper, will yield the following results.

Define adherently periodic, hyperbolicity and 1-ellipticity for periodic orbits, and homoclinic orbits etc. in the natural way for Hamiltonian systems. For example, a periodic point x is hyperbolic if it is a hyperbolic fixed point of its Poincaré map, if x is non-critical; while x must be a hyperbolic fixed point of ϕ_H^t, $t \neq 0$, if x is a critical point. To avoid trivialities, assume $\dim M = 2n \geqslant 4$.

THEOREM 6.1. *Let $X_H \in \mathfrak{X}'_{\mathcal{H}}(M)$ be a C^1 Hamiltonian vector field on M, and let x be an adherently periodic point of X_H. Given neighborhoods U of x in M and $\mathfrak{N}$ of H in $C^2(M, \mathbf{R})$, there is an $H_1 \in \mathfrak{N}$ such that X_{H_1} has a homoclinic orbit in U.*

THEOREM 6.2. *There is a residual set $\mathfrak{B} \in \mathfrak{X}_H^1(M)$ such that for $X_H \in \mathfrak{B}$ and $x \in M$, if some compact component of $H^{-1}(H(x))$ is not hyperbolic, then that component is in the closure of the 1-elliptic periodic orbits.*

THEOREM 6.3. *Let X_H be a C^1 Hamiltonian vector field with C^2 Hamiltonian $H: M \to \mathbf{R}$, and assume X_H has an integral L independent of H. If x is a* *non-wandering point of X_H whose orbit lies in a compact subset of M, then H may be C^2-perturbed to H_1 so that X_{H_1} has infinitely many periodic, homoclinic, and quasi-periodic motions near x.*

Proof of Theorem 6.3. First recall that x is non-wandering if there are sequences $x_j \to x$ in M, $t_j \to \infty$ in $\mathbf{R}$ such that $\phi_H^{t_j}(x_j) \to x$ as $j \to \infty$. Since X_H has H and L as integrals, we have $o(x) \cup o(x_j) \subset H^{-1}(H(x)) \cap L^{-1}(L(x))$ for each j. The proof of the closing lemma [17] enables one to C^2-perturb H to $\bar{H}$ so that $X_{\bar{H}}$ has a periodic orbit $o(p)$ near x. Let $H_1, H_2, \ldots$ be a sequence of C^2 functions converging to H in the C^2 topology so that each X_{H_i} has a periodic orbit $o(p_i)$ and $p_i \to x$ as $i \to \infty$. The existence of the integral L for X_H insures that the Poincaré maps of $o(p_i)$ on a transverse section through x cannot be uniformly hyperbolic. Modifying the proof of assertion (3) in the proof of theorem (5.1) and the subsequent discussion to flows, we may choose the perturbation $\bar{H}$ so that $o(p)$ is quasi-elliptic for $X_{\bar{H}}$. Then we may further perturb using the Kolmogorov-Arnold-Moser theorem, the Birkhoff fixed point theorem, and the techniques of the proof of Proposition 3.1, to obtain Theorem 6.3.

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