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NONDENSITY OF AXIOM A(a) ON S^2

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1. In [**6**], Smale introduced some theorems which give much information on the structure of certain diffeomorphisms of a compact manifold. A basic question which arose was whether Axiom A [**6**, §*I*.6] was dense on any compact manifold. Subsequently, Abraham and Smale showed that Axiom A(a) was not C^1 dense on $T^2 \times S^2$ where T^2 is the two torus and S^2 is the two sphere [**1**]. We show in this paper that Axiom A(a) is not C^2 dense on S^2.

We consider the set of all diffeomorphisms of S^2 with the uniform C^r topology, $1 \leq r \leq \infty$. Recall from [**6**] that for $f \in \mathrm{Diff}^r(S^2)$ a point $x \in S^2$ is said to be *nonwandering* if the following is true. For each neighborhood U of x there is a positive integer n such that $f^n(U) \cap U \neq \varnothing$. The set of nonwandering points will be denoted by $\Omega(f)$. Following Smale, we say that f satisfies Axiom A if (a) $\Omega(f)$ has a hyperbolic structure, and (b) the periodic points of f are dense in $\Omega(f)$. One says that a diffeomorphism g of S^2 is *topologically conjugate* (Ω-conjugate) to f if there is a homeomorphism $h: S^2 \to S^2$ ($h: \Omega(f) \to \Omega(g)$) satisfying $gh = hf$. f is called *C^r structurally stable* ($C^r\Omega$-stable) if there is a C^r neighborhood N of f such that any $g \in N$ is topologically conjugate (Ω-conjugate) to f. The main result we have is the following.

(1.1) THEOREM. *There is an open set N in* $\mathrm{Diff}^2(S^2)$ *such that if $f \in N$, then f does not satisfy Axiom* A(a) *and f is not C^2 structurally stable.*

The basic idea of the proof of this theorem is to modify Smale's "horseshoe" example [**6**, §I.5] to produce a diffeomorphism $\bar{L}$ of S^2 such that, for some $x \in \Omega(\bar{L})$, the stable manifold $W^s(x, \bar{L})$ is tangent to the unstable manifold $W^u(x, \bar{L})$ at x (see [**6**] for definitions). One then shows that this phenomenon is preserved under small C^2 perturbations of $\bar{L}$. From this, nondensity of Axiom A(a) is immediate, and, with some slight argument, nondensity of structural stability also follows.

We should make several remarks. First, in [**1**], Abraham and Smale also prove that Ω-stable diffeomorphisms are not C^1 dense on $T^2 \times S^2$. Second, Williams has used the "DA" examples of Smale to show that structurally stable diffeomorphisms are not C^1 dense on T^2 [**7**]. Third, C. Pugh has shown that Axiom A(b) is C^1 dense on all compact manifolds [**4**]. Finally, the main references for this paper are the papers of Smale, [**5**] and [**6**]. In fact, I would suggest that the reader be reasonably familiar with §§I.5 and I.6 of [**6**] before proceeding.

In §2, we construct the diffeomorphism $\bar{L}$ as a natural extension of a diffeomorphism L of the plane. §3 contains some results about Cantor sets which will be

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needed. §4 is largely motivation for §5 where the main results about C^2 perturbations of L are proved. In §6 we prove that all sufficiently small C^2 perturbations of L do not satisfy Axiom A(a), and in §7 we sketch a proof that these small perturbations of L are not C^2 structurally stable. We conclude §7 with some remarks about the Ω-instability of these small perturbations of L.

I wish to express my thanks to the many mathematicians with whom I discussed this paper. Particular thanks go to M. Hirsch, C. Pugh, and S. Smale for much encouragement and many valuable conversations, and to R. Williams, who read a preliminary version of the paper and made many valuable suggestions.

2. To prove Theorem (1.1), we shall define a C^∞ diffeomorphism $\bar{L}$ of S^2 such that there is a C^2 neighborhood N of $\bar{L}$ such that no $f \in N$ satisfies Axiom A(a). We first construct a diffeomorphism L of the plane R^2 so that $L(x) = x$ outside some compact subset H of R^2. Then L induces the diffeomorphism $\bar{L}$ of S^2 as follows. Let $\phi: R^2 \to U$ be a C^∞ diffeomorphism where U is a coordinate patch on S^2. Let $\bar{L}(x) = \phi \circ L \circ \phi^{-1}(x)$ for $x \in U$ and $\bar{L}(x) = x$ for $x \notin U$.

We now construct L.

Consider the square $Q = \{(x_1, x_2) \in R^2 : |x_1| \le 1, |x_2| \le 1\}$. Let $L: Q \to R^2$ be such that

(2.1) $L(Q) \cap Q$ has two components $A_1 \subset (x_1 < 0)$ and $A_2 \subset (x_1 > 0)$.

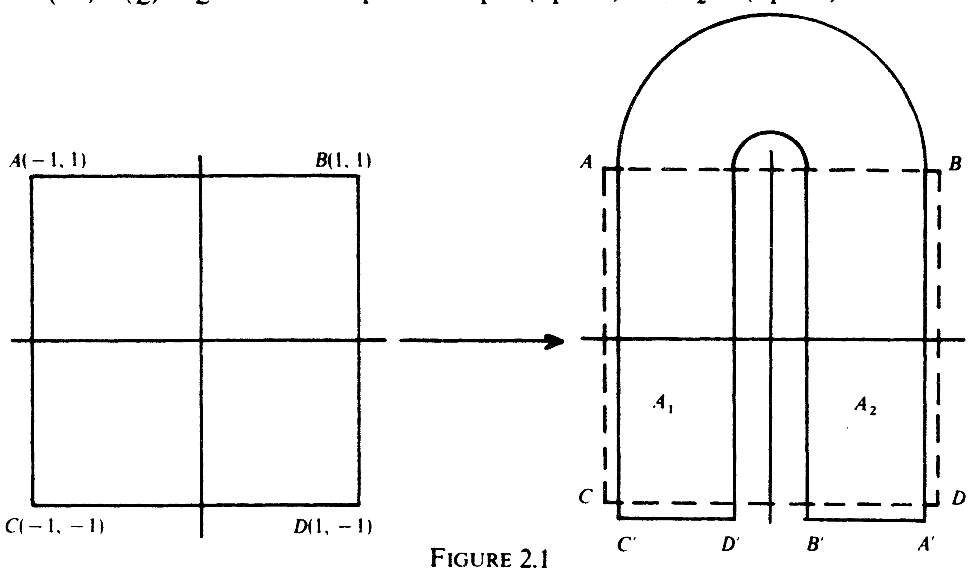

FIGURE 2.1

(2.2) $L(AB) \cup L(CD) \subset (x_2 < 0)$.

(2.3)
$$L_*|_{L^{-1}(A_1)} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \qquad \lambda < 1/2,$$
$$L_*|_{L^{-1}(A_2 \cup (EFA'B'))} = \begin{pmatrix} -\lambda & 0 \\ 0 & -1/\lambda \end{pmatrix}$$
where $HA' = L(A)$, $B' = L(B)$, etc., and $EF = A_2 \cap CD$.

(2.4) $l\pi_1(A_1) = l\pi_1(A_2) > \text{dist}(A_1, A_2) > 2l\pi_1(L(A_1) \cap A_1)$ where l means horizontal length and $\pi_1 : R^2 \to (x_2 = 0)$ is the vertical projection.

(2.5) $\text{dist}(A_2, BD) = \text{dist}(A_1, AC) = \beta$ where β is small.

For instance, suppose $4\alpha\beta + \text{dist}(A_1, A_2) < l\pi(A_1) - (\beta + 2\alpha\beta)$. Suppose also that estimates analogous to those in (2.4) and (2.5) are valid for $L^{-1}(A_1)$ and $L^{-1}(A_2)$.

Define L on $EFA'B'$ such that there is a subrectangle Q_1 with sides parallel to the coordinate axes such that

(2.6) $L(Q_1)$ is fibered by concentric semicircles which are the images of the vertical line segments in Q_1.

(2.7) The images of the top and bottom sides of Q_1 are on the same horizontal line.

(2.8) $(L|_{Q_1})_*(E_1)$ is perpendicular to $(L|_{Q_1})_*(E_2)$ where E_1 is the horizontal tangent space and E_2 is the vertical tangent space. We identify these with $(x_2 = 0)$ and $(x_1 = 0)$, respectively.

(2.9) The image under L of the horizontal line segment through the midpoint of Q_1 connecting both vertical sides of Q_1 is vertical. Call this image ρ_0.

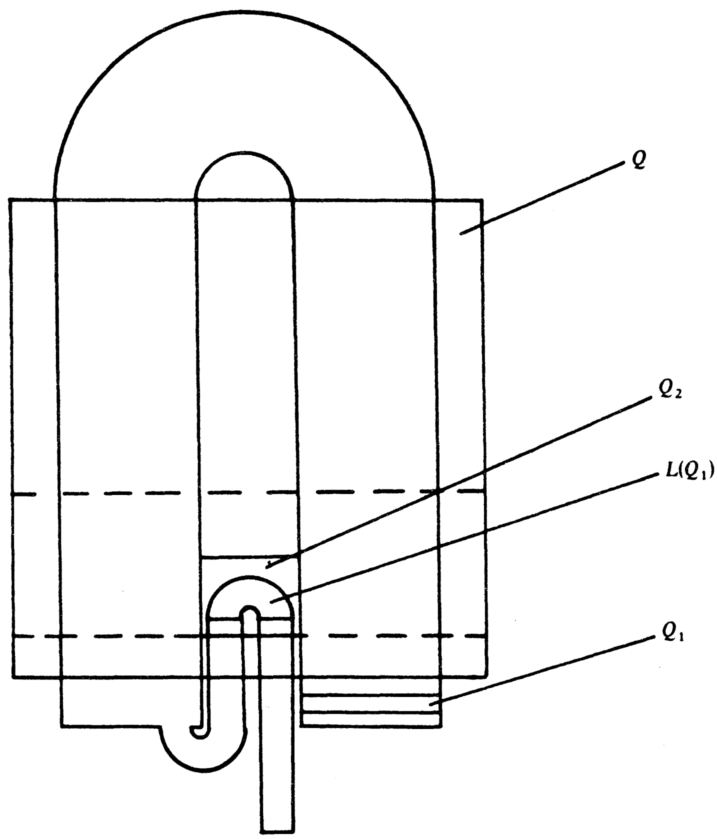

FIGURE 2.2

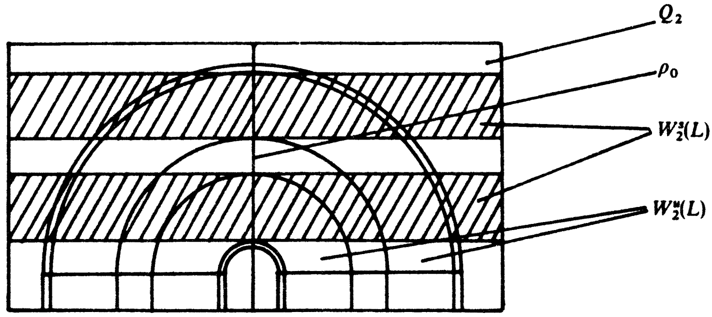

FIGURE 2.3

Let $Q = W_{-1}^s(L) = W_{-1}^u(L)$ and, for $i > -1$, let $W_i^s(L) = L^{-1}(W_{i-1}^s(L) \cap L(Q))$ and $W_i^u(L) = L(W_{i-1}^u(L)) \cap Q$.

Let Q_2 be a small rectangle bounded on the left by part of the boundary of A_1, bounded on the right by part of the boundary of A_2, and such that $L(Q_1) \subset$ interior Q_2.

Define $\bar{W}_i^u(L) = L^2(W_{i-2}^u(L)) \cap Q_2$, $i \geq 2$, and $\bar{W}_i^s(L) = W_i^s(L) \cap Q_2$, $i \geq 2$. *Let* $\bar{W}^s(L) = \bigcap_{i \geq 2} \bar{W}_i^s(L)$ *and* $\bar{W}^u(L) = \bigcap_{i \geq 2} W_i^u(L)$.

We also assume that

(2.10) $\bar{W}_2^s(L) \cap \rho_0 = \bar{W}_2^u(L) \cap \rho_0$.

The effect of conditions (2.4), (2.5), and (2.10) is to insure that the sets $\bar{W}^s(L) \cap \rho_0$ and $\bar{W}^u(L) \cap \rho_0$ are k-thick Cantor sets in the sense of definition (3.6), with $k > 1$. This is necessary for the preservation of tangencies between stable and unstable manifolds of small perturbations of L, as will become clear in the sequel.

Observe that $L|_Q$ is the same as in Smale's original horseshoe example, so any parts of the stable and unstable manifolds of L which depend only on $L|_Q$ are the same as in his example. This is true, in particular, for the set $\bar{W}^s(L)$ defined above.

Now extend L to be a diffeomorphism of the plane so that outside some large compact set H, L is the identity. Assume that all of the picture in Figure 2.2 is contained in the interior of H. Then, letting $\bar{L} \in \mathrm{Diff}^\infty(S^2)$ be as above, we proceed to show that for some C^2 neighborhood N of $\bar{L}$, any $f \in N$ has a nonwandering point x whose stable and unstable manifolds are tangent at x.

3. We need some results about Cantor sets.

(3.1) DEFINITION. By a *two component Cantor set* F in the interval $I \subset R^1$, we mean one obtained as the intersection of a decreasing sequence of closed sets $F_i \subset I$, $i \geq 2$, where (a) F_2 has two components, and (b) for $i \geq 2$, each component of F_i contains precisely two components of F_{i+1}. We call such a sequence a *defining sequence* for F. If $\{F_i\}_{i \geq 2}$ is a defining sequence for the two component Cantor set F, the *two-gap*, g_2, of F or F_i is the component of $I - F_2$ which is between the two components of F_2. An *i-gap*, g_i, is a component of $I - F_i$ between two components of F_i such that $g_i \subset F_{i-1}$. Let F be a two component Cantor set in I, g_i be an i-gap, and c_i be the smaller of the two components of F_i adjacent to

g_i. Let $lg_i = \sup g_i - \inf g_i$, and $lc_i = \sup c_i - \inf c_i$. Let $k > 0$. Say that a two component Cantor set $F \subset I$ is a k-Cantor set in I if there is a defining sequence $\{F_i\}_{i \geq 2}$ for F satisfying

(3.2) $\sup F_2 = \sup I$, $\inf F_2 = \inf I$.

(3.3) For any i-gap g_i of F_i, $lc_i/lg_i > k$, $i \geq 2$.

(3.4) If c is a component of F_i, $i \geq 2$, then $c - F_{i+1}$ is an open interval in c.

Thus a k-Cantor set is one which has a defining sequence $\{F_i\}_{i \geq 2}$ such that F_{i+1} is obtained by removing open intervals from the components of F_i in such a way that (3.3) is satisfied. For instance, the standard Cantor middle third set is a k-Cantor set in the unit interval for any $0 < k < 1$. When we speak of a defining sequence for a k-Cantor set, we mean one which satisfies conditions (3.2), (3.3), and (3.4).

(3.5) LEMMA. *If F is a k-Cantor set in I, G is a k-Cantor set in J, $k > 1$, $I \cap J \neq \varnothing$, and neither F nor G is completely contained in a gap of the other, then $F \cap G \neq \varnothing$.*

PROOF. Let $\{F_i\}_{i \geq 2}$ and $\{G_i\}_{i \geq 2}$ be defining sequences for F and G, respectively. If $F \cap G = \varnothing$, then there is an $m \geq 2$ such that $F_m \cap G_m = \varnothing$. Let g be the smallest gap of F_m or G_m which contains a component of the other. Suppose for definiteness g is a gap of F_m, and g contains the component c of G_m. In the interval $I \cap J$, either there is a gap g_r of G_m to the right of c such that $lg_r \geq lg$ or $\max G_m \in g$. If g_r exists, assume it to be the first such gap; if not, let $g_r = \max G_m$. Similarly, either there is a first gap g_l of G_m to the left of c such that $lg_l \geq lg$ or $\min G_m \in g$. Adjust the definition of g_l accordingly. It follows from the hypothesis of the lemma that at least one of g_l and g_r exists. If they both exist, let $\min\{lg_l, lg_r\}$ be the smaller of the two lengths; if one does not exist, let $\min\{lg_l, lg_r\}$ be the length of the existing gap. Let c_1 be the interval from the right endpoint of g_l to the left endpoint of g_r. Now it follows from the choices of g, g_l, and g_r that $c_1 \subset g$ so that $lc_1 < lg$. Further, if $i \leq m$ is the first integer such that both gaps g_l and g_r appear in G_i, then there are a c_i and g_i for G, as defined above, satisfying $lc_i \leq lc_1$ and $\min\{lg_l, lg_r\} \leq lg_i$. In fact, g_i can be chosen as either g_l or g_r, and c_i will then be its adjacent component which is contained in c_1. Thus

$$lc_i \leq lc_1 < lg \leq \min\{lg_l, lg_r\} \leq lg_i < lc_i/k$$

which is a contradiction.

We will need to generalize the notion of k-Cantor set to the situation in which the endpoints of the components of the defining sequence are not in the Cantor set.

(3.6) DEFINITION. Let $F \subset I$ be a two component Cantor set with defining sequence $\{F_i\}$. Let g_i, c_i be as above. For $m \geq i$, let g_{im} be the component of $I - F_m$ which contains g_i. Let c_{im} be the union of the components of F_m which are contained in c_i. Let $lc_{im} = \sup c_{im} - \inf c_{im}$ and $lg_{im} = \sup g_{im} - \inf g_{im}$. Let $k, k_1, k_2 > 0$. We say that F is (k_1, k_2)-thick if there is a defining sequence $\{F_i\}$ for F such that for all i-gaps, g_i, $i \geq 2$, and all $m \geq i$, $k_1 < lc_{im}/lg_{im} < k_2$. We say that F is k-thick if there is a defining sequence $\{F_i\}$ for F such that for all i-gaps, $i \geq 2$, and all $m \geq i$, $k < lc_{im}/lg_{im}$. Then we have the following lemma, the proof of which is similar to that of Lemma (3.5).

(3.7) LEMMA. *Let F and G be two k-thick Cantor sets in I. Suppose $k > 1$, neither F nor G is completely contained in a gap of the other, and* $\max(\min F, \min G) < \min(\max F, \max G)$. *Then $F \cap G \neq \varnothing$.*

4. To motivate the use of thick Cantor sets in establishing the stability of the tangency condition in §2, and to motivate the proof of Lemma (5.1), we consider a perturbation problem in a one-dimensional setting. The result of this section will not be used in the sequel, so the reader who wishes to skip the section may do so.

Let V be a closed bounded interval contained in the reals R^1. We assume that all functions which we consider in this section are defined on V and map V into itself. All closed intervals I, J, etc., which we consider are assumed to be contained in the interior of V.

Let $I \subset V$ be a closed interval, and let f_1 and f_2 be two contracting (derivative everywhere between 0 and 1) C^r diffeomorphisms which map I into itself such that $f_1(I) \cap f_2(I) = \varnothing$. Let $F_1(f_1,f_2) = I$ and, for $i \geq 2$, $F_i(f_1,f_2) = f_1(F_{i-1}(f_1,f_2)) \cup f_2(F_{i-1}(f_1,f_2))$. Then $F(f_1,f_2) \equiv \bigcap_{i \geq 2} F_i(f_1,f_2)$ is a two component Cantor set in I. We say that $F(f_1,f_2)$ is *defined* on I. By a C^r perturbation or approximation $F(h_1,h_2)$ of $F(f_1,f_2)$ we mean a Cantor set defined as above on an interval $J \subset V$ where

(4.1) the endpoints of J are close to those of I, and

(4.2) h_1 and h_2 are C^r close on V to f_1 and f_2, respectively.

Now we ask the following question.

(4.3) Given $F(f_1,f_2)$, can one find perturbations $F(h_1,h_2)$ and $F(\bar{h}_1,\bar{h}_2)$ arbitrarily close to $F(f_1,f_2)$ such that $F(h_1,h_2) \cap F(\bar{h}_1,\bar{h}_2) = \varnothing$?

We do not intend to discuss this question in detail, but rather to consider only those aspects of it which relate to the diffeomorphism L of §2. In this connection we have the following proposition.

(4.4) PROPOSITION. *Let $k > k' > 1$. Assume that $F(f_1,f_2)$ is defined as above, that $F(f_1,f_2)$ is a k-Cantor set in I, and that f_1 and f_2 are linear contracting diffeomorphisms of I into itself, i.e. the second derivatives $f_1''(x)$ and $f_2''(x)$ are identically zero on I. Then any $F(h_1,h_2)$ which is sufficiently C^2 close to $F(f_1,f_2)$ is a k'-Cantor set on its interval of definition.*

We observe that combining this proposition and Lemma (3.1), we obtain that, in general, the answer to question (4.3) is no.

PROOF. Let $c = \frac{1}{2}\inf\{\min(f_1'(x), f_2'(x)) : x \in I\}$. Let $g_i(h)$ be an i-gap of any approximation $F(h_1,h_2)$ of $F(f_1,f_2)$, and $c_i(h)$ be its adjacent component. Clearly, if h_1 and h_2 are close enough to f_1 and f_2, then $lg_2(h)/lc_2(h) < 1/k < 1/k'$.

Fix $i > 2$. Then there is a sequence $n_3, n_4, \ldots, n_i$ where each $n_j = 1$ or 2, $3 \leq j \leq i$, such that

$$h_{n_i} \circ h_{n_{i-1}} \circ \cdots \circ h_{n_3}(g_2(h)) = g_i(h)$$

and

$$h_{n_i} \circ h_{n_{i-1}} \circ \cdots \circ h_{n_3}(c_2(h)) = c_i(h).$$

For $3 \le j \le i$, let $g_j'(h) = h_{n_j} \circ \dots h_{n_3}(g_2(h))$ and $c_j'(h) = h_{n_j} \circ \dots \circ h_{n_3}(c_2(h))$.
Let $\alpha_j = lg_j'(h)/lg_{j-1}'(h)$ and $\beta_j = lc_j'(h)/lc_{j-1}'(h)$.
Then $lg_i(h) = \alpha_i \cdot \alpha_{i-1} \dots \alpha_3 lg_2(h)$ and $lc_i(h) = \beta_i \cdot \beta_{i-1} \dots \beta_3 lc_2(h)$.
Let $m_j = l(g_{j-1}(h) \cup c_{j-1}'(h))$.

Now, α_j is the derivative of h_{n_j} at some point of g_{j-1}', and β_j is the derivative of h_{n_j} at some point of c_{j-1}'. If $\varepsilon > 0$ is given, and h_1 and h_2 are C^2 close to f_1 and f_2, then the mean value theorem yields (since f_1 and f_2 are linear)

$$(\alpha_j - \beta_j)/m_j < \varepsilon.$$

Thus, for $3 \le j \le i$,

$$\alpha_j/\beta_j < (\beta_j + m_j\varepsilon)/\beta_j.$$

Thus,

$$\frac{lg_i(h)}{lc_i(h)} < \left(\prod_{j=3}^{\infty} 1 + \frac{m_j\varepsilon}{c}\right)\frac{lg_2(h)}{lc_2(h)}.$$

Now one can easily check that $\sum_{j=3}^{\infty} m_j < \infty$. Hence, choosing ε appropriately, the proposition follows.

5. In what follows, all of our approximations will be with respect to the C^2 metric d on $\mathrm{Diff}^2(H)$. We shall apply the results of §3 to obtain some results about diffeomorphisms C^2 near L.

Recall $\overline{W}^s(L) = \bigcap_{i\ge 2} \overline{W}_i^s(L)$ and $\overline{W}^u(L) = \bigcap_{i\ge 2} \overline{W}_i^u(L)$. Then, if γ is a C^1 compact arc in Q_2 which is C^1 near ρ_0, $\overline{W}^s(L) \cap \gamma$ and $\overline{W}^u(L) \cap \gamma$ are (k_1, k_2)-thick Cantor sets for some $1 < k_1 < k_2$. The next lemma asserts that this is true for a perturbation of L. Note that for f close to L we may define $\overline{W}_i^s(f)$, $\overline{W}_i^u(f)$, $\overline{W}^s(f)$ and $\overline{W}^u(f)$ as we did for L. We observe that if $x \in \overline{W}^s(f) \cap \overline{W}^u(f)$, $x \in \Omega(f)$, since it is an accumulation point of homoclinic points of a fixed point of f (see [6] for definitions). For f close to L, γ C^1 near ρ_0, let $\overline{W}_{i\gamma}^s(f) = \overline{W}_i^s(f) \cap \gamma$. Let $g_{i\gamma}^s(f)$ be an i-gap of $\overline{W}_{i\gamma}^s(f)$, $c_{i\gamma}^s(f)$ be an adjacent component to $g_{i\gamma}^s(f)$. Make similar definitions for $g_{im\gamma}^s(f)$, $c_{im\gamma}^s(f)$, $\overline{W}_{i\gamma}^u(f)$, $\overline{W}_\gamma^u(f)$, etc. For γ C^1 near ρ_0, let $|\gamma - \rho_0|$ denote the C^1 distance between γ and ρ_0.

(5.1) LEMMA. *Let $1 < k_1 < k_2$ be such that $\overline{W}_{\rho_0}^s(L)$ and $\overline{W}_{\rho_0}^u(L)$ are (k_1, k_2)-thick Cantor sets. Let $1 < k_1' < k_1 < k_2 < k_2'$. Then there is an $\alpha > 0$ and a C^2 neighborhood N of L such that for any compact arc γ in Q_2 with $|\gamma - \rho_0| < \alpha$, and any $f \in N$, $\overline{W}_\gamma^s(f)$ and $\overline{W}_\gamma^u(f)$ are (k_1', k_2')-thick Cantor sets.*

PROOF. We first observe that by C^1 dependence of the stable and unstable manifolds on f near L (Smale [5]), it is sufficient to prove that for f C^2 near L, $\overline{W}_{\rho_0}^s(f)$ and $\overline{W}_{\rho_0}^u(f)$ are (k_1', k_2')-thick Cantor sets. We prove this for $\overline{W}_{\rho_0}^s(f)$. The proof for $\overline{W}_{\rho_0}^u(f)$ is similar to that for $\overline{W}_{\rho_0}^s(f)$. One does the estimates on $f^{-2}(\overline{W}_{\rho_0}^u(f))$ and then carries them over to $\overline{W}_{\rho_0}^u(f)$.

Let $c = \frac{1}{2}\inf\{|D(L^{-1})_x(v)| : v \text{ is a unit vector in } E_1 \times E_2, x \in H\}$.

We first prove that for $i > 2$, f close to L, $lg_i^s(f)/lc_i^s(f) < 1/k_1'$ where $g_i^s(f)$ is an i-gap of $\overline{W}_{\rho_0}^s(f)$ and $c_i^s(f)$ is an adjacent component. We have that

$$lg_2^s(L)/lc_2^s(L) < 1/k_1 < 1/k_1'.$$

Fix $i > 2$. Define $g_i^{s,\prime}(f) = g_i^s(f)$, $c_i^{s,\prime}(f) = c_i^s(f)$, $g_{j-1}^{s,\prime}(f) = f(g_j^{s,\prime}(f))$, and $c_{j-1}^{s,\prime}(f) = f(g_j^{s,\prime}(f))$, for $3 \le j \le i$.

Let ε_1 be such that if $d(f, L) < \varepsilon_1$, then

$$lg_2^{s,\prime}(f)/lc_2^{s,\prime}(f) < 1/k_1 < 1/k_1'.$$

This can be done since, by taking ε_1 small, independent of i, if $d(f, L) < \varepsilon_1$, the arcs $g_j^{s,\prime}(f)$ and $c_j^{s,\prime}(f)$ are nearly vertical [5], and thus $lg_2^{s,\prime}(f)/lc_2^{s,\prime}(f)$ is close to $lg_2^s(f)/lc_2^s(f)$.

Let ε_2 be such that

$$\prod_{i=0}^{\infty}\left(1 + \frac{\varepsilon_2}{2^i c}\right)\frac{1}{k_1} < \frac{1}{k_1'}.$$

Define $\alpha_j = lg_j^{s,\prime}(f)/lg_{j-1}^{s,\prime}(f)$ and $\beta_j = lc_j^{s,\prime}(f)/lc_{j-1}^{s,\prime}(f)$ for $3 \le j \le i$. Let $m_j(f) = l(g_{j-1}^{s,\prime} \cup c_{j-1}^{s,\prime})$. Thus,

$$lg_i^{s,\prime}(f) = \alpha_i \cdot \alpha_{i-1} \dots \alpha_3 lg_2^{s,\prime}(f)$$

and

$$lc_i^{s,\prime}(f) = \beta_i \cdot \beta_{i-1} \dots \beta_3 lc_2^{s,\prime}(f).$$

Now α_j may be thought of as the derivative of $f^{-1}|_{c_{j-1}^{s,\prime}(f)}$ at some point of $c_{j-1}^{s,\prime}(f)$, and β_j may be thought of as the derivative of $f^{-1}|_{c_{j-1}^{s,\prime}(f)}$ at some point of $c_{j-1}^{s,\prime}(f)$. Since $D^2L^{-1} = 0$ on A_1, for ε_1 smaller, if necessary, $|D^2f^{-1}| < \varepsilon_2$ on A_1. Since the arcs are nearly vertical, we have, with the proper orientations, if $\alpha_j > \beta_j$, then $(\alpha_j - \beta_j)/m_j(f) < \varepsilon_2$ by the one-dimensional mean value theorem.

Thus for $\alpha_j > \beta_j$,

$$\frac{\alpha_j}{\beta_j} < \frac{\beta_j + m_j\varepsilon_2}{\beta_j} = 1 + \frac{m_j\varepsilon_2}{\beta_j}.$$

But again for ε_1 small, $\beta_j > c$ and $m_j < \frac{1}{2}m_{j-1}$ for $3 \le j \le i$.

So $\alpha_j/\beta_j < 1 + m_j\varepsilon_2/c$, for $3 \le j \le i$ and $m_j < (1/2^{j-2})m_2 < 1/2^{j-3}$. Thus,

$$\frac{lg_i^s(f)}{lc_i^s(f)} < \prod_{j=3}^{i}\left(1 + \frac{m_j\varepsilon_2}{c}\right)\frac{lg_2^{s,\prime}(f)}{lc_2^{s,\prime}(f)}$$

$$< \prod_{j=3}^{\infty}\left(1 + \frac{\varepsilon_2}{2^{j-3}c}\right)\cdot\frac{1}{k_1} < \frac{1}{k_1'}.$$

To prove that for $m \ge i$, $lg_{im}^s(f)/lc_{im}^s(f) < 1/k_1'$ for f close to L, we need only to make estimates of the first derivative of f at certain points. Note that by the above argument, we may prove that $lg_i^s(f)/lc_i^s(f) < \lambda$ where $1/k_1 < \lambda < 1/k_1'$ for f C^2 close to L. We observe that for f C^1 close to L, $\sup g_{im}^s(f) - \sup g_i^s(f)$ and $\inf g_i^s(f) - \inf g_{im}^s(f)$ are small enough so that $lg_{im}^s(f)$ is close to $lg_i^s(f)$. Similarly, $lc_{im}^s(f)$ is close to $lc_i^s(f)$. Making these estimates refined enough we may prove $lg_{im}^s(f)/lc_{im}^s(f) < 1/k_1'$.

Now notice that we may similarly prove that for f C^2 close to L, $lc_{im}^s(f)/lg_{im}^s(f) < k_2'$ for all i and all $m \ge i$. This proves Lemma (5.1).

Our goal is to prove that for f C^2 near L, $\overline{W}^u(f)$ and $\overline{W}^s(f)$ have a point of tangency. Using Lemmas (5.1) and (3.2), we can show that for γ C^1 near ρ_0 and f C^2 near L, $\overline{W}^u_\gamma(f) \cap W^s_\gamma(f) \neq \varnothing$. Let us orient the arcs of $\overline{W}^u(L)$ and $\overline{W}^s(L)$ so that for γ on the left of ρ_0, the angles between intersecting arcs of $\overline{W}^u(L)$ and $W^s(L)$ are all less than zero. Then for γ on the right of ρ_0, the corresponding angles of intersection will all be greater than zero. It is clear that for f C^1 close to L, an analogous result is true about angles of intersection of $\overline{W}^u_\gamma(f)$ and $\overline{W}^s_\gamma(f)$ for appropriate γ.

This makes it plausible that the desired points of tangency should exist. However, to prove they actually do exist, we need to know that, for any $i \geq 2$, each component of $f^{-2}(\overline{W}^u_i(f))$ has a nearly constant horizontal width. This is essentially the content of Lemma (5.3). To prove this lemma we will need the following theorem.

(5.2) THEOREM (M. HIRSCH AND C. PUGH [2]). *Let f be a C^2 Anosov diffeomorphism of the two torus T^2. Let the tangent bundle have the continuous hyperbolic splitting $T(T^2) = E^s \oplus E^u$ where $\|(Df|E^s)\| < 1$ and $\|(Df|E^u)^{-1}\| < 1$. Then E^s and E^u are C^1 subbundles of $T(T^2)$, and if f_1 is C^2 close to f, the unit ball bundles of the invariant subbundles E^s_1 and E^u_1 are C^1 close to those of E^s and E^u, respectively.*

I should remark that the statement about C^1 dependence in the above theorem is not actually written down in [2]. However, both M. Hirsch and C. Pugh informed me that it follows from their methods.

I am indebted to C. Pugh for telling me about the above theorem and for a conversation which was very helpful for the proof of the following lemma.

Notice that the definitions of $W^u_i(f)$, $W^s_i(f)$, etc. make sense for any diffeomorph $\tilde{Q}$ of Q which is C^1 close enough to Q and any f C^1 close enough to L. That is, let $\tilde{W}^s_i(f) = f^{-1}(\tilde{W}^s_{i-1}(f) \cap f(\tilde{Q}))$, etc. We shall call any such $\tilde{Q}$ near Q a "square" near Q.

(5.3) LEMMA. *Let $0 < \delta_1 < 1 < \delta_2$. Then there exists a neighborhood N of L and a C^2 "square" $\tilde{Q}$ near Q such that the following is true. Let $f \in N$, $i \geq 2$, γ_1 and γ_2 be C^1 curves nearly horizontal in Q, and define $c^u_{i\gamma_1}(f)$, $c^u_{i\gamma_2}(f)$ to be components of $\tilde{W}^u_i(f) \cap \gamma_1$ and $\tilde{W}^u_i(f) \cap \gamma_2$ contained in the same strip of $\tilde{W}^u_i(f)$. Then*

$$\delta_1 < lc^u_{i\gamma_1}(f)/lc^u_{i\gamma_2}(f) < \delta_2.$$

PROOF. We may assume γ_1 and γ_2 are horizontal in Q. Let f_0 be an Anosov diffeomorphism of the two torus T^2 (see Smale [6]) such that

(5.4) There is a subset Q' of T^2 such that $f_0(Q') \cap Q'$ has two components A'_1, and A'_2.

(5.5) There is a diffeomorphism $d_1 : Q \to Q'$ such that $d_1(A_1) = A'_1$, $d_1(A_2) = A'_2$, $d_1(L^{-1}(A_1)) = f_0^{-1}(A'_1)$, and $d_1(L^{-1}(A_2)) = f_0^{-1}(A'_2)$.

Let e be the isometry $A_2 \to A_2$ which is rotation by π about the midpoint of A_2.

(5.6) There are disjoint neighborhoods U_1 of A_1, U_2 of A_2 and a diffeomorphism $d_2 : U_1 \cup U_2 \to T^2$ such that $d_2|_{U_1} = d_1$, and $d_2|_{U_2} = d_1 \circ e$.

(5.7) $d_1(AC)$ and $d_1(BD)$ are segments of unstable manifolds of f_0, and $d_1(AB)$ and $d_1(CD)$ are segments of stable manifolds of f_0.

(5.8) $f_0 = d_2 \circ L \circ d_1^{-1}$ on a neighborhood of $f_0^{-1}(A_1' \cup A_2')$.

It is easy to see that such an Anosov diffeomorphism exists. Now let f be a slight C^2 perturbation of L. Then for some neighborhood U_3 of $f_0^{-1}(A_1' \cup A_2')$, $d_2 \circ f \circ d_1^{-1}|_{U_3}$ is a slight C^2 perturbation of $f_0|_{U_3}$. Let $\tilde{f}_0$ be an Anosov diffeomorphism of T^2 which is close to f_0 such that $\tilde{f}_0 = d_2 \circ f \circ d_1^{-1}$ on a neighborhood of $f_0^{-1}(A_1' \cup A_2')$ and $\tilde{f}_0 = f_0$ on $T^2 - U_3$. Then let $\tilde{Q}'$ be a new "square" in T^2, near Q', such that $\tilde{f}_0|_{\tilde{Q}'}$ is close to $\tilde{f}_0|_{Q'}$ and $\tilde{Q}'$ is bounded by segments of the stable and unstable manifolds of $\tilde{f}_0$. Now if f is C^2 close to L, $\tilde{f}_0$ will be C^2 close to f_0. Hence Theorem 3.2 gives that, for $i \geq 2$,

$$\frac{lc^u_{id_1(\gamma_1)}(\tilde{f}_0)}{lc^u_{id_1(\gamma_2)}(\tilde{f}_0)} \quad \text{is close to} \quad \frac{lc^u_{id_1(\gamma_1)}(f_0)}{lc^u_{id_1(\gamma_2)}(f_0)}.$$

Here we assume $c^u_{id_1(\gamma_1)}(f_0)$ and $c^u_{id_1(\gamma_2)}(f_0)$ are defined with respect to Q', and $c^u_{id_1(\gamma_1)}(\tilde{f}_0)$ and $c^u_{id_1(\gamma_2)}(\tilde{f}_0)$ are defined with respect to $\tilde{Q}'$. Now let $\tilde{Q} = d_1^{-1}(\tilde{Q}')$ and the lemma follows.

6. Now we conclude the proof of the nondensity of Axiom A(a) by showing that for f C^2 close enough to L, $\overline{W}^s(f)$ and $\overline{W}^u(f)$ have a point of tangency. Recall that such a point will belong to $\Omega(f)$ since it will be an accumulation point of homoclinic points of a fixed point of f.

In this section we will make the simplifying assumption that for all perturbations f of L which we consider, $\overline{W}_n^s(f) = \overline{W}_n^s(L)$ for $n \geq 2$. This avoids technical difficulties and indicates the main ideas needed in the proof. To make the proof rigorous we would have to enlarge each $\overline{W}_n^s(f)$ to obtain a C^2 foliation of Q_2 and proceed as below with respect to these foliations. Since, for f C^1 close to L, all of these foliations can be made uniformly C^1 close to the natural horizontal foliation of Q_2, we are justified in making the estimates with respect to this foliation.

Let f be close to L and let $n \geq 2$. Note that $\overline{W}_n^u(f)$ has 2^{n-1} components and thus the boundary of $\overline{W}_n^u(f)$ consists of 2^n curves. Label these curves $\xi_n^1(f), \xi_n^2(f), \ldots, \xi_n^{2^n}(f)$ so that $\xi_n^1(f)$ is below $\xi_n^2(f)$, $\xi_n^2(f)$ is below $\xi_n^3(f)$, etc. For each $i = 1, 2, \ldots, 2^n$, let $\psi_n^i(f)$ be a point of $\xi_n^i(f)$ at which $\xi_n^i(f)$ assumes its maximum with respect to the horizontal foliation of Q_2. Assume that f is C^1 close enough to L so that for each i, $\psi_n^i(f)$ is in the interior of the curve $\xi_n^i(f)$, i.e. $\psi_n^i(f)$ is not an endpoint of $\xi_n^i(f)$. Note that $\xi_n^i(f)$ is tangent to the horizontal foliation of Q_2 at $\psi_n^i(f)$. For $i = 1, 3, 5, \ldots, 2^n - 1$, let $c_n^i(f)$ be the closed rectangular strip in Q_2

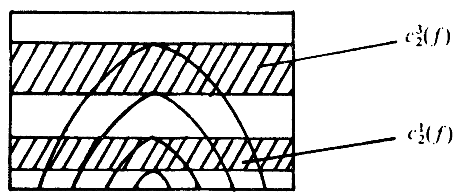

FIGURE 6.1

whose boundary consists of the horizontal line segment in Q_2 through $\psi_n^i(f)$, the horizontal line segment in Q_2 through $\psi_n^{i+1}(f)$ and parts of the vertical edges of Q_2. Then let $F_n(f) = \bigcup_{i=1,3,\ldots,2^n-1} c_n^i(f)$. Note that if f is close to L, $\bigcap_{n\geq 2} F_n(f) \equiv F(f)$ is the product of a horizontal line segment in Q_2 and a Cantor set. Let $F_{\rho_0}(f) = F(f) \cap \rho_0$.

(6.1) LEMMA. *There is a C^2 neighborhood N of L such that if $f \in N$, then $F_{\rho_0}(f)$ is a k-thick Cantor set where $k > 1$.*

PROOF. This follows from Lemma (5.1), Lemma (5.3), and the construction of $F_{\rho_0}(f)$.

To conclude the proof of the first part of Theorem (1.1) we see that for some small C^2 neighborhood N of L, if $f \in N$, then $F_{\rho_0}(f)$ and $\overline{W}^s_{\rho_0}(f)$ are k-thick Cantor sets for some $k > 1$. By restricting N further if necessary, we may assume the hypotheses of Lemma (3.7) are satisfied by $F_{\rho_0}(f)$ and $\overline{W}^s_{\rho_0}(f)$. Thus $F_{\rho_0}(f) \cap \overline{W}^s_{\rho_0}(f) \neq \emptyset$. If $x \in F_{\rho_0}(f) \cap \overline{W}^s_{\rho_0}(f)$, then the stable manifold of f through x is a horizontal line segment which has a point of tangency with some unstable manifold of f.

7. Here we sketch a proof that for f C^2 close enough to L, f is not structurally stable. We also make some remarks about the Ω-instability of f.

Let γ_1 and γ_2 be continuous arcs in the plane which have a single point x of intersection. Suppose there is a disk D about x such that $D - \gamma_1$ has two components and $D - \gamma_2$ has two components. Say that the intersection is *one-sided* if the following is true. If γ_2 meets the component V of $D - \gamma_1$, then $(\gamma_2 - \{x\}) \cap D \subset V$. Thus we have a picture as in Figure 7.1.

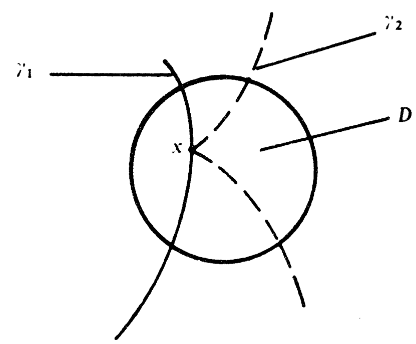

FIGURE 7.1

It is clear that if γ_1 and γ_2 are smooth arcs with nonvanishing tangent vectors, then a point of one-sided intersection is a point of tangency. In this case, we say that γ_1 and γ_2 have a point of *one-sided tangency*. Notice that if h is a homeomorphism defined in a neighborhood of x, then $h(\gamma_1)$ and $h(\gamma_2)$ have a one-sided intersection at $h(x)$. Further, if γ_1 and γ_2 have a transversal intersection at x, this intersection is *not* one-sided.

These remarks together with the Kupka-Smale theorem [**6**, Theorem 6.7] imply that if the stable and unstable manifolds of a hyperbolic fixed point of a diffeomorphism g have a point of one-sided tangency, then g cannot be structurally stable.

Now, by the results of §6, if f is C^2 close enough to L, $\overline{W}^s(f)$ has a point of one-sided tangency with $\overline{W}^u(f)$. Since $\overline{W}^s(f)$ is completely determined by the action of f on Q, a perturbation g of f which agrees with f except in a small neighborhood of Q_1 will have $\overline{W}^s(g) = \overline{W}^s(f)$. Applying techniques of Smale (I.7 of [**6**]) to $f|_Q$, we have that f has a fixed point in Q whose stable manifold contains a dense subset of $\overline{W}^s(f)$ and whose unstable manifold contains a dense subset of $\overline{W}^u(f)$.

Therefore, arbitrarily C^2 close to f, we can find a diffeomorphism g which agrees with f outside a small neighborhood of Q_1 and has a fixed point whose stable and unstable manifolds have a one-sided tangency. Since the set of all structurally stable diffeomorphisms is open in $\mathrm{Diff}(S^2)$, we have that f is not structurally stable.

It does not follow immediately from the one-sided tangency of the stable and unstable manifolds of a fixed point of a diffeomorphism g that g is not C^2 Ω-stable. However, it is not hard to see that such a g is not C^1 Ω-stable, and that if g_1 is Kupka-Smale, C^2 close to g, and Ω-conjugate to g, then the conjugating homeomorphism cannot be close to the identity. Furthermore, with a bit more work, it can be shown that, for $r \geq 2$, any C^r diffeomorphism g which is sufficiently C^2 close to L is not $C^r\Omega$-stable.

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