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ON HOMOCLINIC POINTS

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ABSTRACT. Results of R. C. Robinson and D. Pixton on the existence of homoclinic points for diffeomorphisms on the two-sphere are extended. An application to area preserving diffeomorphisms on surfaces is given.

The purpose of this note is to extend results of Robinson [7] and Pixton [5] concerning the existence of homoclinic points for diffeomorphisms on two-dimensional manifolds.

The basic problem is this. Suppose $y \in \mathrm{Cl}\ W^u(p, f) \cap (W^s(p, f) - \{p\})$ where p is a hyperbolic periodic point of a C^r diffeomorphism f of a manifold $r \geq 1$, $W^u(p, f)$ is the unstable manifold of p while $W^s(p, f)$ is the stable manifold of p. Is there a small C^r perturbation g of f such that p is a hyperbolic periodic point of g and $y \in W^u(p, g) \cap W^s(p, g)$? Following Poincaré, such a point y in $W^u(p, g) \cap W^s(p, g)$ is called a homoclinic point for g. We will also say that y is (p, g)-homoclinic. Homoclinic points generally yield interesting phenomena. In particular, as Smale realized [8], [2, Appendix], they usually give the existence of infinitely many periodic points.

From [5] and [7], the above question has a positive answer on the two-sphere if $W^u(p, f) \cap W^s(p, f) = \varnothing$. Here we shall consider any two-dimensional manifold M and a C^r diffeomorphism $f\colon M \to M$ having a hyperbolic periodic saddle point p. We use the Whitney C^r topology for perturbations of f. Assume that $W^u(p, f)$ and $W^s(p, f)$ already have a nonempty transversal intersection, say y_1. Let $W_1^u(p, f)$ be the component of $W^u(p, f) - \{p\}$ containing y_1, and let $W_1^s(p, f)$ be the component of $W^s(p, f) - \{p\}$ contining y_1. We wish to take another point y in $W_1^s(p, f)$ and give a sufficient condition for y to become (p, g)-homoclinic for a small C^r perturbation g of f. Let $W_0^u(p, f)$ be the component of $W^u(p, f) - \{p\}$ not meeting $W_1^u(p, f)$.

Let $\Omega(f)$ denote the nonwandering set of f and let $\alpha(y, f)$ denote the α-limit set of y. We recall that $x \in \Omega(f)$ if and only if there are sequences $x_i \to x$ and $n_i \to \infty$ with $f^{n_i}(x_i) \to x$ as $i \to \infty$ while $x \in \alpha(y, f)$ if and only if there is a sequence $n_i \to -\infty$ with $f^{n_i}(y) \to x$ as $i \to \infty$.

THEOREM 1. *With the above notations, assume y is in $\Omega(f)$, p is not in $\alpha(y, f)$, and $W_0^u(p, f)$ has some nonempty transversal intersection with $W^s(q, f)$ for*

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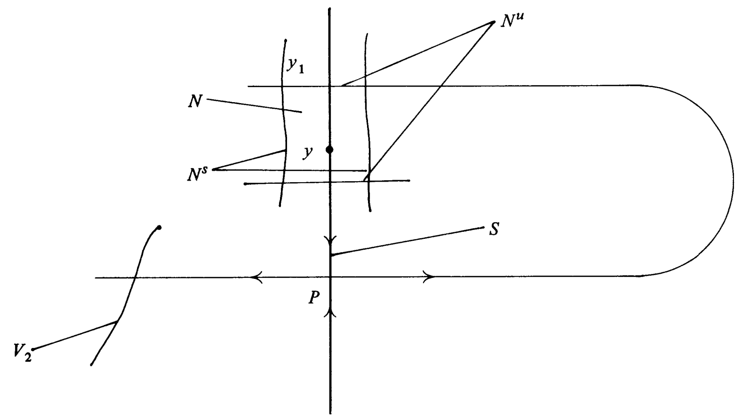
some hyperbolic periodic point q of f. Then f may be C^r perturbed to g so that y is (p, g)-homoclinic.

PROOF. Let S be a compact arc in $W^s(p, f)$ containing p and y_1 in its interior. Choose a small arc V_1 in $W_1^u(p, f)$ such that y_1 is in its interior and $V_1 \cap S = \{y_1\}$. Let n_0 be the least integer so that $f^{n_0}(p) = p$. Let V_2 be a small arc in $W^s(q, f)$ having a nonempty transversal intersection with $W_0^u(p, f)$ in its interior. Observe that q must be a saddle point or a sink. From the λ-lemma [4], there are arcs in

$$S' = \bigcup_{k \leq 0} f^{kn_0}(S) \quad \text{and} \quad V_2' = \bigcup_{k \leq 0} f^{kn_0}(V_2)$$

which are uniformly C^1 close to S. Hence, replacing y_1 by another element of its orbit if necessary, we may find a box-like closed neighborhood N of y whose boundary $\partial N = N^u \cup N^s$ where N^u consists of an arc in V_1 and an arc in $f^{n_0}V_1$ while N^s consists of an arc in S' and an arc in V_2'.

The following figure describes N and ∂N.



Since $p \notin \alpha(y, f)$, we have that $S \cap \alpha(y, f) = \emptyset$ by the invariance of $\alpha(y, f)$. Since $\alpha(y, f)$ is closed we have that $\alpha(y, f) \cap N = \emptyset$ provided the arcs in N^s are close enough to S. Choosing these arcs even closer we may arrange that $f^{-n}(y) \notin N$ for all $n > 0$ and $f^{-n}(N^u) \cap \operatorname{int} N = \emptyset$ for all $n \geq 0$.

For $z \in M$, let $o_-(z, f) = \{f^{-n}(z): n > 0\}$, and let $o(z) = \{f^n(z): n = 0, \pm 1, \pm 2, \dots\}$. We claim

(1) there is a sequence $x_i \in W_1^u(o(p), f)$, $i = 1, 2, \dots$, converging to y so that $o_-(x_i, f) \cap \operatorname{int} N = \emptyset$ for all large i.

Here $W_1^u(o(p), f) = \bigcup_{z \in o(p)} W_1^u(p, f)$.

Once (1) is established, standard methods, as in Robinson [7], enable one to perturb f to g so that p is a hyperbolic saddle periodic point for g and $W^u(o(p), g)$ has a transversal intersection with $W^s(p, g)$ at y. Then it follows from Corollary (1.3) in [4] that $W^u(p, g)$ has transversal intersections with $W^s(p, g)$ arbitrarily near y. In fact, it is known that such points y_i may be found whose orbits $o(y_i)$ are near y only at y_i. From this, g may be further perturbed to g_1 so that y becomes (p, g_1) homoclinic.

We now prove (1). The method is a variant of the one introduced in [7].

For each integer $n \geqslant 0$, let $D_n = \bigcup_{1 \leqslant j \leqslant n} f^j(N)$. Since $o_-(y, f) \cap N = \varnothing$, we have that $y \notin D_n$ for each n. Let x_n be the point of D_n closest to y. Clearly,

$$x_n \in \partial D_n \subset \bigcup_{1 \leqslant j \leqslant n} \partial(f^j N) = \bigcup_{1 \leqslant j \leqslant n} \left[f^j(N^u) \cup f^j(N^s) \right].$$

We may choose a neighborhood U of y so that $f^n(N^s) \cap U = \varnothing$ for $n \geqslant 0$, since for $n > 0$, $f^{-n}(y) \notin N$, and $f^n(N^s) \cap N = \varnothing$ for n large. Since y is nonwandering for f, there are sequences $y_i \to y$ and $n_i \to \infty$ so that $f^{n_i}(y_i) \to y$ as $i \to \infty$. Thus, for i large, $\{y_i, f^{n_i}(y_i)\} \subset N$. Hence $f^{n_i}(N)$ accumulates on y, so $x_{n_i} \to y$ as $i \to \infty$. Let $n_1 > 0$ be such that for $i \geqslant n_1$, $x_{n_i} \in U$. Then $x_{n_i} \in \bigcup_{1 \leqslant j \leqslant n_i} f^j(N^u) \subset W_1^u(o(p), f)$.

Suppose $o_-(x_{n_i}, f) \cap \operatorname{int} N \neq \varnothing$ for some $i \geqslant n_1$. Then there is an integer $k_i > 0$ so that $f^{-k_i}(x_{n_i}) \in \operatorname{int} N$ or $x_{n_i} \in \operatorname{int} f^{k_i}(N)$. Since $\bigcup_{n \geqslant 0} f^{-n}(N^u) \cap \operatorname{int} N = \varnothing$, we see that $0 < k_i < n_i$. But then $x_{n_i} \in f^{k_i}(\operatorname{int} N) \subset \operatorname{int} D_{n_i}$, which is impossible since $x_{n_i} \in \partial D_{n_i}$. Thus, for $i \geqslant n_1$, $o_-(x_{n_i}, f) \cap \operatorname{int} N = \varnothing$, and the proof is completed.

REMARKS 1. Notice that the α-limit set condition on y will be fulfilled if $y \in W^u(q_1, f)$ for some hyperbolic periodic point q_1 not in the orbit of p.

2. If y actually is a transversal homoclinic point for (p, f) then $W^u(y, f)$ is a limit of infinitely many unstable manifolds of different hyperbolic periodic orbits. Thus, y is a limit of points y_i in $W^s(y, f)$ so that $p \notin \alpha(y_i, f)$. Theorem 1 should be thought of as a sort of converse to this.

There are analogous results when f is area preserving. Indeed, if M has a smooth 2-form ω with $\omega(p) \neq 0$, $f^*\omega = \omega$, and $\int_M \omega < \infty$, then the perturbation g of f in Theorem 1 may be chosen so that $g^*\omega = \omega$ as well. For this one uses generating functions as in [1], [3, §2]. Also, in this case, the point y (and all points in $W^u(p, f) \cup W^s(p, f)$) will automatically be nonwandering, so that hypothesis may be dropped. Moreover, one has the following result.

THEOREM 2. *Let p be a hyperbolic periodic point of a diffeomorphism f on an orientable two-dimensional manifold M having a transversal homoclinic point. Suppose there is a smooth 2-form ω on M with $\omega(p) \neq 0$, $f^*\omega = \omega$, and $\int_M \omega < \infty$. Let q be another hyperbolic periodic point of f, and let $y \in W^u(q) \cap W^s(p)$. Then f may be C^r perturbed to g so that $g^*\omega = \omega$ and y is a limit of (p, g) homoclinic points.*

PROOF. By [6], we first perturb f to f_1 so that y is a transversal intersection of $W^u(q, f_1)$ and $W^s(p, f_1)$. For f_1 close enough to f, (p, f_1) still has a transversal homoclinic point.

By Smale's homoclinic point theorem [8], [2, Appendix], p is a limit of a sequence of hyperbolic saddle periodic points p_i of f_1 such that $W^s(p_i, f_1)$ has nonempty transversal intersections with $W^u(q, f_1)$, say y_i, near y. Further, it is easily seen that the p_i's may be chosen so that both components of $W^u(p_i, f_1) - \{p_i\}$ meet $W^s(p_i, f_1)$. Observe that the y_i's are nonwandering points for i large. Indeed, since $\omega(p) \neq 0$, we have $\omega(f_1^n(y)) \neq 0$ for $n > 0$ large, so $\omega(y) \neq 0$ as $f_1^*\omega = \omega$. Hence for i large, $\omega(y_i) \neq 0$. For any such y_i, if U is a small neighborhood of y_i, we have $\int_U \omega > 0$. Since $\int_{\cup_{n>0} f_1^n U} \omega \leqslant \int_M \omega < \infty$, there are integers $0 \leqslant n_1 < n_2$ so that $f_1^{n_1}(U) \cap f_1^{n_2}(U) \neq \varnothing$, whence $f_1^{n_2 - n_1}(U) \cap U \neq \varnothing$. Thus, y_i is nonwandering. By Theorem 1 and the remarks about generating functions preceding the statement of Theorem 2, f_1 may be perturbed to make y_i homoclinic, and Theorem 2 is proved.

REMARK. If M is compact, and f is area preserving, then Poincaré expected that generically $W^u(p) \cap W^s(p)$ would be dense in $W^u(p)$ for any hyperbolic periodic point p. Takens has proved this in the C^1 topology [9]. However, the problem remains unsolved in the C^r topology, $r \geqslant 2$.

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