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PREFACE

On June 19 - 27, a summer school on dynamical systems was held at Bressanone, Italy under the auspices of C.I.M.E. It was organized by A. Moro and C. Marchioro. There were three series of lectures given which appear in this volume, more or less in the original form.*

The purpose of these lectures was to present some recent and active lines of research work in dynamical systems in a form accessible to interested students and researchers; i.e. they are not directed to the expert. In the limited space of eight lectures it is clearly impossible to develop a theory with complete proofs. Rather, it was the goal to give in an informal style a coherent picture of three areas in dynamical systems leading up to current research work and to open problems. Here are some comments on the three sets of lectures:

The lectures by S. Newhouse describe the qualitative theory of dynamical systems as it was initiated by S. Smale in the sixties. It deals with the topological description of the flow for typical or generic systems. Structural stability is a central theme in these lectures. Emphasis is placed both upon the characterization of structurally stable systems and reasons why some systems fail to be structurally stable. We mention, in particular, Newhouse's

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* The original plan, to have the lectures published in 1978 was not realized. We thank Roberto Conti for his permission to have these manuscripts published with Birkhäuser, Boston.

description of the "shadowing lemma" which for the case of hyperbolic systems was derived by Anosov in order to establish the structural stability of those systems. In the last section, Newhouse describes systems having infinitely many attracting periodic orbits on a compact manifold, a property which is inconsistent with structural stability. The discovery of this surprising and unintuitive phenomenon, namely that in a generic one-parameter family a dynamical system may have infinitely many attracting periodic orbits is due to S. Newhouse. In the meantime it has been realized that this situation occurs in many examples such as the forced van der Pol equation of nonlinear circuit theory. In addition to the topological theory, Newhouse also gives a glimpse into ergodic theory and describes some recent results of Ruelle.

Guckenheimer's lectures deal with bifurcation theory. Historically the term bifurcation theory refers to systems depending on a parameter for which a stationary or a periodic orbit bifurcates into two or several such orbits as the parameter passes a critical value. In fluid dynamics the parameter is usually the Reynold's number and the problem is the bifurcation of a nonstationary solution from the "laminar" stationary flow. In Guckenheimer's lecture the concept is taken in a much more general sense, namely bifurcation theory refers to the study of systems depending on one or more parameters and the changes of topological character of the flow which occur as the parameter changes. This may involve more complicated phenomena than bifurcation of periodic orbits and Guckenheimer describes, for instance, such fascinating phenomena as they occur for the Lorenz attractor and bifurcations for the van der Pol equation. Another major topic of

Guckenheimer's lecture is the topological study of the iterates of a mapping of an interval into itself. This seemingly simple question hides a number of intriguing and complicated phenomena which in recent years have been studied by many mathematicians and physicists, and Guckenheimer himself has made several contributions to this subject. Section 6 of his lectures contains a description of "kneading sequences" which were introduced by Milnor and Thurston into the subject. In Section 7, Guckenheimer describes some applications to various population models. He illustrates how rather simple models can lead to complicated flows.

While the lectures of Guckenheimer and Newhouse relate to each other in spirit, those of the undersigned deal with a very different topic, namely with integrable Hamiltonian systems. This is a very special class of nonlinear systems possessing sufficiently many integrals so that they can be solved more or less explicitly. For this reason, integrable systems played a role in the last century before existence theorems were available and before qualitative methods had been developed. In the last ten years there has been renewed interest in integrable systems since a number of partial differential equations have been discovered which play the role of integrable systems of infinitely many degrees of freedom. An example of this type is the Korteweg-de Vries equation occurring in fluid mechanics. Moreover, a number of new finite dimensional systems have been recognized as integrable ones. As an example we mention the n-particle system of Calogero: it describes the motion of n particles of equal mass moving on the line under the influence of an inverse square potential. This example is treated in detail in these lectures and its relation to Lie algebra is revealed.

In the other lectures Jacobi's geodesic flow on an ellipsoid and another mechanical problem, going back to C. Neumann, are discussed and a surprising relation of these mechanical systems and the Korteweg-de Vries equation is shown. Although this subject has a long history there is no systematic theory; it consists of a number of examples, coming from physics, geometry or other fields and special techniques. Recent work has shown close connection to algebraic geometry. Much remains to be done in this field.

It is our hope that through this publication these ideas are made available to a larger circle than the audience at the summer Institute at Bressanone, and possibly find their way to some readers who take an interest in this subject and contribute to it.

Jürgen K. Moser

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Lectures on Dynamical Systems *

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Contents

* Partially supported by N.S.F. Grant MCS 76-05854.

Introduction.

A basic question in the theory of dynamical systems is to study the asymptotic behaviour of orbits. This has led to the development of many different subjects in mathematics. To name a few, we have ergodic theory, hamiltonian mechanics, and the qualitative theory of differential equations.

A particularly baffling and interesting problem is to describe systems with non-trivial recurrence. For example, consider a smooth area preserving diffeomorphism f of the two dimensional disk D^2. According to the Poincaré recurrence theorem, almost all points in D^2 are recurrent. That is, for x off a set of Lebesgue measure zero, the orbit of x accumulates on x infinitely often. However, except for simple cases, we have no global model that describes all the motion. On the other hand, if we consider the mapping g on the two-dimensional torus T^2 induced by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then again almost all points are recurrent. However, in this case not only do we have a fairly good picture of the total motion, but this picture persists for any g' which is C^1 close to g. This example gives an indication of some remarkable progress which has been made in describing non-trivial recurrence during the last twenty years.

The main feature possessed by the toral mapping g above which is not shared by f is what is called hyperbolicity. In its present form, this concept arose in the work of Anosov on geodesic flows on negatively curved Riemannian manifolds [2]. It was subsequently realized by Smale that hyperbolicity could be used to describe other systems with non-trivial recurrence, and this led him to define Axiom A diffeomorphisms.

The recurrent orbits in Axiom A diffeomorphisms lie in certain sets which Smale called <u>hyperbolic basic sets</u>. These sets have been studied by many authors. Perhaps the most significant result about hyperbolic basic sets is that they can be modeled by certain symbolic spaces (called subshifts of finite type), and this gives one very precise information about their orbit structures.

If one thinks about the structure of hyperbolic basic sets a bit, one realizes that they are special cases of certain sets which we call h-closures which exist for many diffeomorphisms. At the present time we have relatively little information about the fine structure of non-hyperbolic h-closures, but hopefully we will understand more about them in the future.

Our goal in these lectures is to introduce the reader to some of the results in this fascinating area of mathematics. Considerations of time and space have forced us to choose a rather limited set of topics to present here. Our intention has been to describe a variety of results with a special emphasis on the theory of attractors. While many references are given in the ensuing sections, we recommend that the reader consult the recent survey of Bowen [9], and the lectures of Ruelle [49] for different perspectives.

1. Periodic points, flows, diffeomorphisms, and generic properties.

In this section, we shall begin to motivate the concept of hyperbolicity. First, let us consider the relationship between flows and diffeomorphisms.

Let M be a compact C^∞ manifold. A C^k vector field X on M is a C^k mapping from M into the tangent bundle TM of M so that $X(x) \in T_xM$ for each $x \in M$. Here T_xM is the tangent space to M at x. We shall always assume $k \geq 1$. The vector field X induces a C^k mapping $\phi : \mathbb{R} \times M \to M$ where $\mathbb{R}$ is the real line such that for each $x \in M$ and $t,s \in \mathbb{R}$ we have

(1) $\phi(0,x) = x$

(2) $\phi(t + s,x) = \phi(t,\phi(s,x))$

(3) $\dfrac{\partial\phi}{\partial t}(t,x) = X(\phi(t,x))$

Conditions (1) and (2) imply that the mapping $t \to \phi(t,\cdot)$ is a homomorphism from $\mathbb{R}$ to the group $\mathrm{Diff}^k M$ of C^k diffeomorphisms of M.

The mapping ϕ is called the flow or one parameter group generated by X. The mapping $x \longrightarrow \phi(1,x)$ is called the time-one map of the flow ϕ.

It is a fact that every C^k diffeomorphism f of a manifold M arises as the time-one map of a C^k flow ϕ on a manifold $\tilde{M}$ with $\dim \tilde{M} = \dim M + 1$. The flow ϕ is called the suspension of f and is defined as follows. Let $M_1 = [0,1] \times M$, and define the equivalence relation $\sim$ on M_1 by $(t,x) \sim (t_1,x_1)$ if and only if $x = x_1$ and $t = t_1$ or $x_1 = f(x)$, $t = 1$, and $t_1 = 0$, or $x = f(x_1)$, $t = 0$, and $t_1 = 1$. The quotient space $M_1/\!\sim\, = \tilde{M}$ inherits a differentiable structure and a flow $\phi : \mathbb{R} \times \tilde{M} \to \tilde{M}$ with

$$\phi(s,\{(t,x)\}) = \{(s + t - [s + t],\ f^{[s+t]}(x))\}$$

where $[s + t]$ is the greatest integer in $s + t$.

Examples: 1. Let $0 < \alpha < 1$ and let S^1 be the circle which we think of as $\mathbb{R}/Z$ with $\mathbb{R}$ the reals and Z the integers. Let $f(x) = x + \alpha \mod 1$. Then $\tilde{M}$ is diffeomorphic to the two torus $T^2 = \mathbb{R}^2/Z^2$ and ϕ may be thought of as the flow induced by the differential equations $\begin{array}{l}\dot{x} = 1\\ \dot{y} = \alpha\end{array}$ on R^2.

2. Let $D^2 = \{(x,y)\} \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk in $\mathbb{R}^2$ and let $f : D^2 \to D^2$ be a C^k diffeomorphism from D^2 into its interior which preserves orientation. The suspension construction applied to f gives us a flow ϕ on $S^1 \times D^2$ (actually defined for forward time only so we should say semi-flow) whose orbits come in at the boundary. We could make this a global flow (via differentiable change of coordinates) embedding $S^1 \times D^2$ in S^3, the 3-sphere, extending the vector field of ϕ to all of S^3, and taking the flow of this extended vector field.

If ϕ is the suspension of f, then they have essentially the same qualitative features, and f gives us the advantage of one less dimension. As we shall see, this enables us to describe interesting 3-dimensional flows with 2-dimensional diffeomorphisms.

For most of these lectures, we shall be concerned with diffeomorphisms. We begin by describing certain generic properties of diffeomorphisms. Fix an integer $k \geq 1$, and let $\mathrm{Diff}^k M$ be the set of C^k diffeomorphisms of M. We give $\mathrm{Diff}^k M$ a topology as follows. Fix a finite covering of M by open coordinate charts $(U_1,\phi_1),\ldots,(U_n,\phi_n)$ of M with $\phi_i : U_i \to \mathbb{R}^m$ a C^k diffeomorphism from U_i onto a bounded open set V_i in the Euclidean space $\mathbb{R}^m$, $m = \dim M$. Choose the pairs (U_i,ϕ_i) so that $\phi_i \circ \phi_j^{-1}$ and its partial derivatives of order less than or equal to k are uniformly continuous. -If $\alpha = (\alpha_1,\ldots,\alpha_r)$ is a multi-index of non-negative integers, we set $|\alpha| = \alpha_1+\ldots+\alpha_r$. For $f,g \in \mathrm{Diff}^k M$, let

$$\psi(f,g) = \sup\{|D_x^\alpha(\phi_i g\phi_j^{-1} - \phi_i f\phi_j^{-1})| : x \in V_j,\ f\phi_j^{-1}x \in U_i,\ g\phi_j^{-1}x \in U_i,\ |\alpha| \leq k,\ 1 \leq i \leq n,\ \text{and}\ 1 \leq j \leq n\}.$$

where $D_x^\alpha(\xi)$ is a partial derivative at x of ξ of order $|\alpha|$. Given $\varepsilon > 0$, let $B_f^k(\varepsilon) = \{g \in \mathrm{Diff}^k M : \psi(g,f) < \varepsilon\}$. The sets $B_f^k(\varepsilon)$ form a neighborhood base for a topology on $\mathrm{Diff}^k M$ called the uniform C^k topology. This topology is independent of the choices of the charts $\{(U_i,\phi_i)\}$.

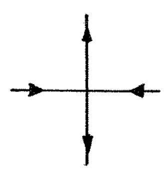
Let us consider the local structure near a periodic point. A point $p \in M$ is periodic for f if there is an integer $n \geq 1$ so that $f^n(p) = p$. The least such n is called the *period* of p. A periodic point p of period n is *hyperbolic* if the derivative $T_p f^n$ of f^n at p has no eigenvalues of absolute value 1. Note that $T_p f^n : T_p M \to T_p M$ is a linear automorphism, and any two local representatives of f^n define conjugate automorphisms. Thus, the eigenvalues of $T_p f^n$ are well-defined.

For convenience of notation, we will frequently indentify $T_p M$

with $\mathbb{R}^m$ and think of $T_p f^n$ as a linear isomorphism of $\mathbb{R}^m$.

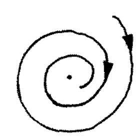
Theorem (1.1). *(Hartman and Grobman). Suppose* p *is a hyperbolic fixed point of a* C^k *diffeomorphism* $f : M \to M$. *There is a neighborhood* U *of* p *in* M *and homeomorphism* $h : U \to \mathbb{R}^m$ *such that* $h(p) = 0$ *and* $hfh^{-1} = T_p f$ *where both sides are defined.*

Thus, via a continuous change of coordinates, f looks like the linear mapping $T_p f$. The structure of $T_p f$ is given by linear algebra. There is a direct sum decomposition $\mathbb{R}^m = E^s \oplus E^u$ into invariant subspaces so that $T_p f|E^s$ has eigenvalues of norm less than one while $T_p f|E^u$ has eigenvalues of norm greater than one. For some norm on $\mathbb{R}^m$, $||T_p f|E^s|| < 1$ and $||T_p f^{-1}|E^u|| < 1$. Some hyperbolic linear automorphisms are sketched in the next figure.



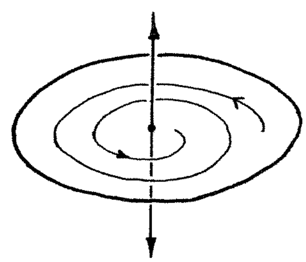
$f(x,y) = (\mu x, \lambda y)$

$0 < \mu < 1 < \lambda$



$f(x,y) = (\alpha x + \beta y, -\beta x + \alpha y)$

$0 < \alpha^2 + \beta^2 < 1$



$f(x,y,z) = (\alpha x + \beta y, -\beta x + \alpha y, \lambda z)$

$0 < \alpha^2 + \beta^2 < 1,\ \lambda > 1$

Figure 1.1

The map h in theorem (1.1) is called a C^o linearization. For C^r linearization theorems, $r \geq 1$, and other normal forms, see [58], [60]. For a hyperbolic periodic point p of period $n > 1$, f^n looks like $T_p f^n$ near p.

An elegant proof of theorem (1.1) which even works in Banach spaces is due independently to Palis [40] and Pugh [45]. The idea is as follows.

One may assume $f : \mathbb{R}^m \to \mathbb{R}^m$ and $f(0) = 0$. Let $L = T_o f$. With a suitable choice of bump function α (a function which is one on a neighborhood of 0 and zero off a slightly bigger neighborhood), one may replace f by $\alpha f + (1 - \alpha)L$ and assume that

(1) $f(x) = Lx$ for x outside some neighborhood of 0

(2) the Lipschitz constant of $f-L$ is small.

Then one tries to find $h = id + u$ where id is the identity map and u is a bounded continuous function.

Consider

(3) $(L + \phi_1)(h) = (h)(L + \phi_2)$

with ϕ_1 and ϕ_2 Lipschitz functions with small C^o and Lipschitz sizes, and $h = id + u$. We get

$$L + Lu + \phi_1(id + u) = L + \phi_2 + u(L + \phi_2)$$

or

$$Lu - u(L + \phi_2) = \phi_2 - \phi_1(id + u)$$

or

$$u - L^{-1}u(L + \phi_2) = L^{-1}[\phi_2 - \phi_1(id + u)].$$

The operator $H : u \longrightarrow u - L^{-1}u(L + \phi_2)$ is invertible, so we get $u = H^{-1}L^{-1}(\phi_2 - \phi_1(id + u)) = \psi(u)$. For ϕ_1 and ϕ_2 small, ψ is a contraction, so (3) has a unique solution h_{ϕ_1,ϕ_2}. Then h_{ϕ_1,ϕ_1} and id solve $(L + \phi_1)h = h(L + \phi_1)$, so $h_{\phi_1,\phi_1} = \text{id}$. Also, $h_{\phi_2,\phi_2} = \text{id}$. Since

$$(L + \phi_1) \circ h_{\phi_1,\phi_2} \circ h_{\phi_2,\phi_1}$$

$$= h_{\phi_1,\phi_2} \circ (L + \phi_2) \circ h_{\phi_2,\phi_1} = h_{\phi_1,\phi_2} \circ h_{\phi_2,\phi_1} (L + \phi_1)$$

we have $$h_{\phi_1,\phi_2} \circ h_{\phi_2,\phi_1} = \text{id} \text{ by uniqueness.}$$

Similarly, $h_{\phi_2,\phi_1} \circ h_{\phi_1,\phi_2} = \text{id}$, so h_{ϕ_1,ϕ_2} is a homeomorphism. If $f = L + \phi_1$, then $h_{\phi_1,0}$ is the required linearization for theorem (1.1).

Now let d be a distance function on M induced by a Riemann metric. Given a point $x \in M$, let $W^s(x,f) = \{y \in M : d(f^n x, f^n y) \to 0$ as $n \to \infty\}$, and let $W^u(x,f) = W^s(x,f^{-1})$. One calls $W^s(x,f)$ the stable set of x $W^u(x,f)$ the unstable set of x. The next result shows that for a hyperbolic period point these sets have nice structure.

Theorem (1.2) (Stable manifold theorem for a point.) Suppose p *is a hyperbolic periodic point of a* C^k *diffeomorphism* f *with splitting* $T_pM = E^s_p \oplus E^u_p$. *Then* $W^s(p,f)$ *is a* C^k *injectively immersed copy of* $\mathbb{R}^s$ *with* $s = \dim E^s_p$ *and* $W^s(p,f)$ *is tangent at* p *to* E^s_p.

Early versions of this theorem were given by Poincaré and then by Hadamard and Perron. The outline we give here is based on the treatment of Hirsch and Pugh [17]. A proof based on the implicit function theorem is due to Irwin [19]. Related and important results are in Hirsch-Pugh-

Shub [18].

The basic idea of the proof of theorem (1.2) is as follows. Let $s = \dim E_p^s$ and $u = \dim E_p^u$. Replace f by a power of f so we may assume $f(p) = p$.

Choose a neighborhood U of p and a diffeomorphism $\phi : U \to \mathbb{R}^s \times \mathbb{R}^u$ such that $\phi(p) = 0$, $T_o\phi(E_p^s) = \mathbb{R}^s \times \{0\}$, and $T_o\phi(E_p^u) = \{0\} \times \mathbb{R}^u$. Here we identify $T_o(\mathbb{R}^s \times \mathbb{R}^u)$ with $\mathbb{R}^s \times \mathbb{R}^u$. For U small, $\bar{f} \equiv \phi f \phi^{-1} = L + \psi$ where L is a linear hyperbolic map and ψ is C^1 small. Let $B = B_1^s \times B_1^u$ be the product of the unit balls in $\mathbb{R}^s$ and $\mathbb{R}^u$. Now $\bar{f}^{-1}$ maps B as in the next figure.

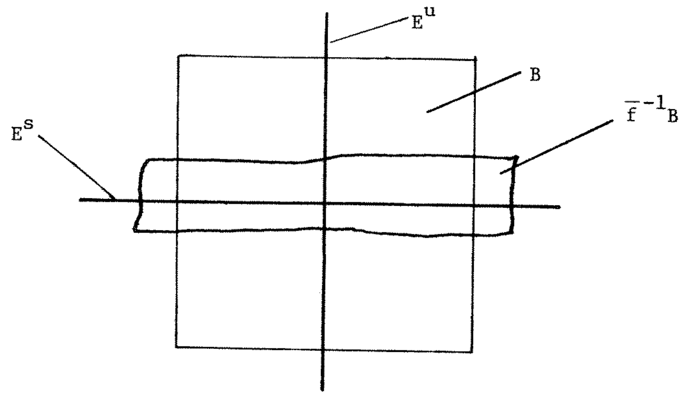


Figure 1.2

Also, $\bar{f}^{-1}(B \cap \bar{f}^{-1}B) \cap B \cap \bar{f}^{-1}B = \bar{f}^{-2}B \cap \bar{f}^{-1}B \cap B$ is a thinner strip in the $\mathbb{R}^u$-direction. It is reasonable and provable that $\bigcap_{n\geq 0} \bar{f}^{-n}B$ is a C^k manifold, and equals the set of points in B whose forward orbits remain there and approach 0. Actually, it turns out better to consider the transformation $\Gamma_{\bar{f}}$ defined by graph $(\Gamma_{\bar{f}}(g)) = \bar{f}^{-1}$ (graph g) where g is a C^k mapping from B^s to B^u of Lipschitz constant m (1.2) is as follows. Let

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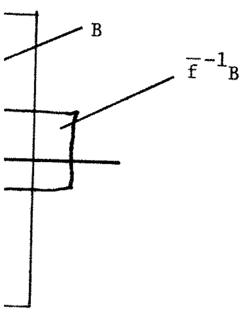

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≤ 1. Then, for any such g, $\Gamma^n_{\bar{f}}(g)$ converges to a unique $\bar{g}$ so that $\Gamma_{\bar{f}}(\bar{g}) = \bar{g}$ and $T_o\bar{g} = 0$. For details, see [17], [18]. We warn the reader that $\Gamma_{\bar{f}}$ is *not* a contraction in any suitable metric. Now, $W^s_{loc}(p,f) = \phi^{-1}$ (graph $\bar{g}$) is called the local stable manifold of p for f. Then, $W^s(p,f) = \bigcup_{n\geq 0} f^{-n}W^s_{loc}(p,f)$ is an expanding union of immersed disks, and hence it is an injectively immersed Euclidean space.

Applying theorem (1.2) to f^{-1} gives a similar structure to $W^u(p,f)$. $W^s(p,f)$ and $W^u(p,f)$ are called the stable and unstable manifolds of p.

It was realized long ago that it is impossible to describe the orbit structures of all differential equations or all diffeomorphisms. Poincaré and Birkhoff emphasized the concept of typical or general systems in which certain exceptional or rare phenomena were to be excluded. There are many precise notions of typicality which can be introduced in the space $\mathrm{Diff}^k M$. One of the most frequently used is the notion of residuality. A subset $\mathcal{B} \subset \mathrm{Diff}^k M$ is called *residual* if it contains a countable intersection of dense open sets. Residual sets are dense, and a countable intersection of residual sets is again residual. Properties which are true for residual sets are called *generic*. It is to be hoped that one day we will be able to understand the orbit structures of elements in a residual set in $\mathrm{Diff}^k M$. At present, we are far from this goal. The next two results describe some useful generic properties.

Let $\phi_1 : N_1 \to M$ and $\phi_2 : N_2 \to M$ be two immersions. We say that ϕ_1 is transverse to ϕ_2 (or that ϕ_1 and ϕ_2 are transverse) if for any $x \in N_1$, $y \in N_2$, such that $\phi_1(x) = \phi_2(y)$ we have $T_{\phi_1(x)}M = T_x\phi_1(T_xN_1) + T_y\phi_2(T_yN_2)$. That is, the tangent space to N_1 at $\phi_1(x)$ and N_2 at $\phi_2(y)$ span the tangent space to M at $\phi_1(x) = \phi_2(y)$.

Sometimes we identify N_1 and N_2 with their images and just say that N_1 and N_2 are transverse.

Theorem (1.3) (Kupka-Smale). There is a residual set $\mathcal{B} \subset \mathrm{Diff}^k M$ *so that each* f *in* $\mathcal{B}$ *has only hyperbolic periodic points and the stable and unstable manifolds of the periodic points are transverse.*

A point $x \in M$ is non-wandering for f if for every neighborhood U of x there is an integer $n > 0$ wuch that $f^n U \cap U \neq \emptyset$. The set of non-wandering points is denoted $\Omega(f)$. It is closed f-invariant set, and it contains all the recurrent behavior of f.

Theorem (1.4) (Pugh [44]) There is a residual set $\mathcal{B} \subset \mathrm{Diff}^1 M$ *so that for* $f \in \mathcal{B}$, *the preiodic points of* f *are dense in* $\Omega(f)$.

It is still unknown if theorem (1.4) holds in $\mathrm{Diff}^k M$, $k > 1$. It does hold if M is the circle, S^1.

Proofs of theorem (1.3) are in [37], [41]. The best proof of theorem (1.4) is in [46].

2. Hyperbolic Sets and Homoclinic Points.

In this section we extend the condition of hyperbolicity to larger sets than a single orbit. The extended condition is fairly restrictive, but it permits us to understand many complicated orbit structures.

We have already defined hyperbolic periodic points, and we have established some of their properties. It is clear how to define a hyperbolic non-periodic orbit $o(x)$. One should require a splitting $T_xM = E^s_x \oplus E^u_x$ so that $T_xf(E^s_x) = E^s_{fx}$, $T_xf(E^u_x) = E^u_{fx}$, and some sort of contraction for $Tf|E^s$ and expansion for $Tf|E^u$. Suitable definitions of the contraction and expansion are that, in some Riemann norm $|\cdot|$ on TM, there is a constant $1 < \lambda$ so that for $y \in o(x)$.

$$|T_yfv| \le \lambda^{-1}|v| \quad \text{if} \quad v \in E^s_y ,$$

and

$$|T_yfv| \ge \lambda|v| \quad \text{if} \quad v \in E^u_y .$$

If $\Lambda \subset M$ is closed invariant set, i.e. $f(\Lambda) = \Lambda$, then Λ is called hyperoblic if all of the orbits in Λ are hyperbolic in a uniform way.

More precisely, we have the following. A Riemann metric on TM induces a norm $|\cdot|$ on each T_xM. We will call this a Riemann norm.

Definition 2.1. *A closed* f*-invariant set* Λ *is hyperbolic if there is a splitting* $T_xM = E^s_x \oplus E^u_x$ *for each* $x \in \Lambda$, *which varies continuously with* $x \in \Lambda$, *a constant* $\lambda > 1$, *and a Riemann norm* $|\cdot|$ *such that*

(1) $T_xf(E^s_x) = E^s_{fx}$, $T_xf(E^u_x) = E^u_{fx}$

(2) $|T_xf(v)| \le \lambda^{-1}|v|$ *for* $v \in E^s_x$, *and* $|T_xf(v)| \ge \lambda|v|$ *for* $v \in E^u_x$.

It can be proved that the bundles E^s and E^u are unique subject to conditions (2.1.1) and (2.1.2). The continuity of $x \longrightarrow E_x^s$ and $x \longrightarrow E_x^u$ also follows from (2.1.1) and (2.1.2). The norm $|\cdot|$ in definition (2.1) is called *adapted* to Λ.

A definition of hyperbolicity which is independent of any particular Riemann matric involves replacing (2.1.2) with

$$(2.1.2)' \qquad |T_x f^n(v)| \le C\lambda^{-n}|v|,\ v \in E_x^s\ ,$$

and

$$|T_x f^n(v)| \ge C^{-1}\lambda^n|v|,\ v \in E_x^u$$

for any $n \ge 0$ and some constants $C > 0$, $\lambda > 1$ independent of n. Then changing the norm merely changes C and λ. We will always use an adapted norm.

Definition (2.1) has the defect that it is hard to establish its existence in examples. There is an equivalent formulation of hyperbolicity which is easier to use.

Let M be Riemannian manifold with norm $|\cdot|$ on TM, and let $T_xM = E_{1x} \oplus E_{2x}$ for $x \in M$. Let $\varepsilon(x)$ be a positive real-valued function on M. Define the $\varepsilon(x)$-sector $S_{\varepsilon(x)}(E_{1x}, E_{2x})$ of (E_{1x}, E_{2x}) by

$$S_{\varepsilon(x)}(E_{1x},E_{2x}) = \{(v_1,v_2) \in E_{1x} \oplus E_{2x} : |v_2| \le \varepsilon(x)|v_1|\} \quad .$$

When the splitting $T_xM = E_{1x} \oplus E_{2x}$ is understood, we write $S_{\varepsilon(x)}$ for $S_{\varepsilon(x)}(E_{1x},E_{2x})$ and $S'_{\varepsilon(x)}$ for $T_xM - S_{\varepsilon(x)}$. If $\lambda > 1$ and $A \subset T_xM$, we say $T_xf|A$ is a λ-expansion if $|T_xf(v)| \ge \lambda|v|$ for all $v \in A$.

Theorem (2.2). Let f *be a* C^1 *diffeomorphism of the compact Riemannian manifold* M, *and let* $\Lambda \subset M$ *be a closed* f*-invariant set. Then* Λ *is hyperbolic for* f *if and only if there are a splitting* $T_xM = E_{1x} \oplus E_{2x}$

for $x \in \Lambda$, *an integer* $m > 0$, *a constant* $\lambda > 1$, *and a positive real-valued function* $\varepsilon : \Lambda \to \mathbb{R}$ *satisfying the following conditions.*

(1) $\sup_{x \in \Lambda} \{\max(\varepsilon(x), \varepsilon(x)^{-1})\} < \infty$

(2) *For each* $x \in \Lambda$, *we have*

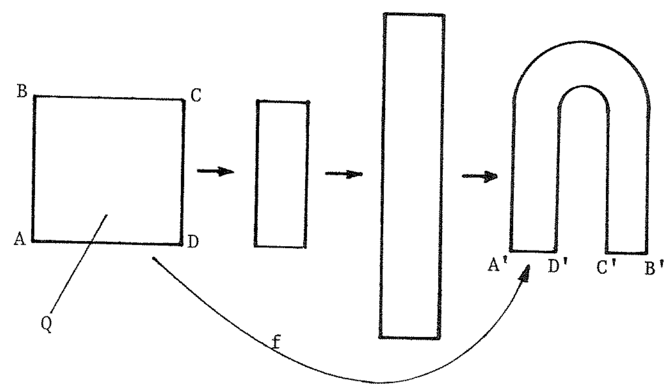
(a) $T_x f^m | S_{\varepsilon(x)} \subset S_{\varepsilon(f^m x)}$ *and both* $T_x f^{-m} | S_{\varepsilon(x)}$ *and* $T_x f^{-m} | S'_{\varepsilon(x)}$ *are* λ *-expansions.*

Theorem 2.2 says that to establish hyperbolicity, one only needs to find a field of cones C_x in $T_x M$ for $x \in \Lambda$ such that $T_x f$ maps C_x to C_{fx}, and for some $m > 0$, $T_x f^m$ expands C_x, and $T_x f^{-m}$ expands $T_x M - C_x$. The fields $x \longrightarrow C_x$ do not even have to be continuous. In most applications, however, they are piecewise continuous.

For a proof of theorem (2.2), see [33], [34].

Let us give a well-known example of a hyperbolic set--the Smale horseshoe diffeomorphism.

Let Q be a square in the plane $\mathbb{R}^2$ and define f from Q into $\mathbb{R}^2$ as described in figure 2.1. The map f first squeezes Q horizontally, then stretches it vertically, and finally wraps the top of Q around as in the figure. Write $f(A) = A'$, $f(B) = B'$, etc.



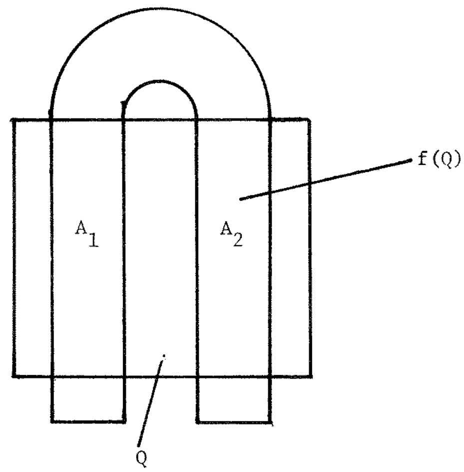


Figure 2.1

Label the two components of $f(Q) \cap Q$ by A_1 and A_2. We assume

$$T_x f = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{for} \quad x \in f^{-1}A_1 ,$$

and

$$T_x f = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha^{-1} \end{pmatrix} \quad \text{for} \quad x \in f^{-1}A_2$$

where $0 < \alpha < \frac{1}{2}$. Then $Q \cap f(Q) \cap f(Q \cap f(Q)) = Q \cap fQ \cap f^2Q$ consists of 2^2 vertical strips, $\bigcap_{0 \le j \le n} f^jQ$ consists of 2^n vertical strips, and $\bigcap_{0 \le j < \infty} f^jQ = C_1 \times I$ where C_1 is a Cantor set and I is an interval.

Similarly, $\bigcap_{-\infty < j \le 0} f^jQ = I \times C_2$ with C_2 a Cantor set, and $\bigcap_{-\infty < j < \infty} f^jQ = C_1 \times C_2$ is a Cantor set. The splitting on $T_x\Lambda$ is given by the horizontal and vertical subspaces, and we may take $\lambda = \alpha^{-1}$, $C = 1$.

The dynamics of $f|\Lambda$ can be conveniently described as follows. Let $\Sigma_2 = \{1,2\}^Z$ be the set of bi-infinite sequences of 1's and 2's with the product (compact-open) topology, and write elements of Σ_2 by $\underline{x}$

where $\underline{x}(i) = 1$ or 2 and $i \in Z$. Define the shift map $\sigma : \Sigma_2 \to \Sigma_2$ by $\sigma(\underline{x})(i) = \underline{x}(i+1)$; that is, σ shifts a sequence $\underline{x}$ one step to the left.

Now if $x \in \Lambda$, then $f^i(x) \in A_1 \cup A_2$, so we may define a sequence $h(x) \in \Sigma_2$ by

$$\begin{aligned} h(x)(i) &= 1 \quad \text{if} \quad f^i(x) \in A_1 \\ &= 2 \quad \text{if} \quad f^i(x) \in A_2 \quad . \end{aligned}$$

Then one can prove that $h : \Lambda \to \Sigma_2$ is a homeomorphism and $hf = \sigma h$, or $hfh^{-1} = \sigma$. From this it follows that the periodic orbits of $f|\Lambda$ are dense in Λ.

For a sequence $\underline{x} \in \Sigma_2$ is periodic if and only if there is a $k > 0$ such that $\underline{x}(i+k) = \underline{x}(i)$ for all i. If $\underline{x} \in \Sigma_2$ and U is any neighborhood of $\underline{x}$, choose a $k > 0$ such that $\underline{y} \in U$ whenever $\underline{y}(i) = \underline{x}(i)$ for $|i| \le k$. The sequence $\underline{z}$ defined by

$$\begin{aligned} \underline{z}(i) &= \underline{x}(i) \quad \text{for} \quad |i| \le k \\ \underline{z}(i) &= \underline{x}(\ell) \quad \text{for} \quad |i| > k \quad \text{where} \quad -k \le \ell \le k \quad \text{and} \\ &\qquad\qquad\qquad i \equiv \ell \bmod (2k+1) \end{aligned}$$

is periodic and lies in U. Thus the periodic orbits of σ are dense in Σ_2, and this implies the same for $f|\Lambda$.

In general, if $f : X \to X$ and $g \quad Y \to Y$ are homeomorphisms, one says that f is topologically conjugate (or topologically equivalent) to g if there is a homeomorphism $h : X \to Y$ satisfying $hf = gh$. The topological equivalence h preserves all the dynamical structure of f. For a given f, one would like to find a simple g to use a model for f.

Remark. Horseshoes actually occur in simple mappings. Fix numbers

$a > 0$ and $0 < b < 1$, and consider the one-parameter family of mappings $f_c(x,y) = (y,-ay^2-bx+c) = (x_1,y_1)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. One easily checks that $g_c(x_1,y_1) = (b^{-1}(-ax_1^2 - y_1 + c),x_1)$ is the inverse of f_c so f_c is a diffeomorphism of $\mathbb{R}^2$. The images of horizontal lines are vertical, and the images of vertical lines are parabolas. Using theorem 2.2, the reader may prove that there is a $c(a,b) > 0$ so that for $c \geq c(a,b)$, the only bounded invariant set for f_c is a set $\Lambda(f_c)$ on which f_c is topologically equivalent to σ on Σ_2.

Let us return to our horseshoe map f. It follows easily from theorem 2.2 that if g is C^1 near f, then the largest g-invariant subset $\Lambda(g)$ of Q is also hyperbolic for g. For this note that $\Lambda(g) = \bigcap_{n\in Z} g^n Q$, and we need only choose $m = 1, \lambda = \frac{\alpha^{-1}}{2}$, $\varepsilon(x) = 1$ for all x, and the tangents to the coordinate lines as E_{1x} and E_{2x}. Also, $g|\Lambda(g)$ remains topologically equivalent to (σ,Σ_2).

Horseshoe type mappings were discovered by Smale [55] in trying to geometrically descirbe a variant of Van der Pol's equation studied by Levinson [20]. They arise in many physical situations near what are called homoclinic points.

If p is a hyperbolic periodic point of a C^1 diffeomorphism f then a point $x \in W^u(p) \cap W^s(p) - \{p\}$ is call a *homoclinic point*. If the intersection of $W^u(p)$ and $W^s(p)$ at x is transverse, the homoclinic point is called transverse.

Theorem (2.3) (Smale homoclinic theorem [55]*). Let* f *be a* C^1 *diffeomorphism with a hyperbolic periodic point* p *having a transverse homoclinic point* x. *There is a integer* $n > 0$ *such that* f^n *has a closed invariant set* Λ *containing* x *and* p *so that* $f^n|\Lambda$ *is topologically equivalent to the shift automorphism* (σ,Σ_2). *Moreover,* Λ *is a hyper-*

one-parameter family of mappings
^{2} to $\mathbb{R}^2$. One easily checks
is the inverse of f_c so f_c
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having a transverse homo-
such that f^n *has a closed*
that $f^n|\Lambda$ *is topologically*
Moreover, Λ *is a hyper-*

bolic set for f^n.

Corollary (2.4). Each transverse homoclinic point of a diffeomorphism f *is in the closure of the hyperbolic periodic points of* f.

Let us sketch a proof of the homoclinic theorem.

Let p be a hyperbolic periodic point of the diffeomorphism f. Replacing f by a power of f, we assume $f(p) = p$. Let x be a transverse homoclinic intersection of $W^u(p)$ and $W^s(p)$. Write $s = \dim W^s(p)$ and $u = \dim W^u(p)$. Let D^s be an s-disk in $W^s(p)$ with $\{p,x\} \subset D^s - \partial D^s$, and let D^u be a u-disk in $W^u(p)$ with $p \in D^u - \partial D^u$ and $x \in f(D^u - \partial D^u) - (D^u - \partial D^u)$. In suitable coordinates about D^s, we may think of a small tubular neighborhood of D^s as $N^s_\delta = D^s \times \delta D^u = \{(\xi, \delta\eta);\ \xi \in D^s,\ \eta \in D^u\}$ where $\delta > 0$ is small. Thus we have the following figure.

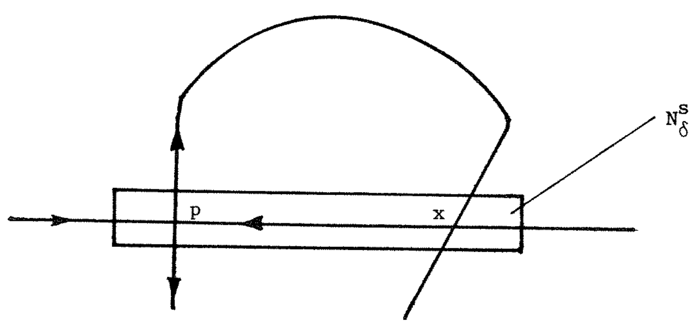


Figure 2.2.

Let us use $C(z,F)$ to denote the connected component of a point z in a set F.

Notice that if $\delta > 0$ is small, then iterates $f^n N^s_\delta$ tend to accumulate along $W^u(p)$ as in figure 2.3.

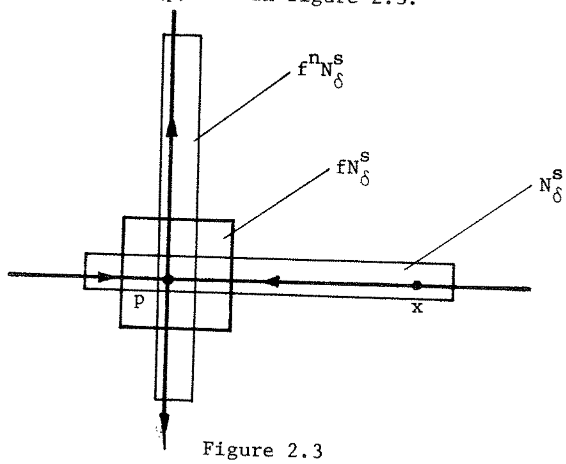


Figure 2.3

Let us set $A_1^{n,\delta} = C(x, f^nN^s_\delta \cap N^s_\delta)$, and $A_2^{n,\delta} = C(x, f^nN^s_\delta \cap N^s_\delta)$. If we adjust the boundary of D^u appropriately, then for large n and small δ, $A_1^{n,\delta}$ and $A_2^{n,\delta}$ look somewhat like the A_1 and the A_2 of the horseshoe diffeomorphism. In figure 2.4, we indicate several possibles for $f^nN^s_\delta$. Note that $W^u(p)$ and $W^s(p)$ need not be transverse everywhere.

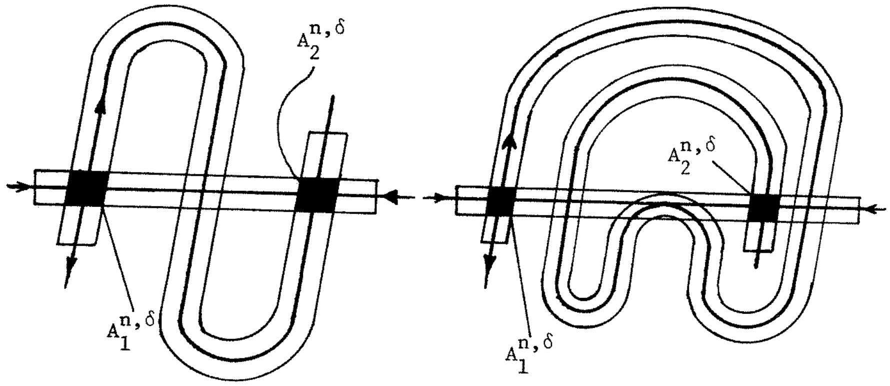
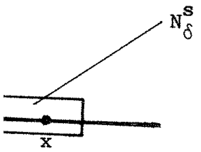


Figure 2.4

Let $\pi_s : N_\delta \to D^s$ and $\pi_u : N_\delta \to \delta D^u$ be the projections. For

2.3.



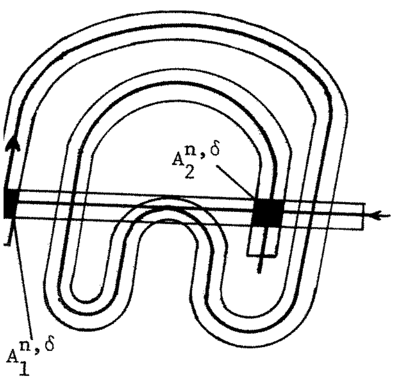
$A_2^{n,\delta} = C(x, f^n N^s_\delta \cap N^s_\delta)$. If we adjust

. for large n and small δ, $A_1^{n,\delta}$

nd the A_2 of the horseshoe

ate several possibles for $f^n N^s_\delta$.

e transverse everywhere.



be the projections. For

$F \subset N_\delta$, set $d_u(F) = \sup_{z \in D^s} \{\text{diam}\ \pi_s^{-1}(z) \cap F\}$ and $d_s(F) = \sup_{z \in \delta D^u} \{\text{diam}\ \pi_u^{-1}(z) \cap F\}$.

Assume n is large and δ is small. Then a little thought shows that there are constants $c > 0$, $\lambda > 1$ such that for any finite sequence $(i_1, \ldots, i_m)$ with $i_k = 1$ or 2,

(1) $\bigcap_{0 \le k \le m} f^{nk}(A_{i_k}^{n,\delta})$ is a disk homeomorphic to $D^s \times D^u$ and

$$d_s\Big(\bigcap_{0 \le k \le m} f^{nk}(A_{i_k}^{n,\delta})\Big) < c\lambda^{-m}.$$

Similarly,

(2) $\bigcap_{-m \le k \le 0} f^{nk}(A_{i_k}^{n,\delta})$ is a disk homeomorphic to $D^s \times D^u$ with

$$d_u\Big(\bigcap_{-m \le k \le 0} f^{nk}(A_{i_k}^{n,\delta})\Big) < c\lambda^{-m}.$$

Some typical sets $\bigcap_{0 \le k \le m} f^{nk}(A_{i_k}^{n,\delta})$ are shaded in figure 2.5a and typical sets $\bigcap_{-m \le k \le 0} f^{nk}(A_{i_k}^{n,\delta})$ are shaded in figure 2.5b.

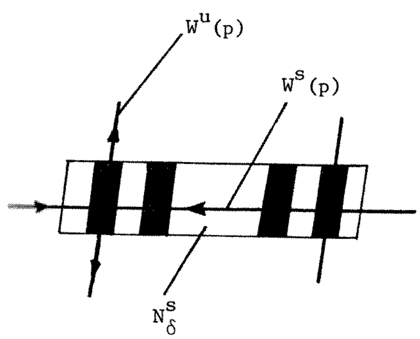


Figure 2.5a

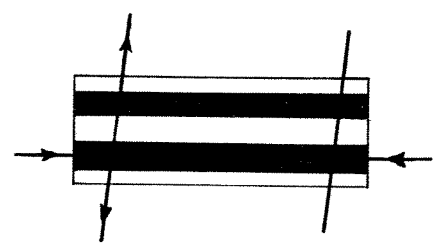


Figure 2.5b

Now it follows that if $(i_{-m}, i_{-m+1}, \ldots, i_0, \ldots, i_m)$ is any finite sequence of 1's and 2's then

(3) $\bigcap_{-m \le k \le m} f^{-nk}(A^{n,\delta}_{i_k})$ is a disk of diameter less than $C_1\lambda^{-m}$ where $C_1 > 0$ is a positive constant.

Hence, for $\underline{i} = (i_k)_{k\in Z} \in \Sigma_2$, $\bigcap_{k\in Z} f^{-nk}(A^{n,\delta}_{i_k})$ is a single point. Set $h(\underline{i}) = \bigcap_{k\in Z} f^{-nk}(A^{n,\delta}_{i_k})$. So $f^{nk}h(\underline{i}) \in A^{n,\delta}_{i_k}$ for all k. Hence $f^{nk}(f^n h(\underline{i})) \in A^{n,\delta}_{i_{k+1}} = A^{n,}_{\sigma(\underline{i})_k}$ for all k, so $f^n h(\underline{i}) = h\sigma(\underline{i})$. If we let $\Lambda = \bigcap_{k\in Z} f^{nk}(A^{n,\delta}_1 \cup A^{n,\delta}_2)$, then $h : \Sigma_2 \to \Lambda$ is 1 - 1, onto, and conjugates σ with $f^n|\Lambda$. Statement (3) shows that h is continuous. Thus, h is a homeomorphism since Σ_2 is compact.

The proof of (1) and (2) as well as the hyperbolicity of Λ involve the same estimate as the following basic result known as the λ-lemma.

Proposition (2.5) (λ)-lemma) [39]. *Let* f *be a* C^1 *diffeomorphism with a hyperbolic periodic point* p, *and let* D^u *be a* u-*disk in* $W^u(p)$. *Let* Δ *be* u-*disk meeting* $W^s(p)$ *transversely at some point* x. *Then* $\bigcup_{n\ge 0} f^n(\Delta)$ *contains* u-*disks arbitrarily* C^1 *close to* D^u.

<u>Proof</u>. Let (u,v) be coordinates on $\mathbb{R}^s \times \mathbb{R}^u$. We assume p is fixed by f. The general case is then obtained by replacing f by some f^n and Δ by some $f^r\Delta$. Since $W^u(p)$ and $W^s(p)$ are C^1 manifolds, we can use the implicit function theorem to produce a coordinate chart (U,ϕ) centered at p so that they become flat. Thus, U is a neighborhood of p and $\phi : U \to \mathbb{R}^s \times \mathbb{R}^u$ is a C^1 diffeomorphism so that $\phi(p) = (0,0)$, $\phi^{-1}((v=0)) \subset W^s(p)$, and $\phi^{-1}((u-0)) \subset W^u(p)$. Replacing

x by $f^{n_1}(x)$ and Δ by $f^{n_1}(\Delta) \cap U$ for some $n_1 > 0$, we say assume $x \in \phi^{-1}((v = 0))$ and $\Delta \subset U$. Write $\overline{f} = \phi f \phi^{-1}$ so that for (u,v) near $(0,0)$, $\overline{f}$ is given by $u_1 = Au + f_1(u,v)$, $v_1 = Bv + f_2(u,v)$ with $|A| < 1$ and $|B^{-1}| < 1$ (the coordinates ϕ may be chosen this way by linear algebra). We assume the partial derivatives $f_{1u}(0,0)$, $f_{1v}(0,0)$, $f_{2u}(0,0)$ and $f_{2v}(0,0)$ are all zero.

Choose $\lambda < 1$ so that $|A| < \lambda$ and $|B\eta| > \lambda^{-1}|\eta|$ for all $\eta \neq 0$ in $\mathbb{R}^u$. Then choose $\delta_1 > 0$ so that whenever $|(u,v)| < \delta_1$, we have $|A + f_{1u}(u,v)| < \lambda$ and $|(B + f_{2v}(u,v))\eta| > \lambda^{-1}|\eta|$ if $\eta \neq 0$. Set $\overline{x} = \phi(x)$ and $\overline{\Delta} = \phi(\Delta)$. Pick $n_2 > 0$ so that $n \geq n_2$ implies $|\overline{f}^n(\overline{x})| < \delta_1$. Let $K > 0$ be such that if (ξ,η) is a vector in $\mathbb{R}^s \times \mathbb{R}^u$ tangent to $\overline{f}^{n_2}\overline{\Delta}$, then $\frac{|\xi|}{|\eta|} \leq \frac{K}{2}$. Since $W^s(p)$ is invariant, $f_2(u,0) \equiv 0$, so we may choose $\delta_2 > 0$ so that $|v| < \delta_2$ and $|u| < \delta_1$ imply $\lambda^{-1} - |f_{2u}(u,v)|K > 1$.

Suppose (ξ,η) is a vector in $\mathbb{R}^s \times \mathbb{R}^u$ with $\eta \neq 0$ and $\frac{|\xi|}{|\eta|} \leq K$, and $z = (u,v)$ is a point with $|u| < \delta_1$ and $|v| < \delta_2$. Letting $T_z\overline{f}(\xi,\eta) = (\xi',\eta')$, we have

$$\frac{|\xi'|}{|\eta'|} = \frac{|(A + f_{1u})\xi + f_{1v}\eta|}{|f_{2u}\xi + (B + f_{2v})\eta|} \leq \frac{\lambda|\xi| + |f_{1v}|\,|\eta|}{\lambda^{-1}|\eta| - |f_{2u}|\,|\xi|}$$

$$\leq \frac{\lambda K|\eta| + |f_{1v}|\,|\eta|}{\lambda^{-1}|\eta| - |f_{2u}|K|\eta|} = \frac{\lambda K + |f_{1v}|}{\lambda^{-1} - |f_{2u}|K}$$

$$\leq \lambda \cdot K + K_1.$$

where $K_1 = \sup_{|(u,v)| \leq \delta_1} |f_{1v}(u,v)|$.

If $f^i(z) \in \{(u,v) : |u| < \delta_1, |v| < \delta_2\}$ for $0 \leq i \leq m - 1$, and we write $T_z\overline{f}^m(\xi,\eta) = (\xi^m,\eta^m)$, then

$$\frac{|\xi^2|}{|\eta^2|} \le \lambda(\lambda K + K_1) + K_1 = \lambda^2 K + \lambda K_1 + K_1,$$

and

$$\frac{|\xi^m|}{|\eta^m|} \le \lambda^m K + K_1 \sum_{i=0}^{m-1} \lambda^i < \lambda^m K + \frac{K_1}{1-\lambda}$$

$$< K + \frac{K_1}{1-\lambda} \equiv K_2.$$

Now iterates of points in $\bar{f}^{n_2}\bar{\Delta}$ near $(v = 0)$ first stay near $(v = 0)$, then they stay near $(0,0)$, and finally they stay near $(u = 0)$. In the first case, vectors tangent to iterates of $\bar{f}^{n_2}\bar{\Delta}$ stay in the sector $S_{K_2}(\mathbb{R}^u, \mathbb{R}^s)$. In the second case, all the partial derivatives f_{1u}, f_{1v}, f_{2u}, f_{2v} are small, so the tangent vectors converge to $\mathbb{R}^u$ exponentially. In the last case, $|f_{1v}|$ is small, so an estimate similar to that of the first case shows the tangent vectors remain close to $\mathbb{R}^u$.

Now choose $n_3 > 0$ so that $f^{-n_3}D^u \subset \phi^{-1}(\{0\} \times \mathbb{R}^u)$. The above estimate shows that $\bigcup_{n\ge 0} f^n(\Delta)$ contains u-disks arbitrarily C^1 close to $f^{-n_3}D^u$, and hence to D^u itself.

Returning to the proof of the homoclinic theorem, the reader may use estimates like those of Proposition (2.5) to show that if $\delta > 0$ is small, and n is large, then some sector $S_K(T(\delta D^u), TD^s)$ over $A_1^{n,\delta} \cup A_2^{n,\delta}$ is invariant and expanded by Tf^n. Also, the complement of $S_K(T(\delta D^u), TD^s)$ is invariant and expanded by Tf^{-n}. Having done this statements (1) and (2) are proved by induction on m. Also, it is immediate from theorem (2.2) that $\Lambda = \bigcap_k f^{nk}(A_1^{n,\delta} \cup A_2^{n,\delta})$ is hyperbolic for f^n.

For more details see [55] or [25].

3. Homoclinic classes, shadowing lemma, and hyperbolic basic sets.

In this lecture we will derive some simple consequences of the homoclinic theorem and the λ-lemma, and we will study the structure of hyperbolic sets.

Let $H(f)$ be the set of hyperbolic periodic points of f and assume $H(f) \neq \emptyset$. For $p \in H(f)$, $o(p)$ is the orbit of p, and we let $W^u(o(p)) = \bigcup_{x \in o(p)} W^u(x)$ be the unstable manifold of the orbit of p. Similarly, we let $W^s(o(p)) = \bigcup_{x \in o(p)} W^s(x)$ be the stable manifold of the orbit of p.

Define a relation $\sim$ on $H(f)$ by saying $p \sim q$ if $W^u(o(p))$ has a non-empty transverse intersection with $W^s(o(p))$ and $W^u(o(q))$ has a non-empty transverse intersection with $W^s(o(p))$. This relation is clearly reflexive and symmetric. It follows from the λ-lemma that it is transitive. For if $p_1 \sim p_2$ and $p_2 \sim p_3$, let z be a point of transverse intersection of $W^u(o(p_1)) \cap W^s(o(p_2))$, and let z' be a point of transverse intersection of $W^u(o(p_2))$ and $W^s(o(p_3))$. Say $z \in W^u(f^{j_1}p_1) \cap W^s(f^{j_2}p_2)$ and $z' \in W^u(f^{j_3}p_2) \cap W^s(f^{j_4}p_3)$. Let τ be so that $f^\tau(p_i) = p_i$ for $i = 1,2,3$. From the λ-lemma for f^τ, we have that $W^u(f^{j_1}p_1)$ contains disks which C^1 accumulate on $W^u(f^{j_2}p_2)$, so $f^{j_3-j_2}\, W^u(f^{j_1}p_1)$ contains disks which C^1 accumulate on $W^u(f^{j_3}p_2)$. Hence, $W^u(f^{j_3-j_2+j_1}p_1)$ has non-empty transverse intersections with $W^s(f^{j_4}p_3)$. Similarly, $W^u(o(p_3))$ has non-empty transverse intersections with $W^s(o(p_1))$. We say p is *homoclinically related* or h-*related* to q if $p \sim q$, and we call the equivalence class of p its homoclinic class or h-class. Denote the h-class of p by $H_p(f)$. Note that the homoclinic theorem (2.3) gives that every transverse homoclinic point of a $p \in H(f)$ is a limit of a sequence $q_1, q_2, \ldots$ in $H(f)$ where each q_i is h-related to p. Clearly $H_p(f) \supset o(p)$ and $H_p(f)$ is f-invariant. It is easy to show, as we will shortly, that $H_p(f) \underset{\neq}{\supset} o(p)$ if and only if p has a transverse homoclinic point. In the latter case, the closure of $H_p(f)$ equals the closure of the orbits of the transverse homoclinic points of p.

Proposition (3.1) (Birkhoff [3]). Let $f : X \to X$ *be a homeomorphism of the complete metric space* X. *Assume the topology for* X *has a countable base, and for every open set* $U \subset X$, $\bigcup_{n\geq 0} f^n U$ *is dense in* X. *Then there is a point* $x \in X$ *whose forward and backward orbits are dense in* X.

Proof. Let $\{V_\alpha\}_{\alpha\in A}$ be a countable open base for the topology of X. Since $\bigcup_{n\geq 0} f^n V_\alpha$ is dense and open, so is $\bigcup_{n\leq 0} f^n V_\alpha$, and, consequently, so is $\bigcup_{n\leq 0} f^n V_\alpha \cap \bigcup_{n\geq 0} f^n V_\alpha$. By the Baire Category Theorem, $\bigcap_{\alpha\in A} [\bigcup_{n\leq 0} f^n V_\alpha \cap \bigcup_{n\geq 0} f^n V_\alpha] \equiv \mathcal{B}$ dense in X. But any $x \in \mathcal{B}$ has both its forward and backward orbits dense in X.

Note that the preceding proof actually gives that the set of points whose forward and backward orbits are dense is residual. A homeomorphism $f : X \to X$ which has dense orbit is called topologically transitive.

Proposition (3.2). For any $p \in H(f)$, *the set* Closure $H_p(f) = C\ell\, H_p(f)$ *is a closed,* f-*invariant set on which* f *is topologically transitive.*

Proof. For hyperbolic periodic points r_1 and r_2, let us write $r_1 > r_2$ if $W^u(r_1) - \{r_1\}$ has a non-empty transverse intersection with $W^s(r_2) - \{r_2\}$. By the λ-lemma applied to f^τ where τ is a common period of r_1 and r_2, we see that $r_1 > r_2$ implies that $W^u(r_1)$ C^1 accumulates on $W^u(r_2)$. Thus, $r_1 > r_2$ and $r_2 > r_3$ imply that $r_1 > r_3$. Also, if $r_1 > r_2$, then $f^j r_1 > f^j r_2$ for any integer j.

Let U_1 and U_2 be non-empty open subsets of $C\ell\, H_p(f)$. We must show that the forward orbit of U_2 meets U_1. If $H_p(f) = o(p)$, there is nothing to prove, so assume there are points $q_1 \in U_1 \cap H_p(f)$ and $q_2 \in U_2 \cap H_p(f)$ such that $o(q_1) \neq o(q_2)$.

Since $q_1 \sim q_2$, there are an integer α and a point $\bar{q}_2 \in o(q_2)$ such that $f^\alpha q_1 > \bar{q}_2 > q_1$. Then, $f^{2\alpha} q_1 > f^\alpha \bar{q}_2 > f^\alpha q_1 > \bar{q}_2 > q_1$. Continuing, we get $f^{j\alpha} q_1 > \bar{q}_2$ for each $j \geq 1$. Letting j be the period of q_1 gives $q_1 > \bar{q}_2 > q_1$. Looking at figure (3.1), we see that this implies that $\bar{q}_2$ is a limit of a sequence $r_1, r_2, \ldots$ of transverse homoclinic points of q_1. By the homoclinic theorem these homoclinic points r_i are in $C\ell\, H_{q_1}(f) = C\ell\, H_p(f)$. For large i, and some fixed k, we have r_i is in $f^k U_2$. Clearly, the forward orbit of $f^j r_i$ meets U_1 for arbitrarily large j. Thus, $U_1 \cap \bigcup_{n>0} f^n U_2 \neq \emptyset$ as required.

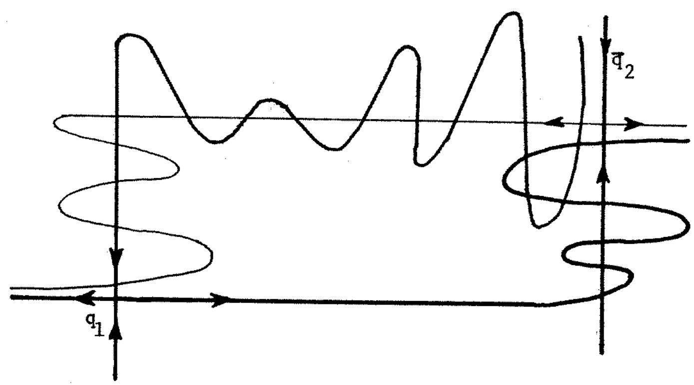


Figure 3.1

Let us call the closure $C\ell\, H_p(f)$ of an h-class of f an h-*closure*.

The h-*closures* for a diffeomorphism form sets with dense orbits and periodic points dense. In general, very little is known about their fine structure. For instance, it is not known when they have positive Lebesgue measure. In the case where f is a generic C^1 area preserving diffeomorphism of a compact two manifold, each h-closure has Hausdorff dimension 2 [31]. We will see that when an h-closure is hyperbolic, then it has a rich structure. Also, we will examine open sets of diffeomorphisms which have non-hyperbolic h-closures.

Out next theorem states that the orbits forward and backward asymptotic to a hyperbolic set behave nicely. Let d be the distance function induced by a Riemann metric. For a point $x \in M$ and a number $\varepsilon > 0$, let $W^s_\varepsilon(x) = W^s_\varepsilon(x,f) = \{y \in M : d(f^n x, f^n y) \le \varepsilon$ for $n \ge 0\}$, and let $W^u_\varepsilon(x) = W^u_\varepsilon(x,f) = W^s_\varepsilon(x,f^{-1})$. One calls $W^s_\varepsilon(x)(W^u_\varepsilon(x))$ the stable (unstable) set of size ε of x.

Theorem (3.3) (Hirsch and Puch [17]). *Let* $f : M \to M$ *be a* C^k *diffeomorphism,* $k \ge 1$, *let* Λ *be a hyperbolic set for* f *with splitting* $T_xM = E^s_x \oplus E^u_x$, $x \in \Lambda$, *and endow* TM *with an adapted metric. Then, for* $\varepsilon > 0$ *small and* $x \in \Lambda$,

(1) $W^s_\varepsilon(x,f)$ *and* $W^u_\varepsilon(x,f)$ *are* C^k *disks through* x *varying continuously with* x *in the* C^k *topology.*

(2) $W^s_\varepsilon(x,f)$ *is tangent at* x *to* E^s_x *and* $W^u_\varepsilon(x,f)$ *is tangent at* x *to* E^u_x.

It follows from the definitions of $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ that $fW^s_\varepsilon(x) \subset W^s_\varepsilon(fx)$ and $f^{-1}W^u_\varepsilon(x) \subset W^u_\varepsilon(f^{-1}x)$. Also, theorem (3.3) implies that $W^s(x) = \bigcup_{n\ge 0} f^{-n}W^s_\varepsilon(f^n x)$ and $W^u(x) = \bigcup_{n\ge 0} f^n W^u_\varepsilon(f^{-n}x)$ for $x \in \Lambda$. Therefore, as in the case of periodic points, $W^s(x)$ and $W^u(x)$ are C^k injectively immersed copies of Euclidean spaces.

A homeomorphism $f : X \to X$ of a compact metric space X is called *expansive* if there is an $\varepsilon > 0$ so that $d(f^n x, f^n y) \leq \varepsilon$ for all n, implies $x = y$. Any such ε is called an expansive constant for f.

Proposition (3.4). If Λ is a hyperbolic set for f, then $f|\Lambda$ is expansive.

<u>Proof</u>. The splitting $T_\Lambda M = E^s \oplus E^u$ is continuous, and for small ε, $W^s_\varepsilon(x)$ is a disk tangent to E^s_x at x, and $W^u_\varepsilon(x)$ is a disk tangent to E^u_x at x. Thus, $W^u_\varepsilon(x) \cap W^s_\varepsilon(x) = \{x\}$. But, if $d(f^n x, f^n y) \leq \varepsilon$ for $n \geq 0$, then $y \in W^s_\varepsilon(x)$, and if $d(f^n x, f^n y) \leq \varepsilon$ for $n \leq 0$, then $y \in W^u_\varepsilon(x)$. Thus, $y \in W^u_\varepsilon(x) \cap W^s_\varepsilon(x) = \{x\}$.

If a hyperbolic set Λ has the property that there is an $\varepsilon > 0$ such that $W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \subset \Lambda$ for all $x, y \in \Lambda$, then one says that Λ has a *local product structure*. The name comes from the fact that if the condition holds and $x \in \Lambda$, then, for small ε, there is a neighborhood U_x of x in Λ which is homeomorphic to the product $(W^s_\varepsilon(x) \cap \Lambda) \times (W^u_\varepsilon(x) \cap \Lambda)$. Of course, compactness allows one to choose ε independent of x.

Let $a < b$ be integers or $a = -\infty$ or $b = +\infty$. For $\delta > 0$, a δ-pseudo-orbit for f is a sequence $\{x_i\}_{a<i<b}$ such that $d(fx_i, x_{i+1}) \leq \delta$ for all i. The orbit $o(x) = \{f^i x : i \in Z\}$ ε-shadows the pseudo-orbit $\{x_i\}$ if $d(f^i x, x_i) \leq \varepsilon$ for all i. Most of the properties of hyperbolic sets may be proved from theorem (3.3) and the following result.

Theorem (3.5) (Shadowing Lemma). Suppose Λ is a hyperbolic set for f with a local product structure. For every $\varepsilon > 0$ there is a $\delta > 0$ so that every δ-pseudo-orbit in Λ can be ε-shadowed by an orbit in Λ.

With a slightly different formulation theorem (3.5) was first proved by Anosov [2]. The formulation we give is due to Bowen [7].

Proof. There are constants $\varepsilon_0 > 0$ and $\lambda > 1$ with the following properties. If $0 < \varepsilon_1 < \varepsilon_0$, then

(a) for $x,y \in \Lambda$, $W^u_{\varepsilon_1}(x) \cap W^s_{\varepsilon_1}(y) \subset \Lambda$ is at most one point,

(b) $y \in W^u_{\varepsilon_1}(x)$ implies $d(f^{-n}y, f^{-n}x) \leq \lambda^{-n}\varepsilon_1$ for $n \geq 0$,

and

(c) $y \in W^s_{\varepsilon_1}(x)$ implies $d(f^n y, f^n x) \leq \lambda^{-n}\varepsilon_1$ for $n \geq 0$.

From the local product structure we may find a constant $c > 1$ such that if $z \in W^u_{\varepsilon_0}(x) \cap W^s_{\varepsilon_0}(y)$, then $d(z,x) \leq cd(x,y)$ and $d(z,y) \leq cd(x,y)$.

Pick an integer $N > 0$ such that $\lambda^{-N}2c < 1$. Observe that it is enough to shadow f^N psuedo-orbits. For if we can do this and $\varepsilon > 0$ is arbitrary, pick ε_1 such that $d(x,y) < \varepsilon_1$ implies $d(f^j x, f^j y) < \frac{\varepsilon}{2}$ for $0 \leq j \leq N$. Choose $\delta_1 > 0$ such that each δ_1-pseudo-orbit for f^N can be ε_1-shadowed. Then choose $\delta > 0$ so that if $\{x_i\}$ is a δ-pseudo-orbit for f, then $\{x_{Ni}\}$ is a δ_1-pseudo-orbit for f^N, and for each i and $0 \leq j \leq N$, one has $d(f^j x_i, x_{i+j}) < \frac{\varepsilon}{2}$. Now let y be a point in Λ such that $d(f^{Ni}y, x_{Ni}) \leq \varepsilon_1$ for all i. Then for $0 \leq j \leq N$, $d(f^{Ni+j}y, x_{Ni+j}) \leq d(f^{Ni+j}y, f^j x_{Ni}) + d(f^j x_{Ni}, x_{Ni+j}) \leq \varepsilon$.

We now show f^N pseudo-orbits can be shadowed. Let $\varepsilon > 0$. Let $\delta > 0$ be such that $\frac{3c\delta}{1-\lambda^{-N}} < \varepsilon$. Assume δ small enough that $x,y \in \Lambda$ and $d(x,y) < 2\delta$ imply $W^u_{\varepsilon_0}(x) \cap W^s_{\varepsilon_0}(y)$ is a unique point.

We work with pseudo-oribts indexed on all of $\mathbb{Z}$.

Let $\{x_i\}_{-\infty<i<\infty}$ be a δ-pseudo-orbit for f^N in Λ, and let $b > 1$ be an integer.

We will produce a point $w_b \in \Lambda$ such that $d(f^{jN}w_b, x_j) \leq \varepsilon$ for $|j| \leq b$. Once this is done, we let w be an accumulation point of $\{w_b\}$, to Bowen [7].

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$^{-n}\varepsilon_1$ for $n \geq 0$,

ε_1 for $n \geq 0$.

constant $c > 1$ such that

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δ_1-pseudo-orbit for f^N

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$d(f^{jN} w_b, x_j) \leq \varepsilon$ for

umulation point of $\{w_b\}$,

and then $d(f^{jN}w, x_j) \leq \varepsilon$ for all j as required.

Now, we have $d(f^N(x_{b-1}), x_b) < \delta$. Set $z_0 = x_b$. Assume, inductively, that for $j \geq 1$, we have a point z_j such that

$(1)_j$ $d(z_j, x_{b-j}) < \delta$

and

$(2)_j$ $d(f^{Nk}z_j, z_{b-j+k}) < 3c\delta(\sum_{i=0}^{k-1} \lambda^{-Ni})$ for $1 \leq k \leq j$.

If we set $z_{j+1} = f^{-N}[W^u_{\varepsilon_0}(f^N x_{b-(j+1)}) \cap W^s_{\varepsilon_0}(z_j)]$, then we claim that z_{j+1} satisfies $(1)_{j+1}$ and $(2)_{j+1}$.

First observe that by $(1)_j$, $d(f^N x_{b-(j+1)}, z_j) \leq d(f^N x_{b-(j+1)}, x_{b-j}) + d(x_{b-j}, z_j) < 2\delta$, so z_{j+1} can be defined.

Also, by the choice of c, $d(f^N z_{j+1}, f^N x_{b-(j+1)}) \leq 2c\delta$. Thus, $d(z_{j+1}, x_{b-(j+1)}) \leq \lambda^{-kN} 2c\delta < \delta$ which is $(1)_{j+1}$.

Now, $d(f^N z_{j+1}, z_j) \leq cd(f^N x_{b-(j+1)}, z_j) \leq c \cdot 2\delta$.

Since $f^N z_{j+1} \in W^s_{\varepsilon_0}(z_j)$, we have $d(f^{(k+1)N} z_{j+1}, f^{kN} z_j) \leq \lambda^{-kN} d(f^N z_{j+1}, z_j) \leq \lambda^{-kN} 2c\delta$ for $k \geq 0$.

For $1 \leq k \leq j+1$,

$$d(f^{Nk} z_{j+1}, x_{b-(j+1)+k}) \leq d(f^{Nk} z_{j+1}, f^{N(k-1)} z_j) + d(f^{N(k-1)} z_j, x_{b-j+(k-1)})$$

If $k = 1$, this gives $d(f^{Nk} z_{j+1}, x_{b-(j+1)+k}) \leq 2c\delta + \delta \leq 3c\delta$. If $2 \leq k \leq j+1$, it gives

$$d(f^{Nk} z_{j+1}, x_{b-(j+1)+k}) \leq \lambda^{-(k-1)N} 2c\delta + 3c\delta \sum_{i=0}^{k-2} \lambda^{-Ni} < 3c\delta \sum_{i=0}^{k-1} \lambda^{-Ni}$$

which is $(2)_{j+1}$.

Now, setting $w_b = f^{Nb} z_{2b}$, $(1)_{2b}$ and $(2)_{2b}$ give us $d(f^{jN} w_b, x_j) \leq \frac{3c\delta}{1-\lambda^{-N}} < \varepsilon$ for $|j| \leq b$ as required.

Corollary (3.6). For $\varepsilon > 0$ *small, set* $W^u_\varepsilon(\Lambda) = \bigcup_{x \in \Lambda} W^u_\varepsilon(x)$ *and*

$W^s_\varepsilon(\Lambda) = \bigcup_{x\in\Lambda} W^s_\varepsilon(x)$. *There is a neighborhood* U *of* Λ *such that*

(a) $\bigcap_{n\le 0} f^n U \subset W^s_\varepsilon(\Lambda)$

(b) $\bigcap_{n\ge 0} f^n U \subset W^u_\varepsilon(\Lambda)$

and

(c) $\bigcap_{n\in Z} f^n U = \quad .$

<u>Proof</u>. Statement (c) follows from (a) and (b) and the local product structure. We prove only (a) since the proof of (b) follows replacing f by f^{-1}. Let $\varepsilon > 0$ be small and choose $\delta_1 \in (0,\varepsilon)$ so that every δ_1-pseudo-orbit in Λ can be $\frac{\varepsilon}{2}$-shadowed by an orbit in Λ. Choose $\delta \in (0, \frac{\delta_1}{2})$ so that $d(x,y) \le \delta$ implies $d(fx,fy) \le \frac{\delta_1}{2}$. Let $U = B_\delta(\Lambda) = \{y \in M : \mathrm{dist}(y,\Lambda) \le \delta\}$. If $x \in \bigcap_{n<0} f^n U$, then for each $i \ge 0$ there is an $x_i \in \Lambda$ such that $d(f^i x, x_i) \le \delta$. Then $d(fx_i, x_{i+1}) \le d(fx_i, f^{i+1}x) + d(f^{i+1}x, x_{i+1}) \le \frac{\delta_1}{2} + \delta \le \delta_1$, so $\{x_i\}_{i\ge 0}$ is a δ_1-pseudo-orbit.

Let $\{f^i y\}$ be an orbit in Λ $\frac{\varepsilon}{2}$-shadowing it. Then, for $i \ge 0$, $d(f^i y, f^i x) \le d(f^i y, x_i) + d(x_i, f^i x) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $x \in W^s_\varepsilon(y)$.

A neighborhood U of Λ as in Corollary (3.6) is called a *fundamental* neighborhood for Λ.

Note that (3.6a) implies that any point x whose forward orbit stays near Λ is actually forward asymptotic to a point of Λ.

The next result asserts that the orbit structures of hyperbolic sets persist under perturbation.

Theorem (3.7) (Stability). Let Λ *be a hyperbolic set for* f *with a local product structure. There are neighborhoods* U *of* Λ *in* M *and* N *of* f *in* $\mathrm{Diff}^1 M$ *so that if* $g \in N$, *then* $\Lambda(g) = \bigcap_n g^n U$ *is a hyperbolic set for* g *with a local product structure. Moreover, there is a homeomorphism* $h_g : \Lambda \to \Lambda(g)$ *such that* $gh_g = h_g f$ *and* h_g *varies* *continuously with* g.

Proof. Let $U \supset \Lambda$ be a small fundamental neighborhood for Λ as in Corollary (3.6). If g is C^1 near f, then $\Lambda(g) = \bigcap_{n \in Z} g^n(U)$ is a hyperbolic set for g by theorem (2.2). Also, since $\Lambda(g) \subset \text{int } U$, it has a local product structure. To produce the homeomorphism h, we first let $\varepsilon > 0$ be an expansive constant for $f|\Lambda(f)$. Let $\delta_1 \in (0,\varepsilon)$ be such that every δ_1 pseudo-orbit for f in $\Lambda(f)$ can be $\frac{\varepsilon}{4}$-shadowed by an orbit. Then, let $\delta \in (0, \frac{\delta_1}{3})$ be such that $d(x,y) \leq \delta$ implies $d(fx,fy) \leq \frac{\delta_1}{3}$. Let $U = B_\delta(\Lambda(f))$, and let g be such that $d(gx,fx) \leq \frac{\delta_1}{3}$ for all x. If $x \in \Lambda(g) = \bigcap_n g^n(U)$, then for each i there is an $x_i \in \Lambda(f)$ such that $d(x_i, g^i x) \leq \delta$. Since $d(fx_i, x_{i+1}) \leq d(fx_i, fg^i x) + d(fg^i x, gg^i x) + d(g^{i+1}x, x_{i+1}) \leq \delta_1$, $\{x_i\}$ is a δ_1-pseudo-orbit. Let $\phi(x) \in \Lambda(f)$ be such that $d(f^i\phi(x), x_i) \leq \frac{\varepsilon}{4}$ for all i.

Then, $d(f^i\phi x, g^i x) \leq d(f^i\phi x, x_i) + d(x_i, g^i x) \leq \frac{\varepsilon}{2}$ for all i. By the choice of ε, $\phi(x)$ is unique with this property. Let us prove that ϕ is continuous. If ϕ is not continuous at $x \in \Lambda(g)$, there is a $\delta_0 > 0$ and a sequence $y_k \in \Lambda(g)$ such that $y_k \to x$ as $k \to \infty$ and $d(\phi x, \phi y_k) \geq \delta_0$ for all k. Let (y_{n_k}) be a subsequence such that ϕy_{n_k} converges to a point $z \in \Lambda(f)$. Then, $d(\phi x, z) \geq \delta_0$. But for all i,

$$d(f^i\phi x, f^i z) \leq d(f^i\phi x, g^i x) + d(g^i x, g^i y_{n_k}) + d(g^i y_{n_k}, f^i\phi y_{n_k})$$

$$+ d(f^i\phi y_{n_k}, f^i z) \leq \frac{\varepsilon}{2} + d(g^i x, g^i y_{n_k}) + \frac{\varepsilon}{2} + d(f^i\phi y_{n_k}, f^i z) \quad .$$

Fixing i and letting $k \to \infty$ gives $d(f^i\phi x, f^i z) \leq \varepsilon$ for each i. This contradicts the choice of ε.

Thus, we have proved that for each $x \in \Lambda(g)$ there is a unique

$\phi(x) \in \Lambda(f)$ whose f-orbit stays within $\frac{\varepsilon}{2}$ of the g-orbit of x, and $x \longrightarrow \phi(x)$ is continuous. Interchanging f and g gives a continuous function $h : \Lambda(f) \to \Lambda(g)$ such that $d(g^i hx, f^i x) \leq \frac{\varepsilon}{2}$ for all i. Then, $h \circ \phi = id_{\Lambda(g)}$ and $\phi \circ h = id_{\Lambda(f)}$, so h is a homeomorphism. Clearly, $gh = hf$. We let the reader verify that $g \longrightarrow h_g$ is continuous.

A hyperbolic set Λ for f which has a local product structure and has the additional property that $f|\Lambda$ has a dense orbit is called a *hyperbolic basic set*.

Such sets have been studied a great deal. Their orbit sturctures can be modeled very well by certain generalizations of the 2-shift described above.

Let $J = \{1,\ldots,N\}$, and let $\Sigma_N = J^Z$ be the set of bi-infinite sequences of the elements of J with the compact open topology. One defines the shift on Σ_N as before by $(\underline{a})(i) = \underline{a}(i+1)$ for $\underline{a} \in \Sigma_N$. This is the full N-shift. If $A = (A_{ij})$ is an $N \times N$ matrix whose entries are 0's and 1's, we may consider the subset $\Sigma_A \subset \Sigma_N$ defined by

$$\Sigma_A = \{\underline{a} \in \Sigma_N : A_{\underline{a}(i),\underline{a}(i+1)} = 1 \text{ for all } i\}.$$

Thus a sequence $(\ldots a_{-1}a_0a_1,\ldots)$ is in Σ_A if and only if each of its 2-blocks a_ia_{i+1} yields 1 when used as indices for the matrix A. The set Σ_A is a closed σ-invariant set and $\sigma|\Sigma_A$ (or sometimes Σ_A itself) is called a *subshift of finite type*.

Theorem (3.8). Let Λ *be a hyperbolic basic set for a* C^1 *diffeomorphism* f. *Then there are a matrix* A *of* 0*'s and* 1*'s and a finite-to-one continuous surjection* $\pi : \Sigma_A \to \Lambda$ *so that the following diagram commutes*

$$\begin{array}{ccc}\Sigma_A & \xrightarrow{\ \sigma\ } & \Sigma_A \\ \pi \downarrow & & \downarrow \pi \\ \Lambda & \xrightarrow{\ f\ } & \Lambda\end{array}$$

This important result was proved by Sinai [53] when $\Lambda = M$, and later by Bowen [4] for general basic sets. The proof involves special converings of Λ by local product sets called Markov partitions. This is treated nicely in [7].

The mapping π is $1 - 1$ on many points in Σ_A and the cardinality of $\pi^{-1}(x)$ is bounded by N^2 for each x where N is the order of the matrix A [9]. The space Σ_A codes the action of f on Λ in a very comprehensive way, and can be used to prove many facts about $f|\Lambda$. For instance the minimal sets in Λ are zero-dimensional [5], [22], and one can compute the number of periodic points of period n of f [68] for each $n \geq 1$.

Remark: 1. Theorem (3.8) holds for hyperbolic sets with local product structures (i.e. without assuming $f|\Lambda$ has a dense orbit). However, if Λ is a hyperbolic set with a local product structure, one can prove that $\Omega(f|\Lambda) = \Lambda_1 \cup \ldots \cup \Lambda_n$ is a finite union of subsets Λ_i such that $f|\Lambda_i$ has a dense orbit. Here $\Omega(f|\Lambda)$ is the non-wandering set of f restricted to Λ. This decomposition is similar to the one we will give in Proposition (4.2). Since one is usually interested in studying recurrence phenomena of $f|\Lambda$, it is no harm to assume at the outset that $f|\Lambda$ has a dense orbit.

4. Hyperbolic Limit Sets

Let us now describe some diffeomorphisms all of whose recurrence is hyperbolic.

A point $x \in M$ is an α-limit point of f if there are a point $y \in M$ and a sequence of integers $0 < n_1 < n_2 < \dots$ such that $f^{-n_i}(y) \to x$ as $i \to \infty$. Similarly, x is an ω-limit point of f if there are a y and a sequence $0 < n_1 < n_2 < \dots$ such that $f^{n_i}(y) \to x$ as $i \to \infty$.

The set of α-limit points of y is denoted $\alpha(y)$ and the set of ω-limit points of y is denoted $\omega(y)$.

Let $L_\alpha(f)$ be the set of α-limit points of f, and let $L_\omega(f)$ be the set of ω-limit points of f. Define $L(f) = \mathcal{C}\ell(L_\alpha(f) \cup L_\omega(f))$. We call $L(f)$ the limit set of f. It is clearly closed and invariant, and all orbits approach $L(f)$ in the future and past. We shall study the situation when $L(f)$ is hyperbolic.

First we recall the topological notion of the index of a mapping. See [10] for a general treatment of this theory. Let Δ be the closed unit ball in $\mathbb{R}^n$ and $\partial\Delta = S^{n-1}$ be the (n-1)-sphere. If $f : \Delta \to \mathbb{R}^n$ is a continuous map with no fixed points on $\partial\Delta$, one defines the index of f on Δ to be the degree of the map $x \to \frac{x-f(x)}{|x-f(x)|}$ for $x \in S^{n-1}$. This is denoted $\mathrm{Ind}(f,\Delta)$. If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an orientation preserving homeomorphism, $D = \phi(\Delta)$, and f is a continuous map from D to $\mathbb{R}^n$ without fixed points on ∂D, set $\mathrm{Ind}(f,D) = \mathrm{Ind}(\phi^{-1} f \phi, \Delta)$. We call $\mathrm{Ind}(f,D)$ the index of f on D. It does not depend on the orientation preserving homeomorphism ϕ.

It is a standard fact that if $\mathrm{Ind}(f,D) \neq 0$, then f has a fixed point in D.

For example, consider the mappings $f_1(x,y) = (\frac{1}{2}x, 2y)$, $f_2(x,y) = (\frac{1}{2}x, \frac{1}{2}y)$, $f_3(x,y) = (\frac{1}{2}x, -2y)$ on $\mathbb{R}^2$. If Δ is the unit disk in $\mathbb{R}^2$, one may compute $\mathrm{Ind}(f_1,\Delta) = -1$ and $\mathrm{Ind}(f_2,\Delta) = \mathrm{Ind}(f_3,\Delta) = 1$. More generally, if $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear hyperbolic automorphism, then we may compute its index on the unit disk Δ as follows. If $u = \dim W^u(o,L)$, then $\mathrm{Ind}(f,\Delta) = (-1)^u$ if L preserves the orientation of $W^u(o,L)$, and it is $(-1)^{u+1}$ if L reverses the orientation of $W^u(o,L)$. In any case, $\mathrm{Ind}(f,\Delta) = \pm 1$. If D is any n-disk containing 0 in its interior, then $\mathrm{Ind}(L,D) = \mathrm{Ind}(L,\Delta)$.

Proposition (4.1). If L(f) *is hyperbolic, then the periodic orbits are dense in* L(f).

Proof. Let $E^s \oplus E^u = T_{L(f)}M$ be the continuous splitting of $T_{L(f)}M$ given by the definition of hyperbolicity. Then $L(f) = L_1 \cup \dots \cup L_n$ where $L_i \cap L_j = \phi$ and each L_i is a closed f-invariant set on which $\dim E^u$ and $\dim E^s$ are constant. Let $x \in L_i$ for some $1 \le i \le n$, and let $y \in M$ be such that $x \in \omega(y)$. Since $\omega(y) \subset L(f) = \bigcup_{j=1}^{n} L_j$ and the collection $\{L_j\}$ consists of closed disjoint sets, one sees that $\omega(y) \subset L_i$. Let exp be the exponential map of a Riemann metric on TM adapted to Λ.

For z_1, z_2 near x, let $\tau_{z_1,z_2} : T_{z_1}M \to T_{z_2}M$ be the map induced by parallel translation along the geodesic from z_1 to z_2. Choose a small $\delta > 0$, and let $U = \{(u,v) \in E^s_x \oplus E^u_x : |u| \le \delta \text{ and } |v| \le \delta\}$. Then $\exp_x U$ is a small product neighborhood of x. Write $U_1 = \exp_x U$.

Since $\omega(y) \subset L_i$, there is an integer $N > 0$ so that $f^n y$ is near L_i for all $n \ge N$, and there are integers $N < n_1 < n_2$ so that $f^{n_1}y$ and $f^{n_2}y$ are near x in U_1.

Let C be the connected component of $f^{n_1}y$ in $U_1 \cap f^{n_1 - n_2}U_1$.

Since $f^j y$ remains near L_i for $n_1 \leq j \leq n_2$, it keeps behaving hyperbolically, and we see that C and $f^{n_2-n_1}C$ are as in figure 4.1

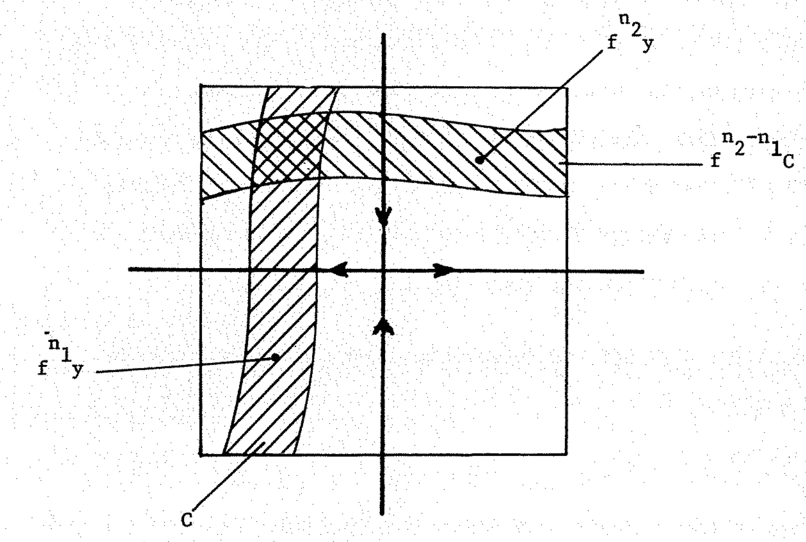


Figure 4.1

Consider the hyperbolic linear map $\Phi = \tau_{z_2 z_1} \circ T_{z_1} f^{n_2-n_1}$ with $z_1 = f^{n_1}y$ and $z_2 = f^{n_2}y$. We have that Φ maps $T_{z_1}M$ to $T_{z_1}M$ and its index on a neighborhood of 0 in $T_{z_1}M$ is ± 1. But from figure (4.1) $f^{n_2-n_1}$ has no fixed points on the boundary of C and its index on C is the same as that of Φ on a small neighborhood of 0 in $T_{z_1}M$. So $f^{n_2-n_1}$ has a fixed point in C. This proves the periodic points are dense in $L_\omega(f)$. The proof for $L_\alpha(f)$ is similar.

Proposition (4.2) (Spectral Decomposition). If $L(f)$ *is hyperbolic, then* $L(f) = \Lambda_1 \cup \ldots \cup \Lambda_n$ *where* $\{\Lambda_i\}$ *is a disjoint collection of closed invariant sets with periodic points dense. Moreover, for each* i, $f|\Lambda_i$ *is topologically transitive, and* Λ_i *has a local product structure.*

Proof. Let Per(f) denote the set of periodic points of f. We know by (4.1) that $L(f) = C\ell\ \mathrm{Per}(f)$. Observe that f has only finitely many h-classes. For if $p_1, p_2, \dots$ is an infinite sequence of points in Per(f) which converge to a point y, then for large i,j, $p_i \sim p_j$ by stable manifold theory. The same argument shows that $C\ell\ H_p(f) \cap C\ell\ H_q(f) \neq \phi$ implies $H_p(f) = H_q(f)$. Now, just take $H_{p_1}(f), \dots, H_{p_n}(f)$ to be the distinct h-classes, and let $\Lambda_i = C\ell H_{p_i}(f)$ be the h-closure of p_i. We only need to prove that Λ_i has a local product structure.

If $\varepsilon > 0$ is small, then any point z in $W^u_\varepsilon(x) \cap W^s_\varepsilon(y)$ with $x,y \in \Lambda_i = C\ell\ H_{p_i}(f)$ is a point of transverse intersection of $W^u_\varepsilon(x)$ and $W^s_\varepsilon(y)$. But $W^u(o(p_i))$ accumulates on $W^u_\varepsilon(x)$ and $W^s(o(p_i))$ accumulates on $W^s_\varepsilon(y)$, so z is a limit of transverse homoclinic points of $o(p_i)$, i.e., transverse intersections of $W^u(o(p_i))$ and $W^s(o(p_i))$. In the proof of proposition (3.2), we saw that such points were limits of the orbits of transverse homoclinic points of p_i. Hence, by theorem (2.3), z is a limit of points in $H_{p_i}(f)$. Thus, $z \in C\ell\ H_{p_i}(f)$.

Corollary (4.3). If L(f) *is hyperbolic, then* $M = \bigcup_{x \in L(f)} W^s(x) = \bigcup_{x \in L(f)} W^u(x)$. *That is, each stable and unstable set in* M *is an immersed submanifold of* M.

Proof. From (4.2), $L(f) = \Lambda_1 \dots \Lambda_n$ where each Λ_i has a local product structure. If $x \in M$, then the ω-limit set of x is in some Λ_i. By (3.6a), $x \in W^s_\varepsilon(y)$ for some $y \in \Lambda_i$. Then $W^s(x) = W^s(y)$. The proof for $W^u(x)$ is similar.

Remark: One can strengthen the condition that L(f) is hyperbolic by requiring that all of the non-wandering set $\Omega(f)$ be hyperbolic. If in addition one assumes the period points of f are dense in $\Omega(f)$, then one gets what is called an Axiom A diffeomorphism. Proposition

(4.2) was first proved by Smale in [56] for Axiom A diffeomorphisms, and Corollary (4.3) was first proved in [16] for Axiom A diffeomorphisms. Proposition (4.1) (with a different proof) and the present treatment of (4.2) were given in [28].

Let us look at some examples.

1. This first example is an extension of the horseshoe map to an Axiom A diffeomorphism f on S^2 whose limit set consists of the previous conjugate of the 2-shift, $\Lambda(f)$, and two hyperbolic fixed points p_1 and p_2. We take p_2 to be a sink (attracting fixed point) as in figure (4.2), and p_1 to be a source (repelling fixed point) "at ∞."

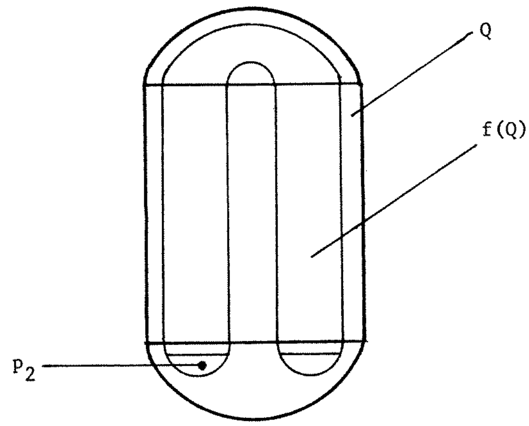


Figure 4.2

2. Anosov diffeomorphisms. These are diffeomorphisms f for which the whole manifold M is hyperbolic. The simplest examples are the following. Let $M = T^n = \mathbb{R}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus, and let $A : Z^n \to Z^n$ be an automorphism. Thus, A can be represented as an $n \times n$ matrix with integer entries and determinant ± 1. Assume A has no eigenvalues of norm 1. We think of A as a map of $\mathbb{R}^n$ to $\mathbb{R}^n$ also. There is a direct sum decomposition $\mathbb{R}^n = E^s \oplus E^u$ and a norm on

$\mathbb{R}^n$ so that $A(E^s) = E^s$, $A(E^u) = E^u$, $|A|E^s| < 1$, and $|A^{-1}|E^u| < 1$. Further, A induces a map $\bar{A} : T^n \to T^n$ and the splitting $E^s \oplus E^u$ projects to an $\bar{A}$-invariant splitting $\bar{E}^s \oplus \bar{E}^u$ with $\bar{A}|\bar{E}^s$ contracting and $\bar{A}|\bar{E}^u$ expanding. Hence, $\bar{A}$ is Anosov. The map $\bar{A}$ is called a linear toral automorphism. A simple example is given by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on $\mathbb{R}^2$

In all known cases, an Anosov diffeomorphism $f : M \to M$ has $L(f) = M$. But, this property has only been proved under additional restrictions on the topology of M. It holds if M is homeomorphic to a torus [23], on certain manifolds called infranil manifolds [23], or if dim E^s or E^u $= 1$ [27]. However, it can be proved that an Anosov diffeomorphism satisfies Axiom A ([17] or [28]). Three other additional problems on Anosov diffeomorphisms are:

a. If $f : M \to M$ is Anosov, is the universal covering space of M diffeomorphic to R^n?

b. Does every Anosov diffeomorphism have a fixed point?

c. Is every Anosov diffeomorphism topologically conjugate to an infranil manifold automorphism (see [11] for definition)?

3. Gradients. Let $\phi : M \to \mathbb{R}$ be a C^2 real-valued function with nondegenerate critical points. Given a C^∞ Riemann metric g on TM, one defines the gradient vector field $\text{grad}_g \phi$ by

$$g_x(\text{grad}_g\, \phi(x), Y) = T_x\phi(Y) \quad \text{for} \quad x \in M, \quad Y \in T_xM \quad .$$

It is easily seen that $\text{grad}_g \phi$ is a C^1 vector field. Its solution curves are the orthogonal trajectories of the level sets of ϕ. Let f be its time-one map.

Then $L(f)$ consists of hyperbolic fixed points and coincides with the critical points of ϕ.

Simple examples on the 2-sphere S^2 and the 2-torus T^2 are pictured below. We have drawn invariant curves.

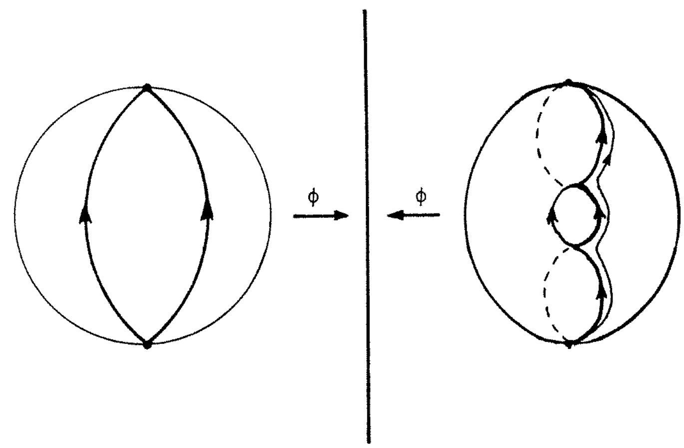


Figure 4.3

4. In the first three examples, the limit set was actually equal to the non-wandering set, so each f satisfied Axiom A. This example of f on S^2 has L(f) finitely many hyperbolic fixed points with $\Omega(f)$ infinite and not hyperbolic.

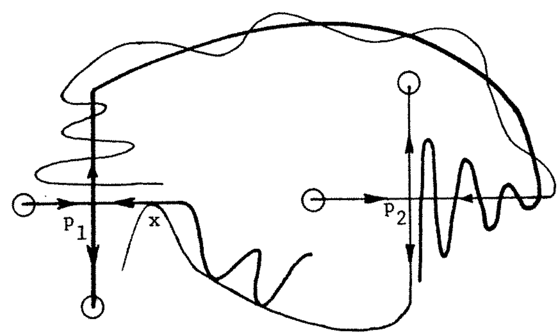


Figure 4.4

The circles represent sources and sinks and there are two saddle points p_1 and p_2. We have that $W^u(p_1) \cap W^s(p_2)$ consists of two orbits of transverse intersections, and $W^u(p_2) \cap W^s(p_1)$ consists of an orbit $o(x)$ of non-transverse intersections. Also, $\Omega(f) = o(x) \cup L(f)$.

General diffeomorphisms with $L(f)$ hyperbolic may be viewed as looking somewhat like these examples with complicated hyperbolic sets replacing the fixed points.

We now briefly consider the concepts of structural stability and Ω-stability. A diffeomorphism $f : M \to M$ is called *structurally stable* if there is a neighborhood N of f in $\mathrm{Diff}^1 M$ such that for each g in N, there is a homeomorphism $h_g : M \to M$ such that $h_g f = g h_g$. This means that the entire orbit structure of f persists under C^1 small perturbation. If f satisfies Axiom A, then one says f satisfies the strong transversality condition if $W^u(x)$ is transverse to $W^s(x)$ for each $x \in M$. Recall that $W^u(x)$ and $W^s(x)$ are manifolds in the Axiom A case.

Theorem (4.4) (Robbin, Robinson [50], [51]). *If* f *satisfies Axiom A and the strong transversality condition, then* f *is structurally stable.*

The extended horseshoe diffeomorphism in example 1 satisfies Axiom A and strong transversality, so it is structurally stable. It is amusing and non-trivial to try to prove directly that this diffeomorphism is structurally stable.

Theorem (4.4) had been proved by Palis and Smale in the case when $\Omega(f)$ is finite [69], and by Anosov for Anosov diffeomorphisms. We remark that if $L(f)$ is hyperbolic and f satisfies the strong transversality condition, then f satisfies Axiom A [28]. Hence, theorem (4.4) holds with these weaker assumptions. It has been conjectured that

Axiom A and strong transversality are necessary and sufficient for structural stability.

A weaker concept of stability than structural stability is that of Ω-stability. One says f in Ω-stable if there is a neighborhood N of f in $\text{Diff}^1 M$ such that for each $g \in N$ there is a homeomorphism $h_g : \Omega(f) \to \Omega(g)$ such that $h_g f = g h_g$.

If f satisfies Axiom A and $\Omega(f) = L(f) = \Lambda_1 \cup \ldots \cup \Lambda_n$ is the spectral decomposition, then a *cycle* is a sequence $\Lambda_{i_0}, \ldots, \Lambda_{i_r}$ such that $\Lambda_{i_0} = \Lambda_{i_r}$ and, for each $0 \leq j < r$, $W^u(\Lambda_{i_j}) \cap W^s(\Lambda_{i_{j+1}}) \neq \phi$.

Theorem (4.5) (Smale [57]). *If* f *satisfies Axiom A and has no cycles, then* f *is* Ω*-stable.*

Again, it is sufficient to assume that L(f). is hyperbolic and there are no cycles [28]. Also, in this case there is the conjecture that Axiom A and no cycles is equivalent to Ω-stability. In the direction of the converses to theorems (4.4) and (4.5) see Pliss [42], Mane [21], and theorem (1.2) in [30].

While there has been relatively little progress on the precise converses to theorems (4.4) and (4.5), it has been shown that somewhat stronger notions of stabilities do characterize Axiom A and strong transversality and Axiom A and no cycles. Say that f is *absolutely* Ω-stable if there are a neighborhood N of f in $\text{Diff}^1 M$ and a constant $k > 0$ so that for $g \in N$ there is a homeomorphism $h_g : \Omega(f) \to \Omega(g)$ such that $g h_g = h_g f$ and the C^0 distance from h_g to the inclusion $i : \Omega(f) \to M$ is less than k times the C^0 distance from f to g. Guckenheimer [67], extending earlier work of Franks [64], proved that an absolutely Ω-stable f must satisfy Axiom A and must have no cycles. The analogous result holds for absolute structural stability [65]. Another characterization of Axiom A and strong transversality is Franks' time-dependent stability [66].

The main importance of the Axiom A and no cycle diffeomorphisms is that, at present, they give the largest open set of diffeomorphisms whose orbit structures are well understood.

We conclude this section with a proof of a form of the Ω-stability theorem slightly more special than that in [28]. A more general version in which $L(f)$ is replaced by the closure of the set of recurrent points (Birkhoff center) is proved by I. Malta in [71].

If Λ is a hyperbolic basic set for an Axiom A diffeomorphism, then $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda$. This is not true if only $L(f)$ is hyperbolic, so we must modify our definition of cycle for the next theorem. If $L(f)$ is hyperbolic with spectral decomposition $L(f) = \Lambda_1 \cup \ldots \cup \Lambda_n$ then a *cycle* is a sequence $\Lambda_{i_0}, \Lambda_{i_1}, \ldots, \Lambda_{i_r}$ such that $\Lambda_{i_0} = \Lambda_{i_r}$, and for $0 \leq j < r$, $(W^u(\Lambda_{i_j}) - \Lambda_{i_j}) \cap (W^s(\Lambda_{i_{j+1}}) - \Lambda_{i_{j+1}}) \neq \emptyset$.

Theorem (4.6) If $L(f)$ *is hyperbolic and has no cycles, then* $L(f) = \Omega(f)$ *and* f *is* Ω*-stable.*

Let $L(f) = \Lambda_1 \cup \ldots \cup \Lambda_n$ be the spectral decomposition of $L(f)$. Write $\Lambda_i \geq \Lambda_j$ if $\Lambda_i = \Lambda_j$ or there is a sequence $\Lambda_{i_0}, \Lambda_{i_1}, \ldots, \Lambda_{i_r}$ such that $\Lambda_{i_0} = \Lambda_i$, $\Lambda_{i_r} = \Lambda_j$, and $(W^u(\Lambda_{i_k}) - \Lambda_{i_k}) \cap (W^s(\Lambda_{i_{k+1}}) - \Lambda_{i_{k+1}}) \neq \emptyset$ for $0 \leq k < r$. The no cycle condition ensures that $\geq$ is a partial ordering on $\{\Lambda_1, \ldots, \Lambda_n\}$.

Extend this to a total ordering on $\{\Lambda_1, \ldots, \Lambda_n\}$ which we still denote by $\geq$. Relabel so that $\Lambda_n \geq \Lambda_{n-1} \geq \cdots \geq \Lambda_1$.

Lemma (4.7). There is a sequence $M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$ *of compact subsets of* M *with the following properties*

(1) $f(M_i) \subset \text{int } M_i$

(2) $\Lambda_i \subset \text{int}(M_i - M_{i-1})$

(3) $\Lambda_i = \bigcap_{-\infty<j<\infty} f^j(M_i - M_{i-1})$

(4) $\bigcup_{j\leq i} W^u(\Lambda_j) = \bigcup_{j\leq i} C\ell\ W^u(\Lambda_j) = \bigcap_{m>0} f^m(M_i)$ *for* $0 \leq i < n$

A sequence $M_n \supset M_{n-1} \supset \dots \supset M_0$ of sets as in lemma (4.7) is called a *filtration* for $L(f)$. The sets M_i, $1 \leq i \leq n-1$, can be chosen to be smooth submanifolds with boundary, but this is not important for our purposes. For more general filtrations see [28], [34], and [71].

Let us defer the proof of (4.7) temporarily.

Proof of theorem (4.6): Let $M = M_n \supset M_{n-1} \supset \dots \supset M_0 = \emptyset$ be a filtration for $L(f)$ as in lemma (4.7), and let $U_i \subset \text{int}(M_i - M_{i-1})$ be a compact neighborhood of Λ_i satisfying the conclusions of Corollary (3.6) and theorem (3.7). Choose an integer $N > 0$ so that $\bigcap_{-N<j<N} f^j(M_i - M_{i-1}) \subset \text{int } U_i$ for each $i = 1,\dots,n$. From the properties of lemma (4.7), it is easy to see that $\Omega(f) \subset \bigcup_{i=1}^{n} U_i$ and that $\Omega(f) \cap U_i \subset \Lambda_i$. Thus, $\Omega(f) = L(f)$. Also, for g C^1 near f, we have $g(M_i) \subset \text{int } M_i$ and $\bigcap_{-N<j<N} g^j(M_i - M_{i-1}) \subset U_i$, so $\bigcap_{-\infty<j<\infty} g^j(M_i - M_{i-1}) = \bigcap_{-\infty<j<\infty} g^j(U_i)$. Set $\Lambda_i = \Lambda_i(f) = \bigcap_{-\infty<j<\infty} f^j(U_i)$, and let $\Lambda_i(g) = \bigcap_{-\infty<j<\infty} g^j(U_i)$. Then, we have $\Omega(g) \subset \bigcup_{i=1}^{n} \Lambda_i(g)$. By theorem (3.7) there are homeomorphisms $h_i : \Lambda_i(f) \to \Lambda_i(g)$ such that $h_i f = g h_i$. Since the periodic points of f are dense in $\bigcup_i \Lambda_i(f)$, the same holds for g on $\bigcup_i \Lambda_i(g)$, so $\Omega(g) = \bigcup_{i=1}^{n} \Lambda_i(g)$. Also, the h_i's provide a conjugacy between $f|\Omega(f)$ and $g|\Omega(g)$.

We now proceed toward the proof of lemma (4.7). We need some preparatory lemmas.

Lemma (4.8) (Smale). Suppose F *is a compact* f*-invariant set and* Q *is a compact neighborhood of* F *such that* $\bigcap_{m\geq 0} f^m(Q) = F$. *Then there is a compact neighborhood* V *of* F *such that* $V \subset \text{int } Q$ *and* $f(V) \subset \text{int } V$.

Proof. Let $A_r = Q \cap fQ \cap \ldots \cap f^rQ$, $r \geq 0$. Then $A_0 \supset A_1 \supset \ldots$ and $\bigcap_{i\geq 0} A_i = F$. Since $f(F) \subset F$, there is an integer $r > 0$ such that $A_r \subset \text{int } Q$ and $f(A_r) \subset \text{int } Q$. But then, $f(A_r) = f(A_r) \cap Q = A_{r+1} \subset A_r$, and an easy induction gives $f^jA_r = A_{r+j}$ for $j \geq 0$. Hence there is an integer $s > 0$ such that $f^s(A_r) \subset \text{int } A_r$. Let W_{s-1} be a compact neighborhood of $f^{s-1}A_r$ such that $W_{s-1} \subset \text{int } Q$ and $f\,W_{s-1} \subset \text{int } A_r$. Continue inductively defining a compact neighborhood W_j of f^jA_r such that $W_j \subset \text{int } Q$ and $fW_j \subset \text{int } W_{j+1}$ for $0 \leq j < s$. Setting $V = W_0 \cup W_1 \cup \ldots \cup W_{s-1}$ gives us $f(V) \subset \text{int } V$ and $V \subset \text{int } Q$ as required.

In the next two lemmas, we assume $L(f) = \Lambda_1 \cup \ldots \cup \Lambda_n$ is hyperbolic with no cycles.

Lemma (4.9). If $\Lambda_i \neq \Lambda_j$, *then the following statements are equivalent*

(a) $C\ell\, W^u(\Lambda_i) \cap \Lambda_j \neq \emptyset$.

(b) $C\ell\, W^u(\Lambda_i) \cap (W^s(\Lambda_j) - \Lambda_j) \neq \emptyset$.

(c) $C\ell\, W^u(\Lambda_i) \cap (W^u(\Lambda_j) - \Lambda_j) \neq \emptyset$.

Proof. We first prove (b) $\Longrightarrow$ (a). If $z \in C\ell(W^u(\Lambda_i) \cap (W^s(\Lambda_j) - \Lambda_j))$, then f^nz approaches Λ_j as $n \to \infty$. Since $f^n C\ell\, W^u(\Lambda_i) = C\ell\, W^u(\Lambda_i)$ for each n, this gives us $C\ell\, W^u(\Lambda_i) \cap \Lambda_j \neq \emptyset$ which is (a).

Now we prove (a) $\Longrightarrow$ (b).

Let U_j be a compact neighborhood of Λ_j such that $\bigcap_{m\leq 0} f^m U_j \subset W^s(\Lambda_j)$ as given by Corollary (3.6). Set $U_j^s = \bigcap_{m\leq 0} f^m U_j$ and let $F = \mathcal{C}\ell(U_j^s - fU_j^s)$. First note that $F \neq \emptyset$. Indeed, if F were empty, then $U_j^s = fU_j^s$ whence $W^s(\Lambda_j) = \Lambda_j$ and $W^u(\Lambda_j)$ is a neighborhood of Λ_j in M. Since $\Lambda_i \neq \Lambda_j$, $W^u(\Lambda_i) \cap W^u(\Lambda_j) = \emptyset$, so (a) could not hold. Thus $F \neq \emptyset$. Next we claim that if V is any neighborhood of F in M, then $\bigcup_{m\geq 0} f^m(V) \cup W^u(\Lambda_j)$ is a neighborhood of Λ_j in M. From this, (a) implies that $W^u(\Lambda_i) \cap V \neq \emptyset$. As V is any neighborhood of F, this implies (b). We must prove the claim. Suppose, by way of contradiction, that $x_1, x_2, \ldots$ is a sequence in M converging to a point of Λ_j such that $x_k \notin \bigcup_{m\geq 0} f^m(V) \cup W^u(\Lambda_j)$ for each $k \geq 1$. Since $x_k \notin W^u(\Lambda_j)$, there is an integer $n_k \geq 0$ such that $f^{-n_k}(x_k) \notin U_j$. Let n_k be the least such integer. Because Λ_j is f-invariant we have that $n_k \to \infty$ as $k \to \infty$. Choose a subsequence of $\{f^{-n_k}(x_k)\}$, say $f^{-n_{k_1}}(x_{k_1})$, $f^{-n_{k_2}}(x_{k_2}), \ldots$ which converges to a point z in M. By the definition of n_k, one has $f^m z \in U_j$ for $m \geq 1$, or $fz \in U_j^s$. Also, $z \in \mathcal{C}\ell\,(M - U_j)$, so $fz \in \mathcal{C}\ell(M - fU_j) \cap U_j^s \subset F$. This means that, for large ν, $f^{-n_{k_\nu}+1}(x_{k_\nu}) \in V$ or $x_{k_\nu} \in f^{n_{k_\nu}-1}(V)$. This contradicts the assumption that $x_{k_\nu} \notin \bigcup_{n\geq 0} f^n(V)$ and proves that (a) $\Longrightarrow$ (b). The proof that (a) $\Longrightarrow$ (c) is similar and will be omitted.

Lemma (4.10). If $\mathcal{C}\ell\, W^u(\Lambda_i) \cap \Lambda_j \neq \emptyset$, *then* $\Lambda_i \geq \Lambda_j$.

<u>Proof</u>. Suppose $\mathcal{C}\ell\, W^u(\Lambda_i) \cap \Lambda_j \neq \emptyset$. If $\Lambda_i = \Lambda_j$, there is nothing to prove, so assume $\Lambda_i \neq \Lambda_j$. By lemma (4.9), there is a point

z_1 in $C\ell\, W^u(\Lambda_i) \cap (W^s(\Lambda_j) - \Lambda_j)$. By Corollary (4.3), we may find a Λ_{i_1} such that $z_1 \in W^u(\Lambda_{i_1})$. Thus, $\Lambda_{i_1} \geq \Lambda_j$. Because there are no cycles, $\Lambda_{i_1} \neq \Lambda_j$. If $\Lambda_{i_1} = \Lambda_i$, we are done. If not, we may again apply lemma (4.8) to get $C\ell\, W^u(\Lambda_i) \cap (W^s(\Lambda_{i_1}) - \Lambda_{i_1}) \neq \emptyset$. Repeating the process, we get Λ_{i_2} such that $\Lambda_{i_2} \geq \Lambda_{i_1}$, $\Lambda_{i_2} \neq \Lambda_{i_1}$, and $C\ell\, W^u(\Lambda_i) \cap \Lambda_{i_2} \neq \emptyset$. If $\Lambda_{i_2} = \Lambda_i$, we are done. If not, we continue. Since there are only finitely many Λ_k's and there are no cycles, the process must terminate with some $\Lambda_{i_\nu} = \Lambda_i$ and $\Lambda_{i_\nu} \geq \Lambda_j$.

<u>Proof of lemma 4.7</u>. Consider the ordered set $\Lambda_n \geq \Lambda_{n-1} \geq \cdots \geq \Lambda_1$. By (4.10), we have $i > j$ implies $C\ell\, W^u(\Lambda_j) \cap \Lambda_i = \emptyset$. This implies that $C\ell\, W^u(\Lambda_1) = \Lambda_1$. Let Q_1 be a compact neighborhood of $C\ell\, W^u(\Lambda_1) = \Lambda_1$ not meeting $\bigcup_{i>1} \Lambda_i$. Then, if $x \in \bigcap_{m>0} f^m Q_1$, it follows that $f^{-m}x \in Q_1$ from $m \geq 0$, so the α-limit set of x, $\alpha(x)$, is in $Q_1 \cap L(f) = \Lambda_1$. Therefore, $x \in W^u(\Lambda_1) \subset \Lambda_1$, so $\bigcap_{m\geq 0} f^m(Q_1) = \Lambda_1$. By (4.8), there is a compact neighborhood M_1 of Λ_1 such that $M_1 \subset \text{int}\, Q_1$ and $f(M_1) \subset \text{int}\, M_1$. Thus, $\bigcap_{m\geq 0} f^m M_1 \subset \bigcap_{m\geq 0} f^m Q_1 = \Lambda_1$. With $M_0 = \emptyset$, (4.7.1), (4.7.2), (4.7.3), and (4.7.4) hold for $i = 1$. Inductively, suppose M_i has been constructed so that (4.7.1), (4.7.2), (4.7.3) and (4.7.4) hold for i. Now we may choose a compact neighborhood Q_{i+1} of $C\ell W^u(\Lambda_{i+1})$ not meeting $\bigcup_{j>i+1} \Lambda_j$. By (4.7.3) for i, $M_i \cap (\bigcup_{j>i} \Lambda_j) = \emptyset$, so if $x \in \bigcap_{m\geq 0} f^m(Q_{i+1} \cup M_i)$, then $\alpha(x) \subset \bigcup_{j\leq i+1} \Lambda_j$. Thus, $x \in \bigcup_{j\leq i+1} W^u(\Lambda_j) \subset \text{int}(Q_{i+1} \cup M_i)$ by (4.7.4) for i. Using (4.8), there is a compact neighborhood M_{i+1} of $\bigcup_{j\leq i+1} C\ell\, W^u(\Lambda_j)$ such that $M_{i+1} \subset \text{int}(Q_{i+1} \cup M_i)$ and $f(M_{i+1}) \subset \text{int}\, M_{i+1}$. Thus, (4.7.1) holds for $i+1$. Since $\Lambda_{i+1} \cap M_i = \emptyset$,

$\Lambda_{i+1} \subset \text{int}\,(M_{i+1} - M_i)$, and, since Λ_{i+1} is invariant, $\Lambda_{i+1} \subset \bigcap_j f^j(M_{i+1} - M_i)$. On the other hand, if $x \in \bigcap_j f^j(M_{i+1} - M_i)$, then $\alpha(x) \subset \Lambda_{i+1}$ and $\omega(x) \subset \Lambda_{i+1}$, so $x \in W^u(\Lambda_{i+1}) \cap W^s(\Lambda_{i+1})$. Since there are no cycles, $W^u(\Lambda_{i+1}) \cap W^s(\Lambda_{i+1}) = \Lambda_{i+1}$, so (4.7.2) and (4.7.3) hold for $i + 1$. Further, $Q_{i+1} \cup M_i$ is a neighborhood of $\bigcup_{j \le i+1} \mathcal{C}\ell\, W^u(\Lambda_j)$ and we have seen that $\bigcap_{m \ge 0} f^m(Q_{i+1} \cup M_i) \subset \bigcup_{j \le i+1} W^u(\Lambda_j)$. Since $\bigcap_{m \ge 0} f^m(Q_{i+1} \cup M_i) = \bigcap_{m \ge 0} f^m(M_{i+1})$, (4.7.4) holds for $i + 1$.

This completes the proof of lemma (4.7).

invariant,

if $x \in \bigcap_j f^j(M_{i+1} - M_i)$,

$W^u(\Lambda_{i+1}) \cap W^s(\Lambda_{i+1})$. Since

$_{i+1}$, so (4.7.2) and (4.7.3)

eighborhood of

$(Q_{i+1} \cup M_i) \subset \bigcup_{j \leq i+1} W^u(\Lambda_j)$.

.4) holds for i + 1.

5. Attractors - topology

Let $f : M \to M$ be a C^1 diffeomorphism. A closed f-invariant set Λ is an *attractor* if there is a compact neighborhood U of Λ such that $f(U) \subset \text{int } U$, $\bigcap_{n \geq 0} f^n(U) = \Lambda$, and $f|\Lambda$ has a dense orbit. Thus, for every $x \in U$, $\omega(x) \subset \Lambda$ where $\omega(x)$ is the ω-limit set of x. The open set $W^s(\Lambda) = \{y \in M : \omega(y) \subset \Lambda\} = \bigcup_{n \geq 0} f^{-n}U$ is called the *basin* of Λ. For a given diffeomorphism it is important to describe its attractors. They give the time evolution of certain open sets in M.

If an attractor Λ is hyperbolic then one has considerable information about its structure. Let us begin with a few examples.

The simplest example is, of course the orbit of a periodic sink. This is the orbit $o(p)$ of a periodic point p of a period n such that $T_p f^n$ has all its eigenvalues of norm less than 1. In this case there is a small neighborhood U of p such that $f^n(U) \subset \text{int } U$, and $W^s(o(p)) = \bigcup_{m \geq 0} f^{-m}(\bigcup_{j=0}^{n-1} f^j U)$ is a union of n open cells which are permuted by f.

At the other end of the spectrum, we have the topologically transitive Anosov diffeomorphisms which were described in the last section.

A third and intriguing example is known as the solenoid.

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane, and let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit 2-disk.

Consider the mapping $f : S^1 \times D^2 \to S^1 \times D^2$ defined by $f(z,w) = (z^2, \frac{z}{2} + \frac{w}{4})$. If we think $S^1 \times D^2$ as the solid torus in $\mathbb{R}^3$, then $f(S^1 \times D^2) \subset \text{int}(S^1 \times D^2)$ and f wraps $S^1 \times D^2$ around itself twice as in figure 5.1.

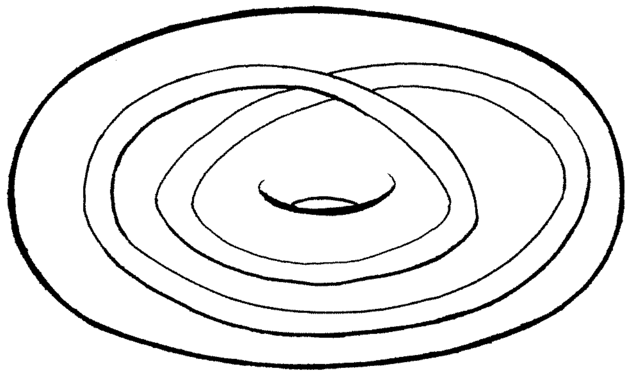


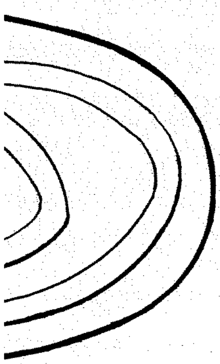
Figure 5.1

Also, $f(\{z\} \times D^2) \subset \{z^2\} \times D^2$, and $f(S^1 \times D^2) \cap \{z^2\} \times D^2$ consists of the two disks $f(\{z\} \times D^2) \cup f(\{-z\} \times D^2) = \{(z^2, \frac{z}{2} + \frac{w}{4}) = |w| \leq 1\}$ $\cup\{(z^2, -\frac{z}{2} + \frac{w}{4}) : |w| \leq 1\}$. Using theorem (2.2) it is easy to verify that $\Lambda = \bigcap_{n \geq 0} f^n(S^1 \times D^2)$ is a hyperbolic set. Also, for each $(z,w) \in \Lambda$, $W^s_{1/2}(z,w)) \subset \{z\} \times D^2 \subset W^s((z,w))$.

It is easy to see that $f|\Lambda$ has a dense orbit using proposition (3.1). For this, it suffices to show that if U and V are open sets in $S^1 \times D^2$ which meet Λ, then for some $n \geq 0$, $f^n(U \cap \Lambda) \cap V \cap \Lambda \neq \phi$. Toward this end, let $g(z) = z^2$ for $z \in S^1$. Note that

(a) If I is an interval in S^1, there is an integer $N > 0$ such that $n \geq N$ implies $g^n(I) = S^1$.

Let $r : S^1 \times D^2 \to S^1$ be the projection. Let $(z,w) \in V \cap \Lambda$. Taking f^{-i} for $i > 0$, expands $W^s_\varepsilon((z,w))$ for small ε, so we may



$^2) \cap \{z^2\} \times D^2$ consists of

$(z^2, \frac{z}{2} + \frac{w}{4}) = |w| \le 1\}$

.2) it is easy to verify

:. Also, for each

orbit using proposition

U and V are open sets

≥ 0, $f^n(U \cap \Lambda) \cap V \cap \Lambda \neq \phi$.

Note that

is an integer $N > 0$

1.

Let $(z,w) \in V \cap \Lambda$.

or small ε, so we may

choose an integer $n_1 > 0$ and a point $z_1 \in g^{-n_1}(z)$ such that $f^{n_1}(\{z_1\} \times D^2) \subset V$. Now pick a point $(z',w') \in U \cap \Lambda$ and a small $\varepsilon > 0$ such that $W^u_\varepsilon((z',w')) \subset U \cap \Lambda$. Since $rW^u_\varepsilon((z',w'))$ is an interval in S^1, we can find an $n_2 > 0$ such that $rf^{n_2}W^u_\varepsilon((z',w')) = S^1$. Then, there is a point ρ in $f^{n_2}W^u_\varepsilon((z',w')) \cap \{z_1\} \times D^2$. Hence, $f^{-n_2}\rho \in U \cap \Lambda$ and $f^{n_1}\rho \in V \cap \Lambda$, so $f^{n_1+n_2}(U \cap \Lambda) \cap V \cap \Lambda \neq \phi$ as required.

The construction of the solenoid (as a hyperbolic attractor) is due to Smale [56].

It leads to a general construction of one dimensional hyperbolic attractors due to Williams [61]. To describe this construction we need some definitions.

Let $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} \phi_1(u) &= e^{-\frac{1}{u^2}}, \quad u \neq 0 \\ &= 0, \quad u = 0 \\ \phi_2(u) &= e^{-\frac{1}{u^2}}, \quad u > 0 \\ &= 0, \quad u \le 0 \end{aligned}$$

Let S be the set of functions $\beta \cdot \phi_1$, $\beta \cdot \phi_2$ where β varies through the real numbers.

The graphs of elements of S have infinite contact with $\{(u,v) \in \mathbb{R}^2 : v = 0\}$ at $(0,0)$.

A compact branched 1-manifold K is compact Hausdorff topological space satisfying the following property. There is a *finite* subset $B \subset K$ such that each point x in B has a neighborhood which is homeomorphic to a finite union of graphs of elements of S, and each $x \in K - B$ has a neighborhood which is homeomorphic to a real open interval.

Typical pictures of branched 1-manifolds are in Figure 5.2.

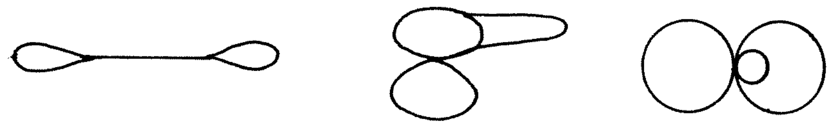


Figure 5.2

The set B in K is called the branch set of K.

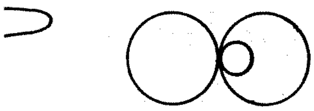
There is a finite open covering $\{U_\alpha\}$ of K such that for each U_α there is a homeomorphism $\psi_\alpha : U_\alpha \to Y_\alpha$ where $Y_\alpha = Y_1^\alpha \cup \ldots \cup Y_n^\alpha$ and $Y_i^\alpha = \{(u, \phi_i^\alpha(u)) \in \mathbb{R}^2\}$ where $\phi_i^\alpha \in S$ for $i = 1,\ldots,n$. The open intervals correspond to $\phi_i^\alpha \equiv 0$.

The family $\{(U_\alpha, \psi_\alpha)\}$ defines a C^r differentiable structure on K as usual by saying that a function $f : K \to \mathbb{R}$ is C^r if it is continuous and for each α, $f \circ \psi_\alpha^{-1}$ extends to a C^r function from neighborhood V_α of $\psi_\alpha(U_\alpha)$ to $\mathbb{R}$. Since the graphs of two elements of S have infinite order contact at any point where they meet, K has a well-defined tangent bundle. One defines Riemann metrics and C^r maps between C^r branched 1-manifolds and other manifolds as usual. Every compact C^r branched 1-manifold can be C^r embedded in $\mathbb{R}^3$ for $r \geq 0$.

We now always assume that K is a compact C^1 branched 1-manifold with a fixed Riemann metric. Let $|\cdot|$ be the induced norm on TK.

An expanding map of K is a C^1 map $g : K \to K$ so that there are constants $c > 0$, $\lambda > 1$ so that $|T_x(g^n)(v)| \geq c\,\lambda^n |v|$ for all $x \in K$, $n \geq 0$, and $v \in T_xK$.

s are in Figure 5.2.



set of K.

of K such that for each

where $Y_\alpha = Y_1^\alpha \cup \dots \cup Y_n^\alpha$ and

= 1,...,n. The open inter-

fferentiable structure on K

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C^r function from neigh-

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mann metrics and C^r maps

manifolds as usual. Every

embedded in $\mathbb{R}^3$ for $r \geq 0$.

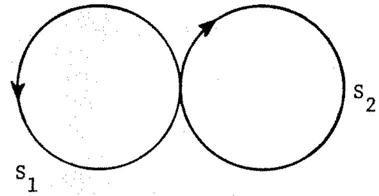
ct C^1 branched 1-manifold

e induced norm on TK.

: $K \to K$ so that there are

$| \geq c\,\lambda^n |v|$ for all

For example, if $g(z) = z^n$, $n > 0$ on $K = S^1$, then g is expanding. Another example is to take K to be a wedge of two circles, say $S_1 \cup S_2$ with $S_1 \cap S_2 = \{p\}$, and let g be the map g described in figure 5.3.



$$S_1 \to S_1 + S_2 + S_1$$

$$S_2 \to S_1 + S_2$$

Figure 5.3

With the indicated orientations, g doubles lengths on S_2, triples lengths on S_1, and $g(p) = p$. This expanding is intimately related to the Anosov diffeomorphism induced by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

If $g : K \to K$ is a map, we define the inverse limit of g to be the set $\hat{K} = \{(a_0, a_1, \dots) : a_i \in K$ and $g(a_{i+1}) = a_i\}$. Then $\hat{K}$ is a closed set in the product of K with itself countably many times and we give $\hat{K}$ the relative topology. One thinks of a point $\underline{a} = (a_0, a_1, \dots)$ of $\hat{K}$ as a point $a_0 \in K$ together with a choice of pre-images a_1, a_2, a_3, etc. One frequently writes $\hat{K} = \underleftarrow{\lim}\, K \xleftarrow{g} K$. There is a natural projection $\pi : \hat{K} \to K$ defined by $\pi(\underline{a}) = a_0$. The map $g : K \to K$ induces a homeomorphism $\hat{g} : \hat{K} \to \hat{K}$ defined by $\hat{g}((a_0, a_1, \dots)) = (ga_0, a_0, a_1, \dots)$ with inverse $\tau((a_0, a_1, \dots)) = (a_1, a_2, \dots)$. In a precise sense, $\hat{g}$ unfolds g into a homeomorphism. Now, one can prove that if $g : S^1 \to S^1$ is the map $g(z) = z^2$, then $\hat{g}$ is topologically equivalent to $f|\Lambda$ where f is the solenoid map above. To see this, note that if $r : S^1 \times D^2 \to S^1$ is the projection, then $rfr^{-1} = g$. If $z \in \Lambda = \bigcap_{n \geq 0} f^n(S^1 \times D^2)$, set $a_n(z) = rf^{-n}z$ for $n \geq 0$. Then, letting

$\phi(z) = (a_0(z), a_1(z), \ldots)$, we get that $\phi : \Lambda \to \hat{S}^1$ is a continuous map. Since $f|\{z\} \times D^2$ is a contraction, we have that $\bigcap_{n\geq 0} f^n(\{a_n\} \times D^2)$ is a single point for any $\underline{a} = (a_0, a_1, \ldots) \in \hat{S}^1$. Let $\{h(a)\} = \bigcap_{n\geq 0} f^n(\{a_n\} \times D^2)$. Then h is also continuous, and $h \circ \phi = id_\Lambda$, $\phi \circ h = id_{\hat{S}^1}$. As $\phi f = \hat{g}\phi$, we see that ϕ is a topological conjugacy.

Williams has given general theorems of this type.

Theorem 5.1 (Williams [61]). Suppose that f *is a* C^1 *diffeomorphism having a 1-dimensional hyperbolic attractor* Λ *with splitting* $T_\Lambda M = E^s \oplus E^u$ *and* $\dim E^u = 1$. *Then there is an expanding map* g *of a branched 1-manifold* K *such that* $f|\Lambda$ *is topologically conjugate to the inverse limit map* $\hat{g}$.

In [61], Williams assumed that the stable manifold foliation was C^1. It is well-known now that this assumption can be removed as follows. Let U be a small fundamental neighborhood of Λ. Then $\bigcap_{n>0} f^n U = \Lambda$. Approximate f by f_1 so that f_1 is C^2. By theorem (3.7), $f_1|\bigcap_n f_1^n(U)$ is topologically conjugate to $f|\Lambda$. By theorem (6.5) in [17], the stable manifold foliation of f_1 on U is C^1.

There is a converse to theorem (5.1).

Theorem (5.2) (Williams [61]). Let $g : K \to K$ *be an expanding map of a branched 1-manifold* K *such that*

(a) every point of K *is non-wandering*

and

(b) each point in K *has a neighborhood whose image by a power* g^m *of* g *is an arc.*

Then there is a diffeomorphism $f : S^4 \to S^4$ *which has a hyperbolic attractor on which it is topologically conjugate to* $\hat{g}$.

The idea of the proof of theorem (5.2) is as follows. First embed K in $\mathbb{R}^3$. via $\phi : K \to \mathbb{R}$ and let N be a "tubular neighborhood" of $\phi(K)$. This is a 2-disk bundle over $\phi(K)$ where the corners at branch points of $\phi(K)$ have been rounded off to look like pants legs as in figure 5.4.

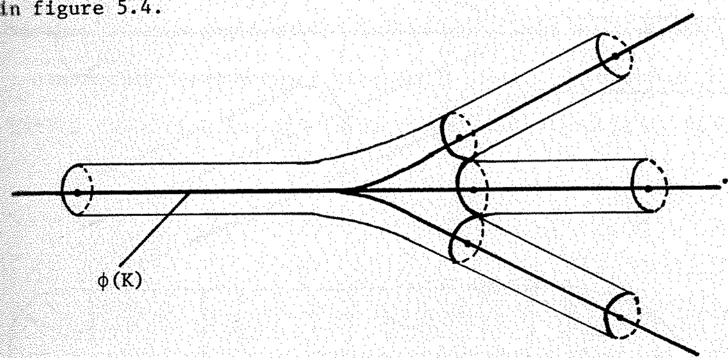


Figure 5.4

One may write N as a union of 2-disks $\{D_x\}$, $x \in \phi(K)$, in which at most two 2-disks have a point in common. If one forms the quotient space by identifying to a point any two 2-disks which intersect, one gets a space K_1 homeomorphic to K. Let $\pi : N \to K_1$ be the identification map.

The map $\phi \circ g : K \to N$ may be approximated by an embedding $g_1 : K \to N$, and the map $g_1 \circ \phi^{-1} : \phi(K) \to N$ extends to a diffeomorphism g_2 from N into its interior so that $g_2(D_x) \subset D_{\phi \circ g \circ \phi^{-1}(x)}$ and g_2 contracts D_x for each $x \in \phi(K)$.

We picture part of the image of g_2 in figure 5.5.

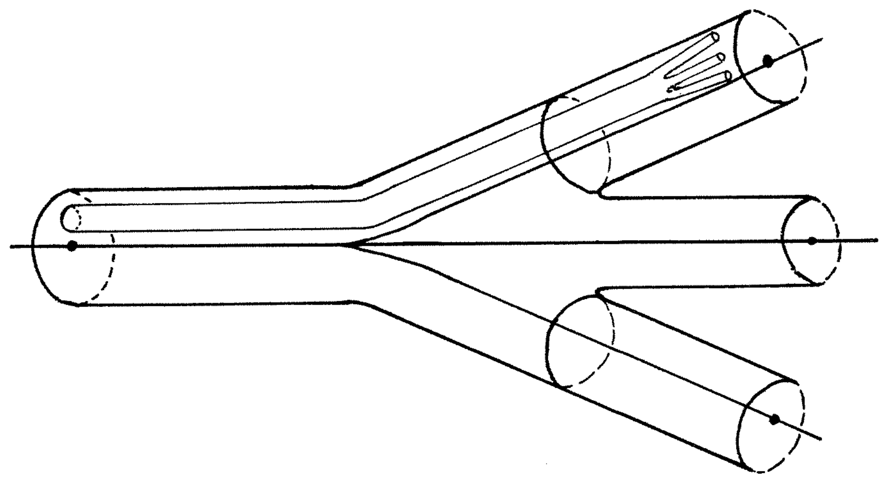
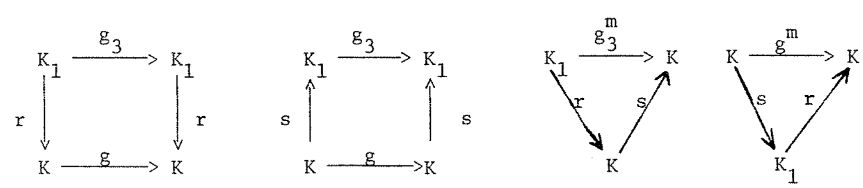


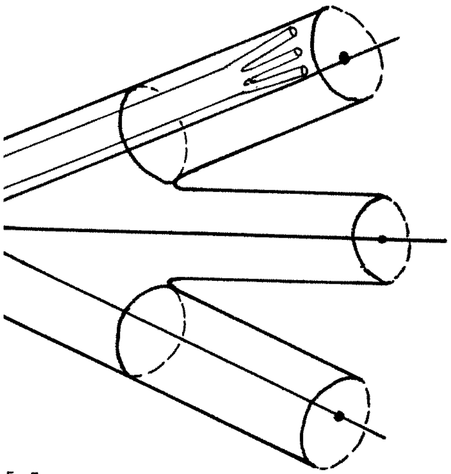
Figure 5.5

Then $\bigcap_{n\geq 0} g_2^n(N) = \Lambda(g_2)$ is a hyperbolic set for g_2. Also, if $\pi x = \pi y$ for $x, y \in N$, then $\pi g_2 x = \pi g_2 y$, so g_2 induces a map $g_3 : K_1 \to K_1$. It is easy to see that g_3 is an expanding map, and that the inverse limit $\hat{g}_3$ is topologically conjugate to $g_2|\Lambda(g_2)$. That $\hat{g}_3$ is also topologically conjugate to $\hat{g}$ requires more work.

We have $K_1 \xrightarrow{g_3} K_1$ and $K \xrightarrow{g} K$ two expanding maps and we wish to know when $\hat{g}_3$ and $\hat{g}$ are topologically conjugate. Williams shows that a sufficient condition is that there exist continuous mappings $r : K_1 \to K$ and $s : K \to K_1$ and an integer $m \geq 1$ such that $gr = rg_3, g_3 s = sg$, $sr = g_3^m$, and $rs = g^m$. The conditions can be expressed as the following commutative diagrams.



In the case of our maps g, g_3, an $m \geq 1$ and maps r,s can be found,



5.5

olic set for g_2. Also, if $\pi x = \pi y$ g_2 induces a map $g_3 : K_1 \to K_1$. It ding map, and that the inverse limit $|\Lambda(g_2)$. That $\hat{g}_3$ is also topo- ore work.

$\xrightarrow{g} K$ two expanding maps and we pologically conjugate. Williams that there exist continuous mappings nteger $m \geq 1$ such that $= g^m$. The conditions can be ex- diagrams.

1

s

$$K_1 \xrightarrow{g_3^m} K, \quad r : K_1 \to K, \quad s : K \to K \qquad\qquad K \xrightarrow{g^m} K, \quad s : K \to K_1, \quad r : K_1 \to K$$

≥ 1 and maps r,s can be found,

so $\hat{g}_3$ is conjugate to $\hat{g}$. See [61] for more details.

Since $g_2 : N \to \mathbb{R}^3$ is homotopic to the inclusion $i : N \to \mathbb{R}^3$, if we choose an embedding $\psi : \mathbb{R}^3 \to S^4$, then $\psi \circ g_2 \circ \psi^{-1} | \psi(N)$ extends to a diffeomorphism of S^4 by standard techniques in differential topology.

Let use give one more example of a 1-dimensional attractor. This is a variant of an example due to Plykin [43]. His was the first example of a 1-dimensional hyperbolic attractor in the two dimensional disk.

Let D be a disk in $\mathbb{R}^2$ with three holes foliated as in figure 5.6.

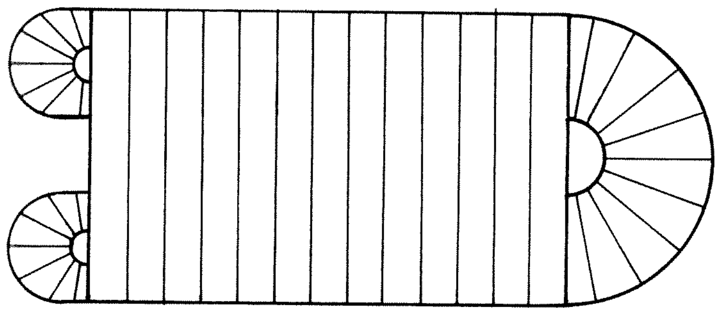


Figure 5.6

We define a diffeomorphism f from D into its interior to preserve the foliation and have f(D) as in figure 5.7.

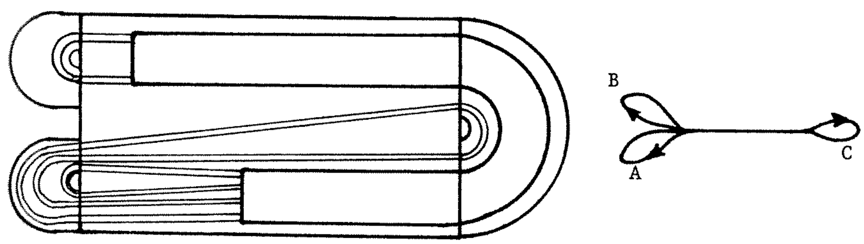


Figure 5.7

The branched manifold is a union of 3 circles and the map (on homology) is

$$A \longmapsto A + C - A$$
$$B \longmapsto A$$
$$C \longmapsto B$$

Remarks 1. One can use this example to show that non-trivial hyperbolic attractors (for flows) appear in arbitrarily small perturbations of constant vector fields on tori T^n of dimension greater than 2 [36]. As a consequence, hyperbolic attractors appear in perturbations of three or more coupled harmonic oscillators, or three or more coupled relaxation oscillators. To be more explicit, recall that a harmonic oscillator has equation $m\ddot{x} + kx = 0$ or $\begin{array}{l}\dot{x} = v \\ m\dot{v} = -kx\end{array}$ where m and k are positive constants. If we have n such oscillators, we obtain the system

$$(1) \qquad \begin{array}{l}\dot{x}_i = v_i \\ m_i\dot{v}_i = -k_i x_i\end{array} \qquad 1 \leq i \leq n$$

on $\mathbb{R}^{2n}$. There is a stable equilibrium at the origin, and all other orbits lie on n-dimensional invariant tori.

A relaxation oscillator is a differential equation of the form $\ddot{x} + f(x)\dot{x} + x = 0$ where, for some constant $k > 0$, $f(x) < 0$ for $|x| < k$ and $f(x) > 0$ for $|x| > k$. For example, if $f(x) = \mu(x^2 - 1)$, $\mu > 0$, one has Vander Pol's equation which comes up in Vacuum tube circuits, (see e.g. [59]). Under certain conditions on f (as in theorem 10.2 in [15]), the system $\begin{array}{l}\dot{x} = v - \int^x f(u)du \\ \dot{v} = -x\end{array}$ has a single asymptotcially stable periodic solution. If one has n such systems, one gets the system

$$(2) \qquad \begin{aligned} \dot{x}_i &= v_i - \int^{x_i} f_i(u)\,du \\ \dot{v}_i &= -x_i \end{aligned} \qquad 1 \leq i \leq n$$

on $\mathbb{R}^{2n}$. This system has a unique invariant attracting n-torus.

If $n \geq 3$, there are small perturbations of both systems (1) and (2) which possess non-trivial hyperbolic attractors. As J. Ford pointed out to us, a recent paper in Science [12] gives related experimental results. In particular, the broad band noise spectrum in figure 2a of [12] may be due to a non-trivial hyperbolic attractor.

2. Williams has extended theorems (5.1) and (5.2) to higher dimensional "expanding" attractors [63], and has given general conditions for topological equivalence of one dimensional attractors [62].

6. Attractors - ergodic theory

We begin with some notions from ergodic theory. Let $f : X \to X$ be a homeomorphism of the compact metric space X. Let $M(f)$ be the set of invariant Borel probability measures μ on X. That is, $\mu \in M(f)$ is a regular Borel non-negative measure on X such that $\mu(X) = 1$, and $\mu(f^{-1}B) = \mu(B)$ for every Borel set B. Let $\mathcal{B}$ be the σ-field of Borel sets in X. The measure μ is called *ergodic* if whenever $B \in \mathcal{B}$ and $f(B) = B$, we have $\mu(B) = 0$ or 1. That is, any invariant μ-measurable set has measure zero or one. An equivalent condition can be given in terms of real valued functions $\phi : X \to R$. Such a function $\phi : X \to R$ is *invariant* if $\phi \circ f = \phi$. This means that ϕ is constant on orbits of f. Then μ is ergodic if and only if any invariant function ϕ in $L^1(\mu)$ is constant almost everywhere. That is, if $\phi \in L^1(\mu)$ satisfies $\phi \circ f = \phi$, then $\phi(x) = \int \phi d\mu$ for μ-almost all x. From the Riesz representation theorem, one may think of $M(f)$ as a subset of the dual space $C(X)^*$ where $C(X)$ is the space of continuous real-valued functions on X, and we set $\mu(\phi) = \int \phi d\mu$ for $\phi \in C(X)$. This gives a topology on $M(f)$ (called the weak or vague topology) so that a sequence $\mu_i \in M(f)$ converges to $\mu \in M(f)$ if and only if $\mu_i(\phi) \to \mu(\phi)$ for each $\phi \in C(X)$. With this topology, $M(f)$ becomes a compact metrizable space. It is also a convex subset of $C(X)^*$, and the extreme points of $M(f)$ are the ergodic invariant measures of f. Note that $M(f)$ is nonempty because if δ_y is the point mass at y, then any weak limit of a subsequence of

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x}\right)_{n \geq 1} \quad \text{is in } M(f).$$

The most basic result of ergodic theory is the following theorem.

Theorem (6.1) (Birkhoff ergodic theorem). Let f *be a homeomorphism of the compact metric space* X*, and let* $\mu \in M(f)$. *For any* $\phi \in L^1(\mu)$, *there is a set* $A \subset X$ *of* μ*-measure* 1 *such that for* $x \in A$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x)$ *exists. Moreover, if we set* $\hat{\phi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x)$ *for* $x \in A$, *then* $\hat{\phi} \in L^1(\mu)$, $\hat{\phi}\circ f = \hat{\phi}$, *and* $\int \hat{\phi} d\mu = \int \phi d\mu$.

If the measure μ in theorem (6.1) is ergodic, then the function $\hat{\phi}$ must be constant μ-almost everywhere, so $\hat{\phi}(x) = \int \hat{\phi} d\mu = \int \phi d\mu$ almost everywhere. Thus, for μ-almost all x, the time-averages $\frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x)$ along the orbit of x approach the space average $\int \phi d\mu$.

If M is an orientable compact manifold, there is a natural Borel measure which can be defined. Let ω be a nowhere vanishing n-form, and define $m(\phi) = \int_M \phi\cdot\omega$ for any $\phi \in C(M)$. That is, define

$m(E) = \int_M \chi_E\cdot\omega$ where χ_E is the characteristic function of a Borel set E:

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E. \end{cases}$$

If M is not orientable, let $\pi : \tilde{M} \to M$ be an orientable 2-to-1 covering. Take the measure m on $\tilde{M}$ and let $\pi_* m$ be the measure on M defined by $\pi_* m(E) = m(\pi^{-1}E)$. Any measure m on M induced by an n-form on M or $\tilde{M}$ will be called Lebesgue measure on M. Dividing by m(M), we will assume m(M) = 1.

Ruelle has proved the following theorem. The support of a measure $\mu \in M(f)$ is the set of points $x \in M$ such that for every open set U containing x, $\mu(U) > 0$.

Theorem 6.2 ([48], [7]). Let m *be Lebesgue measure on* M, *and let* f *be a* C^2 *diffeomorphism having a hyperbolic attractor* Λ. *There is an ergodic* f-*invariant probability measure* μ_Λ *supported on* Λ *with the following property. There is a subset* $A \subset W^s(\Lambda)$ *with* $m(A) = 0$ *such that if* $x \in W^s(\Lambda) - A$ *and* ϕ *is any continuous function on* M, *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \int \phi d\mu_\Lambda \quad .$$

Thus, for m-almost all x in $W^s(\Lambda)$, the time average of ϕ along the orbit of x converges to a definite limit. This result is quite remarkable. For, in a natural sense, Lebesgue measure zero agrees with our intuitive feeling of what is exceptional (or avoidable) in smooth systems. If we think of ϕ as an observable physical quantity evolving along an orbit, then, with probability one, we can compute its expected value. Furthermore, except in the Anosov case, hyperbolic attractors for C^2 diffeomorphisms have Lebesgue measure zero. Therefore, it is surprising that one can say anything about time averages of points in sets of positive Lebesgue measure near these attractors.

If μ_Λ is the measure in theorem (6.2), and U is an open set in $W^s(\Lambda)$ such that $\mu_\Lambda(C\ell\ U) = \mu_\Lambda(U)$, then for m-almost all x in $W^s(\Lambda)$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_U(f^k x) = \mu_\Lambda(U)$. Here χ_U is the characteristic function of U. Thus, for almost all x, the average number of points in $\{x, fx, \ldots, f^{n-1}x\} \cap U$ approaches $\mu_\Lambda(U)$. One can project μ_Λ onto the unstable manifolds $W^u(x)$ for $x \in \Lambda$ to get conditional measures with C^1 densities. Thus, any open set U such that $\partial U \cap W^u(x)$ has u-dimensional measure zero for all x has the property that $\mu_\Lambda(C\ell\ U) = \mu_\Lambda(U)$.

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The convergence of $\frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k x)$ to $\mu_\Lambda(U)$ for such sets U is proved with familiar methods of measure theory.

Let A be the above subset of $W^s(\Lambda)$ with $m(A) = 0$. Given $\varepsilon > 0$, let F be a closed subset of U with $\mu_\Lambda(U) < \mu_\Lambda(F) + \varepsilon$, and let V be an open set in $W^s(\Lambda)$ with $\mathcal{C}\ell\, U \subset V$ and $\mu_\Lambda(V) < \mu_\Lambda(\mathcal{C}\ell\, U) + \varepsilon$.

Let ϕ_1 and ϕ_2 be continuous functions such that $\chi_F \le \phi_1 \le \chi_U$ and $\chi_{\mathcal{C}\ell\, U} \le \phi_2 \le \chi_V$. For $n > 0$ and $x \in W^s(\Lambda) - A$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_F(f^k x) \le \frac{1}{n}\sum_{k=0}^{n-1} \phi_1(f^k x) \le \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k x) \quad .$$

Thus, $\liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k x) \ge \int \phi_1 d\mu_\Lambda \ge \mu_\Lambda(F) \ge \mu_\Lambda(U) - \varepsilon.$

Similarly, $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{\mathcal{C}\ell\, U}(f^k x) \le \int \phi_2 d\mu_\Lambda \le \mu_\Lambda(V) \le \mu_\Lambda(\mathcal{C}\ell\, U) + \varepsilon.$

Since ε is arbitrary, and $\mu_\Lambda(\mathcal{C}\ell\, U) = \mu_\Lambda(U)$, we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k x) = \mu_\Lambda(U) \quad .$$

Before proceeding to the proof of theorem (6.2), let us note that it implies a celebrated theorem of Anosov.

Theorem (6.3). Let f *be a* C^2 *topologically transitive Anosov diffeomorphism and suppose* f *preserves a measure* ν *which is absolutely continuous with respect to Lebesgure measure* m. *Then* ν *is ergodic.*

<u>Proof</u>. Let $\phi \in C(M)$. Let $A \subset M$ be such that $m(A) = 0$ and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k x) = \int \phi d\mu_\Lambda \qquad \text{for } x \in M - A.$$

Since ν is absolutely continuous with respect to m, we have $\nu(A) = 0$

also. Thus,

$$\hat{\phi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x)$$

is constant ν-a.e. Hence $\int \hat{\phi} d\nu = \int \phi d\mu_\Lambda \cdot \nu(M)$. By the bounded convergence theorem and the fact that ν is f-invariant.

$$\int \hat{\phi} d\nu = \int \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi\circ f^k d\nu$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int \phi\circ f^k d\nu = \int \phi d\nu \ .$$

So, $\frac{1}{\nu(M)} \int \phi d\nu = \int \phi d\mu_\Lambda$. Hence,

$$\frac{1}{\nu(M)} \cdot \nu = \mu_\Lambda \quad \text{and} \quad \nu \quad \text{is ergodic} \ .$$

Remarks. 1. Theorem (6.2) and (6.3) hold under weaker assumptions than f being C^2. The proof given here works just as well (with straightforward changes) if f is $C^{1+\alpha}$, $0 < \alpha < 1$. That is, f is C^1 and its derivative is Holder continuous of order α. Anosov points out in [2] that theorem (6.3) holds if the modulus of continuity $\omega(r)$ of Tf satisfies

$$\int_0^b \frac{\omega(r)}{r} dr < \infty \quad \text{for some} \quad b > 0 \ .$$

I have not checked the details, but I expect that the proof here gives (6.2) (and hence (6.3) as well) under this assumption. Theorem (6.2) does not hold for all C^1 f. An example where it fails can be obtained by embedding Bowen's example of a horseshoe with positive measure [8] in a hyperbolic attractor. However, it is not known whether theorem (6.3) is false for C^1 f.

2. The Bernoulli shift $B(p_1,\ldots,p_N)$ is defined as follows. Let

$\sigma : \sum_N \to \sum_N$ be the full N-shift with $\sum_N = \{1,\dots,N\}^Z$. Let $p_1,\dots,p_N$ be positive numbers such that $\sum_{i=1}^{N} p_i = 1$. Define the measure μ on $\{1,\dots,N\}$ by $\mu(\{i\}) = p_i$. Let $\bar{\mu}$ be the product measure on the Borel sets of $\sum_N$. Then $\bar{\mu}$ is invariant under the shift σ, and the pair $(\sigma,\bar{\mu})$ is called the Bernoulli shift $B(p_1,\dots,p_N)$. A measure preserving transformation $T : X \to X$ with probability measure m is called *Bernoulli* if (T,m) is measure-theoretically conjugate to some $B(p_1,\dots,p_N)$. This means there is a measurable transformation $S : X \to \sum_N$, and subsets $A \subset X$, $B \subset \sum_N$ with $m(A) = 1$, $\bar{\mu}(B) = 1$ such that S maps A bijectively onto B, and $\sigma S(x) = ST(x)$ for $x \in A$.

A theorem of Ornstein [38] says that $B(p_1,\dots,p_N)$ is measure theoretically conjugate to $B(q_1,\dots,q_r)$ if and only if

$$\sum_{i=1}^{N} p_i \log p_i = \sum_{i=1}^{r} q_i \log q_i \quad .$$

One can actually show that the pair $(f|\Lambda,\mu_\Lambda)$ is Bernoulli if Λ is infinite and $f|\Lambda$ is topologically mixing. The proof involves representing $f|\Lambda$ as a finite-to-one quotient of a subshift of finite type $\sum_A$ via theorem (3.8), and obtaining μ_Λ from a measure on $\sum_A$ which can be shown to be Bernoulli. For details, see [7].

3. Bowen and Ruelle have proved [7], [70] that if f satisfies Axiom A and is C^2, then m-almost all points x in M are forward asymptotic to attractors. This also holds if $L(f)$ is hyperbolic. Putting this together with theorem (6.2), one sees that if $L(f)$ is hyperbolic, there are finitely many ergodic invariant measures which describe the forward asymptotic behavior of m-almost all points in M.

We now proceed towards the proof of Ruelle's theorem (6.2). We shall follow [7] except that our actual construction of μ_Λ comes from [6] which in turn was motivated by [47].

Let $f : M \to M$ be a C^2 diffeomorphism and let Λ be a hyperbolic attractor for f. Let $\varepsilon > 0$ be small, and let $n \geq 1$ be a positive integer. For $x \in \Lambda$, let

$$W^s_\varepsilon(x,n) = \{y \in M : d(f^j y, f^j x) \leq \varepsilon \text{ for } 0 \leq j < n\}.$$

We might call $W^s_\varepsilon(x,n)$ the stable set of x of xize (ε,n). If $\lambda > 1$ is as in the definition of hyperbolicity, then $W^s_\varepsilon(x,n)$ is nearly the product of $W^s_\varepsilon(x)$ and $W^u_{\lambda^{1-n}\varepsilon}(x)$. Let m be Lebesgue measure on the open set $W^s(\Lambda)$.

Proposition (6.4). There is an ergodic f*-invariant measure* $\mu_\Lambda \in M(f)$ *with the following property. For* $\varepsilon > 0$ *small, there is a constant* $C_\varepsilon > 0$ *such that for any* $x \in \Lambda$ *and* $n > 0$,

$$m(W^s_{3\varepsilon}(x,n)) \leq C_\varepsilon \mu_\Lambda(W^s_\varepsilon(x,n)) \ .$$

We defer the proof of proposition (6.4).

Proof of theorem (6.2). Let $\phi \in C(M)$. Let $\delta > 0$ be fixed and choose $\varepsilon > 0$ so that $d(\phi(x), \phi(y)) < \delta$ whenever $d(x,y) < \varepsilon$. For $n > 0$, set

$$C_n(\phi,\delta) = \{x \in M : |\frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) - \int \phi d\mu_\Lambda| > \delta\} ,$$

and let $E(\phi,\delta) = \{x \in M : x \in C_n(\phi,\delta) \text{ for infinitely many } n\}$

$$= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} C_n(\phi,\delta) \ .$$

We first claim

$$(1) \qquad m(W^s_\varepsilon(\Lambda) \cap E(\phi,3\delta)) = 0 \quad .$$

Assume (1) is proved for the moment. As $E(\phi,3\delta)$ is f-invariant, and f preserves sets of m-measure zero, we have $m(f^{-n}(W^s_\varepsilon(\Lambda)) \cap E(\phi,3\delta)) = 0$ for each $n \geq 0$. But $W^s(\Lambda) = \bigcup_{n\geq 0} f^{-n}W^s_\varepsilon(\Lambda)$, so $m(W^s(\Lambda) \cap E(\phi,3\delta)) = 0$. Letting $3\delta = \frac{1}{j}$ for $j \geq 1$, we get that $m(\bigcup_{j\geq 1} (W^s(\Lambda) \cap E(\phi,\frac{1}{j}))) = 0$. But if $A(\phi) = \bigcup_{j\geq 1}(W^s(\Lambda) \cap E(\phi, \frac{1}{j}))$, then $x \in W^s(\Lambda) - A(\phi)$ implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \int \phi d\mu_\Lambda \ .$$

Now, let $\phi_1,\phi_2,\ldots$ be a countable dense subset of $C(M)$, and let $A = \bigcup_{i=1}^{\infty} A(\phi_i)$. Then, $m(A) = 0$, and if $x \in W^s(\Lambda) - A$ and $\phi \in C(M)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \int \phi d\mu_\Lambda \quad .$$

We now prove (1).

Fix $N \geq 0$ and successively define subsets $R_N, R_{N+1},\ldots$ of $\Lambda \cap C_n(\phi,2\delta)$ as follows. For $n \geq N$, let R_n be a maximal subset of $\Lambda \cap C_n(\phi,2\delta)$ such that

(a) $W^s_\varepsilon(x,n) \cap W^s_\varepsilon(y,k) = \emptyset$ for $x \in R_n$, $y \in R_k$ and $N \leq k < n$

(b) $W^s_\varepsilon(x,n) \cap W^s_\varepsilon(y,n) = \emptyset$ if $x \in R_n$, $y \in R_n$ and $x \neq y$.

If $y \in W^s_\varepsilon(\Lambda) \cap C_n(\phi,3\delta)$ for $n \geq N$, and $y \in W^s_\varepsilon(z)$ with $z \in \Lambda$, then $z \in C_n(\phi,2\delta)$ by the choice of ε. By the maximality of R_n, $W^s_\varepsilon(z,n) \cap W^s_\varepsilon(x,k) \neq \phi$ for some $N \leq k \leq n$ and $x \in R_k$. Then $y \in W^s_\varepsilon(z,n) \subset W^s_\varepsilon(z,k) \subset W^s_{3\varepsilon}(x,k)$. Thus

$$W^s_\varepsilon(\Lambda) \cap C_n(\phi,3\delta) \subset \bigcup_{N\le k\le n} \bigcup_{x\in R_k} W^s_{3\varepsilon}(x,k)$$

and

$$W^s_\varepsilon(\Lambda) \cap \bigcup_{n=N}^{\infty} C_n(\phi,3\delta) \subset \bigcup_{k=N}^{\infty} \bigcup_{x\in R_k} W^s_{3\varepsilon}(x,k) .$$

So (2) $m(W^s_\varepsilon(\Lambda) \cap \bigcup_{n=N}^{\infty} C_n(\phi,3\delta)) \le \sum_{k=N}^{\infty} \sum_{x\in R_k} m(W^s_{3\varepsilon}(x,k))$

$$\le C_\varepsilon \sum_{k=N}^{\infty} \sum_{x\in R_k} \mu_\Lambda(W^s_\varepsilon(x,k))$$

by Proposition (6.4). Now, $R_k \subset C_k(\phi,2\delta)$, so if $x \in R_k$, by the choice of ε, one has $W^s_\varepsilon(x,k) \subset C_k(\phi,\delta)$. Also, by the choice of the R_k's $\bigcup_{k=N}^{\infty} \bigcup_{x\in R_k} W^s_\varepsilon(x,k)$ is a disjoint union. So

(3) $$\sum_{k=N}^{\infty} \sum_{x\in R_k} \mu_\Lambda(W^s_\varepsilon(x,k)) = \mu_\Lambda(\bigcup_{k=N}^{\infty} \bigcup_{x\in R_k} W^s_\varepsilon(x,k)) \le \mu_\Lambda(\bigcup_{k=N}^{\infty} C_k(\phi,\delta))$$

But, by the ergodic theorem (6.1), this last number approaches zero as $N \to \infty$ since μ_Λ is ergodic. Thus, putting (2) and (3) together, we get that

$$m(W^s_\varepsilon(\Lambda) \cap \bigcup_{n=N}^{\infty} C_n(\phi,3\delta)) \to 0 \quad \text{as} \quad N \to \infty,$$

and this implies (1).

We are now ready to produce the measure μ_Λ needed to prove proposition (6.4). Our construction of μ_Λ is based on [6]. This has the advantage that it avoids the use of Markov partitions, and, hence, leads to a shorter proof of theorem (6.2) than one finds in [7] or [70]. However, one pays the price that it is not readily apparent from the construction here that the measure μ_Λ is Bernoulli. Nevertheless, it is of some value to give proofs of theorem (6.2) without Markov partitions. This is because there are certain non-hyperbolic attractors (e.g. those in [18]) for which Markov partitions do not exist, but for which one still has the possibility of having theorem (6.2).

To construct μ_Λ, we first observe that we may assume that f has a fixed point. If not, we choose $n > 0$ so that f^n has a fixed point p in Λ (Proposition (4.1) gives that the periodic points of f in Λ are dense in Λ). If $\Lambda_1 = C\ell\, W^u(p,f^n) \cap \Lambda$, then $f^n\Lambda_1 = \Lambda_1$, and $\Lambda = \bigcup_{k=0}^{n-1} f^k\Lambda_1$. It may be that Λ_1 is a proper subset of Λ (for instance, when Λ is a single periodic non-fixed sink). If we find μ_{Λ_1} for f^n as in Proposition (6.4), then set $\mu_\Lambda = \frac{1}{n}\sum_{k=0}^{n-1} f^k_*\mu_{\Lambda_1}$. Here $f^k_*\mu_{\Lambda_1}(E) = \mu_{\Lambda_1}(f^{-k}E \cap \Lambda_1)$ for any Borel set E. One may check that this μ_Λ works for f on Λ.

Now, we assume that $p \in \Lambda$, and $f(p) = p$. We will see in lemma (7.1) that this forces f to have periodic points of all high periods.

By passing to two 2-to-1 coverings of M, and replacing f by f^4, if necessary, we may assume that M is orientable, each of the bundles E^s and E^u is orientable, and $Tf|E^s$ and $Tf|E^u$ preserve orientation. For amusement the reader should examine this covering for the Plykin example.

Let g be a smooth Riemann metric on M. Then g induces a volume form $\bar{\omega}$ on M which we use to define Lebesgue measure m on M. That is, $\int \psi dm = \int \psi\bar{\omega}$ for every continuous function $\psi \in C(M)$. For each $x \in \Lambda$, $W^u(x)$ is a C^2 submanifold of M, so the metric g restricts to a C^1 metric on $W^u(x)$ and this induces a C^1-u-form on $W^u(x)$. Thus, for each $y \in W^u(x)$, and $x \in \Lambda$, we have a u-form $\omega_y \in \Lambda^u(E^{u*}_y)$ where E^{u*}_y is the dual space of E^u_y. The forms ω_y vary C^1 with y in $W^u(x)$, and continuously with y in Λ.

For each $x \in \Lambda$, we have $f^*\omega_{f(x)} = \psi(x)\omega_x$ where $\psi(x)$ is a positive continuous function. The positivity comes from the fact that the bundle E^u is oriented, and $Tf|E^u$ preserves orientation.

One can check that $(f^n)^*\omega_{f^n(x)} = \psi(f^{n-1}x)\cdot\psi(f^{n-2}x)\ldots\psi(x)\cdot\omega_x$, and $(f^{-1})^*\omega_x = \psi(x)^{-1}\omega_{fx}$. Following Bowen and Ruelle [7], we call $\psi(x) = \mathrm{Jac}(T_xf|E^u_x)$, the Jacobian of $T_xf|E^u_x$. Set $\phi^u(x) = -\log\psi(x) = \log\psi(x)^{-1}$.

Lemma (6.5). ϕ^u is Hölder continuous.

<u>Proof</u>. The bundle $x \longrightarrow E^u_x$ is Hölder continuous since f is C^2 by theorem (6.4) in [17], and the metric g is C^∞ on M. Hence, the maps $x \longrightarrow \omega_x$, and $x \longrightarrow T_xf|E^u_x$ are Hölder continuous. This implies that ϕ^u is Hölder continuous.

Set $S_n\phi^u(x) = \sum_{k=0}^{n-1}\phi^u(f^kx)$ for $n > 0$. Note that,

$$e^{S_n\phi^u(x)} = \prod_{k=0}^{n-1} e^{\phi^u(f^kx)} = \prod_{k=0}^{n-1}\psi(f^kx)^{-1} ,$$

so,

$$(f^{-n})^*\omega_x = e^{S_n\phi^u(x)}\cdot\omega_{f^nx} .$$

Thus, $e^{S_n\phi^u(x)}$ measures how much $T_{f^nx}f^{-n}|E^u_{f^nx}$ contracts the ω_{f^nx}-volume.

Now, we can construct μ_Λ. Let $\mathrm{Per}(n)$ be the set of periodic points of f in Λ of period n. Let δ_x be the point pass at x. Define

$$Z_n = \sum_{p\in\mathrm{Per}(n)} e^{S_n\phi^u(p)}, \text{ and set } \mu_n = \frac{1}{Z_n}\sum_{p\in\mathrm{Per}(n)} e^{S_n\phi^u(p)}\delta_p$$

It is easy to check that $\mu_n \in M(f)$; i.e., μ_n is an f-invariant probability measure on Λ. It counts the elements of $\mathrm{Per}(n)$ with certain weights.

Choose a subsequence μ_{n_i} which converges, and set $\mu_\Lambda = \lim_{i\to\infty} \mu_{n_i}$. This μ_Λ works. Observe that the proof we shall give applies to every convergent subsequence of $\{\mu_n\}$. Thus, we will incidentally show that if $\{\mu_{n_i}\}$ and $\{\mu_{m_i}\}$ are subsequences of $\{\mu_n\}$ such that $\mu_{n_i} \longrightarrow \mu$ and $\mu_{n_i} \longrightarrow \nu$, then for any $\phi \in C(M)$, $\int \phi d\mu = \int \phi d\nu$. That is, $\mu = \nu$. Hence, the sequence $\{\mu_n\}$ actually converges to μ_Λ.

We need to show

(6.6) For any small $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that if $x \in \Lambda$ and $n \geq 1$, then

$$m(W^s_{3\varepsilon}(x,n)) \leq C_\varepsilon \mu_\Lambda(W^s(x,n))$$

and

(6.7) μ_Λ is ergodic.

These facts will be proved in the next section.

7. The measure μ_Λ.

In this section we shall prove (6.6) and (6.7) to complete proofs of Proposition (6.4) and theorem (6.2). We will need several technical lemmas. Our notation will be in the context of section 6. Our first lemma is a strengthening of the shadowing lemma called the specification lemma. Given $\varepsilon > 0$, a positive integer $P > 0$, a set of points $x_1,\dots,x_r$ in Λ, and a finite set of positive integers $n_1,n_2,\dots,n_r$, we say that a point q ε-specifies (x_1,n_1), $(x_2,n_2),\dots,(x_r,n_r)$ with delay P if

(1) $q \in \mathrm{Per}(rP + \sum_{i=1}^{r} n_i)$

(2) for $0 \le j \le n_1$, $d(f^j x_1, f^j q) \le \varepsilon$, and

(3) for $1 < i \le r$ and $0 \le j \le n_i$, we have

$$d(f^j x_i, f^{n_1+\dots+n_{i-1}+(i-1)P+j} q) \le \varepsilon$$

The idea is that the first n_1 iterates of q ε-shadow the first n_1 iterates of x_1, then after a delay of P, the next n_2 iterates of q ε-shadow the first n_2 iterates of x_2. Again after a delay of P, the next n_3 iterates of q ε-shadow the first n_3 iterates of x_3, etc. Finally, after ε-shadowing the first n_r iterates of x_r and a delay of P again, the orbit of q closes up.

Lemma (7.1). (Specification lemma). Let Λ *be a hyperbolic set for* f *containing a fixed point of* f. *Then for any* $\varepsilon > 0$, *there is a* $P(\varepsilon) > 0$ *such that any sequence* $(x_1,n_1),\dots,(x_r,n_r)$ *may be* ε*-specified with delay* $P(\varepsilon)$.

Proof. For $x \in \Lambda$ and $\varepsilon > 0$, write $\hat{W}^u_\varepsilon(x) = W^u_\varepsilon(x) \cap \Lambda$ and $\hat{W}^s_\varepsilon(x) = W^s_\varepsilon(x) \cap \Lambda$. We first prove

(a) For any $\varepsilon > 0$, there is an integer $N(\varepsilon) > 0$ such that whenever $n \ge N(\varepsilon)$ and $x, y \in \Lambda$ we have $f^n \hat{W}^u_\varepsilon(x) \cap \hat{W}^s_\varepsilon(y) \ne \phi$.

Once (a) is proved and $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ be such that any $\delta(\varepsilon)$-pseudo-orbit in Λ can be ε-shadowed by an orbit. Let $P(\varepsilon) = P = N(\delta(\varepsilon))$. Then, given $(x_1,n_1),\ldots,(x_r,n_r)$, choose $y_i \in \hat{W}^u_\delta(f^{n_i}x_i) \cap f^{-P(\varepsilon)}\hat{W}^s_\delta(x_{i+1})$ for $1 \le i < r$ and $y_r \in W^u_\delta(f^{n_r}x_r) \cap f^{-P(\varepsilon)}W^s_\delta(x_1)$. Let γ be the finite δ-pseudo-orbit $x_1, fx_1, \ldots, f^{n_1-1}x_1, y_1, fy_1, \ldots, f^{P-1}y_1, x_2, \ldots, f^{n_r-1}x_r, y_r, fy_r, \ldots, f^{P-1}y_r$. Let $\tau = \sum_{i=1}^{r} n_i + rP$ be the length of γ. Since $d(f^P y_r, x_1) \le \delta$, repeating γ to the left and right gives an infinite δ-pseudo-orbit $\overline{\gamma}$. Let $o(q)$ be an orbit in Λ ε-shadowing $\overline{\gamma}$. Then, for any integer j, $d(f^{\tau+j}q, f^j q) \le 2\varepsilon$. If 2ε is small, expansiveness gives $f^\tau q = q$ and $q \in \text{Per}\,\tau$.

We now prove (a).

Let p be the fixed point of f in Λ. For $\varepsilon > 0$ small, let $\delta = \delta(\varepsilon)$ be such that whenever $d(x,y) \le \delta$, $W^u_{\varepsilon/2}(x) \cap W^s_{\varepsilon/2}(y)$ is a unique point. Since $\hat{W}^u(p)$ and $\hat{W}^s(p)$ are dense in Λ, there is an integer $N(\delta(\varepsilon)) > 0$ so that for $n \ge N(\delta(\varepsilon))$, and $x \in \Lambda$, we have $\text{dist}\,(x, f^n \hat{W}^u_{\delta/3}(p)) < \delta$ and $\text{dist}\,(x, f^{-n}\hat{W}^s_{\delta/3}(p)) < \delta$. Let $L(\varepsilon) > N(\delta(\varepsilon))$ be such that $\varepsilon\lambda^{-L(\varepsilon)} < \frac{\delta(\varepsilon)}{3}$ where $\lambda > 1$ is as in the definition of hyperbolicity. Set $N(\varepsilon) = 2L(\varepsilon)$. Then, if $n_1 \ge N(\varepsilon)$, write $n_1 = n + m$ with n and $m \ge L(\varepsilon)$. Pick $z_n, z_m \in \Lambda$ such that $z_n \in \hat{W}^s_{\delta/3}(p)$, $z_m \in \hat{W}^u_{\delta/3}(p)$, $d(x, f^{-n}z_n) < \delta$ and $d(y, f^m z_m) < \delta$. Then, $W^u_{\varepsilon/2}(x) \cap W^s_{\varepsilon/2}(f^{-n}z_n) \ne \phi$ and $W^s_{\varepsilon/2}(y) \cap W^u_{\varepsilon/2}(f^m z_m) \ne \phi$ by the choice of δ. If we let $w_n \in W^u_{\varepsilon/2}(x) \cap W^s_{\varepsilon/2}(f^{-n}z_n)$, and $w_m \in W^s_{\varepsilon/2}(y) \cap W^u_{\varepsilon/2}(f^m z_m)$, then $d(f^n w_n, z_n) \le \lambda^{-n}\frac{\varepsilon}{2}$, and $d(f^{-m}w_m, z_m) \le \lambda^{-m}\frac{\varepsilon}{2}$.

Hence,

$$d(f^n w_n, f^{-m} w_m) \le d(f^n w_n, z_n) + d(z_n, p) + d(p, z_m) + d(z_m, d^{-m} w_m)$$

$$\le \lambda^{-L(\varepsilon)}\varepsilon + \frac{2\delta(\varepsilon)}{3} < \delta(\varepsilon).$$

So, $W^u_{\varepsilon/2}(f^n w_n) \cap W^s_{\varepsilon/2}(f^{-m} w_m) \ne \phi$. But then, $f^n W^u_\varepsilon(x) \supset f^n W^u_{\varepsilon/2}(w_n) \supset W^u_{\varepsilon/2}(f^n w_n)$ and $f^{-m} W^s_\varepsilon(y) \supset f^{-m} W^s_{\varepsilon/2}(w_m) \supset W^s_{\varepsilon/2}(f^{-m} w_m)$, and we get that

$$f^{n_1} W^u_\varepsilon(x) \cap W^s_\varepsilon(x) = f^{n+m} W^u_\varepsilon(x) \cap W^s(y) \ne \phi.$$

This proves (a) and lemma (7.1).

Lemma (7.2). Let $\varepsilon > 0$ *be an expansive constant for* $f|\Lambda$. *Given any* $\delta > 0$, *there is an* $N(\delta) > 0$ *so that if* $d(f^j x, f^j y) \le \varepsilon$ *for* $|j| \le N(\delta)$, *then* $d(x,y) \le \delta$.

<u>Proof</u>. If not, there is some $\delta_0 > 0$ and there are sequences $(x_n), (y_n)$ in Λ such that $d(f^j x_n, f^j y_n) \le \varepsilon$ for $|j| \le n$ and $d(x_n, y_n) \ge \delta_0$. Choose points $x, y \in \Lambda$ and subsequences $(x_{n_k}), (y_{n_k})$ ao that $x_{n_k} \to x$ and $y_{n_k} \to y$ as $k \to \infty$. Then $d(x,y) \ge \delta_0$, but $d(f^j z, f^j y) \le \varepsilon$ for all j. This contradicts the choice of ε.

We need three more lemmas. We will defer their proofs to the end of this section.

Lemma (7.3). There is a constant $K > 0$ *such that if* $\varepsilon > 0$ *is small and* $x, y \in \Lambda$ *with* $y \in W^s_\varepsilon(x,n)$ *and* $n \ge 1$, *then* $|S_n\phi^u(x) - S_n\phi^u(y)| \le K$.

Lemma (7.4) (Volume Lemma). For any $\varepsilon > 0$ *there is a constant* $C_\varepsilon > 0$ *such that*

$$C_\varepsilon^{-1} e^{S_n\phi^u(x)} \le m(W^s_\varepsilon(x,n)) \le C_\varepsilon e^{S_n\phi^u(x)}$$

for all $x \in \Lambda$ *and* $n > 1$.

Lemma (7.5) *There is a constant* $C_1 > 0$ *such that*

(a) $C_1^{-1} \le Z_n \le C_1$ *for all* $n \ge 1$

and

(b) *for any integers* $n_1, \dots, n_r > 0$,

$$C_1^{-r} \prod_{i=1}^{r} Z_{n_i} \le Z_{n_1 + \dots + n_r} \le C_1^{r} \prod_{i=1}^{r} Z_{n_i}.$$

We now move to the proofs of (6.6) and (6.7).

In view of the volume lemma (7.4), (6.6) follows from

(a) for any $\varepsilon > 0$, there is a constant $b_\varepsilon > 0$ such that for any $x \in \Lambda$ and any $n \ge 1$, $\mu_\Lambda(W^s_\varepsilon(x,n)) \ge b_\varepsilon e^{S_n \phi^u(x)}$.

Let $x \in \Lambda$, let $\varepsilon > 0$ be arbitrary and let $n \ge 1$.

Let $\varepsilon_1 > 0$ be an expansive constant for $f|\Lambda$. By lemma (7.2), there is an $N(\varepsilon) > 0$ such that $d(f^j x, f^j y) \le \varepsilon_1$ for $|j| \le N(\varepsilon)$ and $y \in \Lambda$ implies that $d(x,y) \le \varepsilon$. Let $m \ge 1$ be an integer. For any $z \in \mathrm{Per}(m) \cap \Lambda$, use specification to give a $q(z) \in \mathrm{Per}(2N(\varepsilon) + n + m + 2P(\varepsilon))$ such that $q(z)$ $\frac{\varepsilon_1}{3}$-specifies $(f^{-N(\varepsilon)}x, 2N(\varepsilon) + n)$ and (z,m) with delay $P = P(\frac{\varepsilon_1}{3})$. Let $\tau = \tau(m) = 2N(\varepsilon) + n + m + 2P$. Then, $(f^{N(\varepsilon)}q(z)) \in W^s_\varepsilon(x,n)$. Also, by expansiveness for $z \ne w$ in $\mathrm{Per}(m) \cap \Lambda$, the orbits of z and w get at least ε_1 apart somewhere, so the orbits of $q(z)$ and $q(w)$ get at least $\frac{\varepsilon_1}{3}$ apart somewhere. That is, $q(z) \ne q(w)$.

Let $||\phi^u|| = \sup_{\xi \in \Lambda} \{|\phi^u(\xi)|\}$. By lemmas (7.3) and (7.5), we have

$$\mu_\tau W^s_\varepsilon(x,n) \ge \frac{1}{Z_\tau} \sum_{q(z)} e^{S_\tau \phi^u(q(z))}$$

$$\geq \frac{e^{(-2N(\varepsilon)-2P)||\phi^u||}}{C_1^6 Z_{N(\varepsilon)}^2 Z_P^2 Z_n Z_m} \sum_{q(z)} e^{S_n\phi^u(f^{N(\varepsilon)}q(z)) + S_m\phi^u(f^{2N(\varepsilon)+n+P}q(z))}$$

$$\geq \frac{e^{(-2N(\varepsilon)-2P)||\phi^u||-2K}}{C_1^6 Z_{N(\varepsilon)}^2 Z_P^2 Z_n Z_m} \sum_{z\in \mathrm{Per}(m)} e^{S_n\phi^u(x)} \quad e^{S_m\phi^u(z)}$$

$$= \frac{e^{(-2N(\varepsilon)-2P)||\phi^u||-2K}}{C_1^{11}} e^{S_n\phi^u(x)}$$

If we set $b_\varepsilon = C_1^{-11}\exp((-2N(\varepsilon)-2P)||\phi^u||-2K)$, we have $\mu_\tau W_\varepsilon^s(x,n) \geq b_\varepsilon e^{S_n\phi^u(x)}$. Now letting m_i be such that $\tau(m_i) = n_i = 2N(\varepsilon) + 2P + n + m_i$, and letting $i \to \infty$ gives

$$\mu_\Lambda W_\varepsilon^s(x,n) \geq b_\varepsilon e^{S_n\phi^u(x)} \quad \text{which is (a).}$$

To prove (6.7), i.e. that μ_Λ is ergodic, we first prove

(b) there is a constant $C > 0$ such that for any Borel sets A and B,

$$\liminf_{n\to\infty} \mu_\Lambda(A \cap f^{-n}B) \geq C\mu_\Lambda(A)\cdot\mu_\Lambda(B).$$

Once (b) is established, (6.7) follows easily. For suppose A is invariant and $0 < \mu_\Lambda(A) < 1$. Then, $M - A$ is also invariant and $0 < \mu_\Lambda(M - A) < 1$. But from (b), we get $0 = \mu_\Lambda(A \cap (M - A)) \geq C\mu_\Lambda(A) \cdot \mu_\Lambda(M - A) > 0$ a contradiction.

To prove (b), it suffices to show that if A and B are compact subsets of Λ, and $\delta > 0$, then

(c) $\liminf_{n\to\infty} \mu_\Lambda(B_\delta(B) \cap f^{-n}(B_\Delta(B))) \geq C\mu_\Lambda(A)\cdot\mu_\Lambda(B)$

where C is independent of A, B, and δ.

Here $B_\delta(A) = \{y : d(y,A) \le \delta\}$ and $B_\delta(B) = \{y : d(y,B) \le \delta\}$ as before.

For suppose (c) holds. Then (b) is obvious if $\mu_\Lambda(A)$ or $\mu_\Lambda(B) = 0$. If both $\mu_\Lambda(A) > 0$ and $\mu_\Lambda(B) > 0$, let $0 < \delta_1 < 1$ be arbitrary, and choose $\delta_2 > 0$ so small that

$$C(\mu_\Lambda(A) - \delta_2)(\mu_\Lambda(B) - \delta_2) - 2\delta_2 > (1 - \delta_1)C\mu_\Lambda(A)\cdot\mu_\Lambda(B).$$

Then pick compact sets $A_1 \subset A$, $B_1 \subset B$ and open sets $U_1 \supset A$, $V_1 \supset B$ such that $\mu_\Lambda(U_1 - A_1) < \delta_2$ and $\mu_\Lambda(V_1 - B_1) < \delta_2$. Next take $\delta > 0$ so that $B_\delta(A_1) \subset U_1$ and $B_\delta(B_1) \subset V_1$. Then,

$$\begin{aligned}\mu_\Lambda(A \cap f^{-n}B) &= \mu_\Lambda(U_1 \cap f^{-n}V_1) - \mu_\Lambda(U_1 \cap f^{-n}V_1 - A \cap f^{-n}B)\\ &\ge \mu_\Lambda(B_\delta(A_1) \cap f^{-n}B_\delta(B_1)) - (\mu_\Lambda(U_1 - A) + \mu_\Lambda(f^{-n}V_1 - f^{-n}B))\\ &= \mu_\Lambda(B_\delta(A_1) \cap f^{-n}B_\delta(B_1)) - \mu_\Lambda(U_1 - A) - \mu_\Lambda(V_1 - B)\\ &\ge \mu_\Lambda(B_\delta(A_1) \cap f^{-n}B_\delta(B_1)) - 2\delta_2 .\end{aligned}$$

By (c) and the choice of δ_2, A_1, B_1, we have

$$\begin{aligned}\liminf_{n\to\infty} \mu_\Lambda(A \cap f^{-n}B) &\ge C\mu_\Lambda(A_1)\cdot\mu_\Lambda(B_1) - 2\delta_2\\ &\ge C(\mu_\Lambda(A) - \delta_2)(\mu_\Lambda(B) - \delta_2) - 2\delta_2\\ &\ge (1 - \delta_1)C\mu_\Lambda(A)\cdot\mu_\Lambda(B).\end{aligned}$$

As δ_1 was arbitrary, we get (b).

To prove (c), we again use specification. Let $\delta > 0$, and let A and B be compact sets in Λ. Given any large integers $n,r,s > 0$, let $z_1 \in A \cap \mathrm{Per}(n)$, $z_2 \in \mathrm{Per}(r) \cap \Lambda$, $z_3 \in B \cap \mathrm{Per}(n)$, and $z_4 \in \mathrm{Per}(s) \cap \Lambda$. Then, let $P(\frac{\varepsilon_1}{3})$, let $\bar{z} = (z_1,z_2,z_3,z_4)$, and let $\tau = 2n + r + s + 4P$. We $\frac{\varepsilon_1}{3}$-specify the pairs $(f^{-[n/2]}z_1,n)$, (z_2,r),

$(f^{-[n/2]}z_3, n)$, and (z_4, s) with delay P by a periodic point $q(\bar{z}) \in \mathrm{Per}(\tau) \cap \Lambda$. Note that $q(\bar{z}) = q(\bar{w})$ implies $\bar{z} = \bar{w}$ since if $z_i \neq w_i$ for some $1 \leq i \leq 4$, then the orbits of z_i and w_i get ε_1 apart somwhere. Also, if $[\frac{n}{2}] \geq N(\delta)$, then $d(f^{[n/2]}q(\bar{z}), z_1) \leq \delta$, and $d(f^{n+r+[n/2]+2P}q(\bar{z}), z_3) \leq \delta$. Thus $f^{[n/2]}q(\bar{z}) \in B_\delta(A)$ and $f^{n+r+[n/2]+2P}q(\bar{z}) \in B_\delta(B)$, so $f^{[n/2]}q(\bar{z}) \in B_\delta(A) \cap f^{-n-r-2P}B_\delta(B)$. Now,

$$\mu_\tau(B_\delta(A) \cap f^{-n-r-2P}B_\delta(B))$$

$$\geq \frac{1}{Z_\tau} \sum_{f^{[n/2]}q(\bar{z})} e^{S_\tau \phi^u(f^{[n/2]}q(\bar{z}))}$$

$$= \frac{1}{Z_\tau} \sum_{q(z)} e^{S_\tau \phi^u(q(\bar{z}))}$$

$$\geq \frac{C}{Z_n^2 Z_r Z_s} \sum_{\bar{z}} e^{S_n\phi^u(z_1)+S_r\phi^u(z_2)+S_n\phi^u(z_3)+S_s\phi^u(z_4)}$$

$$= \frac{C}{Z_n^2 Z_r Z_s} \left(\sum_{z_1} e^{S_n\phi^u(z_1)}\right)\left(\sum_{z_2} e^{S_r\phi^u(z_2)}\right)\left(\sum_{z_3} e^{S_n\phi^u(z_3)}\right)\left(\sum_{z_4} e^{S_s\phi^u(z_4)}\right)$$

$$= \frac{C}{Z_n^2}\left(\sum_{z_1} e^{S_n\phi^u(z_1)}\right)\left(\sum_{z_3} e^{S_n{}^{u}(z_3)}\right) = C\mu_n(A)\,\mu_n(B)$$

with C depending only on ε_1 and ϕ^u. Letting $s \to \infty$ (so that τ runs through n_i) gives

$$\mu_\Lambda(B_\delta(A) \cap f^{-n-r-2P}B_\delta(B)) \geq C\mu_n(A)\cdot\mu_n(B).$$

Then, letting $r \to \infty$, we get

$$\liminf_{j\to\infty} \mu_\Lambda(B_\delta(A) \cap f^{-j}B_\delta(B)) \geq C\mu_n(A)\cdot\mu_n(B).$$

Finally, letting $n = n_i$ and $i \to \infty$ gives us (c).

We will now give the proofs of lemmas (7.3), (7.4), and (7.5).

<u>Proof of lemma (7.3)</u>. Let $\lambda > 1$ be as in the definition of hyperbolicity of Λ.

Let ε_o be small enough so that for each $x,y \in \Lambda$, $W^s_{2\varepsilon_o}(x) \cap W^u_{2\varepsilon_o}(y)$ is at most one point. Then choose $\varepsilon_1 \in (0,\varepsilon_o)$ small enough so that for each $x \in \Lambda$, $B_{\varepsilon_1}(x) \cap \Lambda$ is contained in an ε_o-product neighborhood. That is, if $y \in B_{\varepsilon_1}(x) \cap \Lambda$, there are points $z_1 \in W^u_{\varepsilon_o}(x)$ and $z_2 \in W^s_{\varepsilon_o}(x)$ such that $\{y\} = W^s_{\varepsilon_o}(z_1) \cap W^u_{\varepsilon_o}(z_2)$. There is a constant $c > 0$ independent of x and y such that $d(x,y) \le c(d(x,z_1) + d(x,z_2))$ whenever $z_1 \in W^u_{\varepsilon_o}(x)$, $z_2 \in W^s_{\varepsilon_o}(x)$, and $\{y\} = W^s_{2\varepsilon_o}(z_1) \cap W^u_{2\varepsilon_o}(z_2)$.

Now suppose $x,y \in \Lambda$ satisfy $d(f^jx,f^jy) \le \varepsilon_1$ for $0 \le j < n$. Then there is a $w_1 \in W^u_{\varepsilon_o}(f^{n-1}x)$ such that $f^{n-1}y \in W^s_{\varepsilon_o}(w_1)$. Now $d(f^{-j}w_1,f^{n-1-j}x) \le \lambda^{-j}\varepsilon_o$ for $0 \le j < n$, so $z \equiv f^{1-n}w_1 \in W^s_{2\varepsilon_o}(y) \cap W^u_{\lambda^{1-n}\varepsilon_o}(x)$. Similarly, if $d(f^js,f^jy) \le \varepsilon_1$ for $-n < j \le 0$, there is a point $z_2 \in W^s_{\lambda^{1-n}\varepsilon_o}(x) \cap W^u_{2\varepsilon_o}(y)$. Thus, $d(f^jx,f^jy) \le \varepsilon_1$ for $|j| < n$ implies there are points $z_1 \in W^u_{\lambda^{1-n}\varepsilon_o}(x)$ and $z_2 \in W^s_{\lambda^{1-n}\varepsilon_o}(x)$ such that $\{y\} = W^s_{2\varepsilon_o}(z_1) \cap W^u_{2\varepsilon_o}(z_2)$. Hence $d(x,y) \le c(d(x,z_1) + d(x,z_2)) \le 2c\lambda^{1-n}\varepsilon_o$.

Now, let $L > 0$ and $0 < \alpha < 1$ be such that $d(\phi^u x,\phi^u y) \le Ld(x,y)^\alpha$ for $x,y \in \Lambda$, and suppose $y \in W^s_{\varepsilon_1}(x,n)$. Then

$$|S_n\phi^u(x) - S_n\phi^u(y)| = |\sum_{k=0}^{n-1}(\phi^u(f^kx) - \phi^u(f^ky))|$$

$$\leq \sum_{k=0}^{n-1} Ld(f^k x, f^k y)^{\alpha}$$

$$\leq 2L[\sup_{x,y\in\Lambda} d(x,y)^{\alpha} + (2c\varepsilon_o)^{\alpha} + (2c\varepsilon_o\lambda^{-1})^{\alpha} + \ldots + (2c\varepsilon_o\lambda^{-[n/2]})^{\alpha}]$$

$$\leq 2L(\sup_{x,y\in\Lambda} d(x,y)^{\alpha}) + \frac{2L(2c\varepsilon_o)^{\alpha}}{1-\lambda^{-\alpha}} \equiv K$$

Proof of (7.4). Consider $W^s_\varepsilon(x,n)$ with ε small, $x \in \Lambda$, and $n \geq 1$. Let (U,ϕ) be a C^2 coordinate chart about x which contains $W^s_\varepsilon(x,n)$ and flattens $W^u(x)$. That is if $B^\sigma = \{z \in \mathbb{R}^\sigma : |z| \leq 1\}$, $\sigma = s,u$, then $\phi : U \to B^u \times B^s$ is a C^2 diffeomorphism such that $\phi(x) = (0,0)$, $W^u_\varepsilon(x) \subset \phi^{-1}(B^u \times \{0\}) \subset W^u_{2\varepsilon}(x)$ and $W^s_{2\varepsilon}(x,n) \subset U$.

Let (ξ,η) be coordinates on $B^u \times B^s$, and for $z \in U$, write $\phi(z) = (\xi(z),\eta(z))$. If $z \in W^s_\varepsilon(x)$ let $D^u_\varepsilon(z,n)$ be the connected component of $\phi^{-1}(B^u \times \{\eta(z)\}) \cap W^s_\varepsilon(x,n)$ containing z. We depict this in figure 7.1.

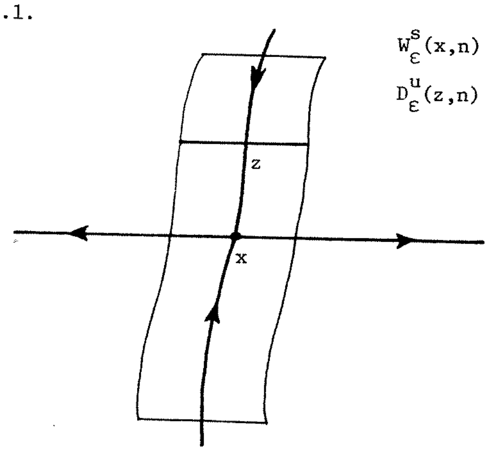


Figure 7.1

Now each $f^j D^u_\varepsilon(z,n)$ $0 \le j < n$, is a C^2 u-disk in M, and the Riemann metric induces a C^1 u-dimensional volume form $w(j,z) = w(j,z,n)$ on it. If $y \in f^j D^u_\varepsilon(x,n)$, then $w(j,x)_y$ coincides with the form w_y used to define ϕ^u, but if $y \notin f^j D^u_\varepsilon(x,n)$, this will not generally be true. The advantage of using the forms $w(j,z,n)$ will be that they come from a smooth coordinate system, ans later we will be able to use Fubini's theorem. Proceeding as in the definition of ϕ^u, if $y \in f^j D^u_\varepsilon(z,n)$, we define $\overline{\psi}(y)$ by the equation

$$f^* w(j+1,z)_{f(y)} = \overline{\psi}(y) w(j,z)_y .$$

Then, we let $\overline{\phi}^u(y) = -\log \overline{\psi}(y)$. As before, if $y \in D^u_\varepsilon(z,n)$, $z \in W^s_\varepsilon(x)$, then $(f^{-n})^* w(0,z)_y = e^{S_n \overline{\phi}^u(y)} w(n,z)_{f^n y}$.

Now we claim there is a constant $K > 0$ such that if $z \in W^s_\varepsilon(x)$ and $y \in D^u_\varepsilon(z,n)$, then

(a) $|S_n \overline{\phi}^u(z) - S_n \phi^u(x)| \le K$ and

(b) $|S_n \overline{\phi}^u(y) - S_n \overline{\phi}^u(z)| \le K$.

Suppose (a) and (b) hold for the moment. It is evident that, for ε small,

(c) $$\bigcup_{z \in W^s_{\varepsilon/2}(x)} D^u_{\varepsilon/2}(z,n) \subset W^s_\varepsilon(x,n) \subset \bigcup_{z \in W^s_{2\varepsilon}(x)} D^u_{2\varepsilon}(z,n).$$

Let m_u be a u-dimensional volume, and m_u be 8-dimensional volume. Clearly, there is a constant $C_{1,\varepsilon} > 0$ so that

(d) $C^{-1}_{1,\varepsilon} \le m_s(W^s_\varepsilon(x)) \le C_{1,\varepsilon}$.

By Fubini's theorem and (c) and (d), the volume lemma (7.4) will be proved if we can find a constant $C_{2,\varepsilon} > 0$ such that for $z \in W^s_\varepsilon(x)$,

(e) $C_{2,\varepsilon}^{-1} \le m_u(D^u_\varepsilon(z,n))\, e^{-S_n\phi^u(x)} \le C_{2,\varepsilon}$.

To prove (3), we first use the change of variables formula for multiple integrals to get

$$\text{(f)} \quad m_u(D^u_\varepsilon(z,n)) = \int_{D^u_\varepsilon(z,n)} w(o,z)_y\,dy = \int_{f^nD^u_\varepsilon(z,n)} e^{S_n\overline{\phi}^u(y)} \cdot w(n,z)_{f^ny}\,d(f^ny)$$

By an estimate similar to the λ-lemma (2.5), the disks $f^jD^u_\varepsilon(z,n)$ converge to $f^jD^u_\varepsilon(x,n)$ in the C^1 topology as j increases. Furthermore, $f^nD^u_\varepsilon(x,n) = W^u_\varepsilon(f^nx)$. Thus, there is a constant $C_{3,\varepsilon} > 0$ such that

$$C_{3,\varepsilon}^{-1} \le m_u(f^nD^u_\varepsilon(z,n)) \le C_{3,\varepsilon} .$$

By (a), (b), and (f),

$$e^{-2K+S_n\phi^u(x)} m_u(f^nD^u_\varepsilon(z,n)) \le m(D^u_\varepsilon(z,n)) \le e^{2K+S_n\phi^u(x)} \cdot m_u(f^nD^u_\varepsilon(z,n)).$$

So, (e) follows taking $C_{2,\varepsilon} = C_{3,\varepsilon}e^{2K}$.

Now the proofs of estimates (a) and (b) are very similar to that of lemma (7.3) except that we replace M by the Grassmann bundle $G^u(M)$ of u-dimensional planes over M. Since f is C^2, if we identify points $y \in f^jD^u_\varepsilon(z,n)$ with their u-dimensional tangent planes, $T_yf^jD^u_\varepsilon(z,n)$, then $\overline{\phi}^u$ becomes a C^1 function ζ^u on $G^u(M)$. Thus, there is a constant $K_1 > 0$ such that

$$|\zeta^u(\overline{x}) - \zeta^u(\overline{y})| \le K_1|\overline{x} - \overline{y}| \quad \text{for } \overline{x},\overline{y} \in G^u(M).$$

Let d_u be the metric on $G^u(M)$. If $z \in W^s_\varepsilon(x)$, then

$$d_u(T_{f^jz}f^jD^u_\varepsilon(z,n), E^u_{f^jx}) \le C\lambda^{-j}$$

for some $C > 0$ and $\lambda > 1$ by a λ-lemma type estimate. Setting $\bar{z} = T_z D^u_\varepsilon(z,n)$ and $\bar{x} = E^u_x$ gives

$$|S_n\zeta^u(\bar{z}) - S_n\zeta^u(\bar{x})| = |\sum_{k=0}^{n-1} \zeta^u(Tf^k\bar{z}) - \zeta^u(Tf^k\bar{x})|$$

$$\leq \sum_{k=0}^{n-1} K_1 C\lambda^{-k} \leq \frac{K_1 C}{1-\lambda^{-1}} .$$

This implies (a) for ζ^u, so (a) follows for $\bar{\phi}^u$.

On the other hand, for $0 \leq j < n$, each disk $f^j D^u_\varepsilon(z,n)$ is the graph of a C^1 function η_j from $W^u_\varepsilon(f^j x)$ to $W^s_\varepsilon(f^j x)$. Moreover, the C^1 sizes the η_j's are all uniformly bounded. This implies that if $y \in D^u_\varepsilon(z,n)$, then

$$d_u(T_{f^j y} f^j D^u_\varepsilon(z,n), T_{f^j z} f^j D^u_\varepsilon(z,n) \leq C_1 d(f^j y, f^j z)$$

$$\leq C_2 \lambda^{-(n-j)} \quad \text{for some constants } C_1, C_2 > 0.$$

Again, as above this gives (b) for ζ^u, and, hence, it gives (b) for $\bar{\phi}^u$.

<u>Proof of lemma (7.5)</u>. Recall that $Z_n = \sum_{p\in Per(n)} e^{S_n\phi^u(p)}$. Let $2\varepsilon > 0$ be an expansive constant for $f|\Lambda$.

We first prove part (a). If $p_1 \neq p_2$ in $Per(n)$, there $0 \leq j < n$ such that $d(f^j p_1, f^j p_2) > 2\varepsilon$. Thus, $W^s_\varepsilon(p_1,n) \cap W^s_\varepsilon(p_2,n) = \phi$. So, $m(M) \geq \sum_{p\ Per(n)} m(W^s_\varepsilon(p,n)) \geq C_\varepsilon^{-1} Z_n$ by the volume lemma (7.4). On the other hand, by specification (lemma (7.1)), for each $x \in \Lambda$, and $n \geq 1$, there is a $p \in Per(n + P(\varepsilon))$ such that $x \in W^s_\varepsilon(p,n)$. Thus

$$W^s_\varepsilon(\Lambda) = \bigcup_{x\in\Lambda} W^s_\varepsilon(x) \subset \bigcup_{p\in Per(n+P(\varepsilon))} W^s_{2\varepsilon}(p,n)$$

and

$$0 < m(W^s_\varepsilon(\Lambda)) \le C_{2\varepsilon} \sum_{p\in Per(n+P(\varepsilon))} e^{S_n\phi^u(p)} .$$

This gives $C_\varepsilon m(M) \ge Z_n \ge C_{2\varepsilon}^{-1} e^{-P||\phi^u||} m(W^s_\varepsilon(\Lambda))$ for $n > P(\varepsilon)$ and (a) clearly follows.

To prove (b), we first prove

(c) there is a constant $D_\varepsilon > 0$ so that for $n_i > P(\frac{\varepsilon}{3}) = P$, we have

$$D_\varepsilon^{-r} \prod_{i=1}^{r} Z_{n_i - P} \le Z_{n_1+\dots+n_r} \le D_\varepsilon^{r} \prod_{i=1}^{r} Z_{n_i+P} .$$

For $p \in Per(\sum_{i=1}^{r} n_i)$ and $1 \le i \le r$, let $z_i(p) \in Per(n_i + P)$ $\frac{\varepsilon}{3}$-specify $(f^{n_o+n_1+\cdot+n_{i-1}}p, n_i)$. For convenience, we let $n_o = 0$. Then let $\overline{z}(p) = (z_1(p),\dots,z_r(p))$. If $p_1 \ne p_2$, then some $j > 0$, $d(f^j p_1, f^j p_2) > \varepsilon$, so for some i and $k, d(f^i z_k(p_1), f^i z_k(p_2)) > \frac{\varepsilon}{3}$, and $\overline{z}(p_1) \ne \overline{z}(p_2)$. Thus, if $\tau = \sum_{i=1}^{r} n_i$, and K is as in lemma (7.3),

$$Z_\tau = \sum_{p\in Per\ \tau} e^{S_\tau\phi^u(p)}$$

$$= \sum_{p\in Per\ \tau} \exp(S_\tau\phi^u\ p) - \sum_{i=1}^{r} S_{n_i+P}\phi^u(z_i) + \sum_{i=1}^{r} S_{n_i+P}\phi^u(z_i))$$

$$\le \exp\ ((K + P||\phi^u||)r) \sum_{z_1,\dots z_r} \prod_{i=1}^{r} \exp(S_{n_i+P}\phi^u(z_i))$$

$$= D_\varepsilon^r \prod_{i=1}^{r} Z_{n_i+P}$$

with $D_\varepsilon = \exp(K + P||\phi^u||)$. Similarly, if $z_i \in Per(n_i - P)$, and

$\bar z = (z_1,\ldots,z_r)$, there is a $p(\bar z) \in \mathrm{Per}\,\tau$ such that

$$f^{n_o+\cdots+n_{i-1}}p(z) \quad \varepsilon/3\text{-specifies} \quad (z_i, n_i - P),$$

and $\bar z \neq \bar w$ implies $p(\bar z) \neq p(\bar w)$.

Hence,

$$\prod_{i=1}^{r} Z_{n_i-P} = \sum_{z_1,\ldots,z_r} \exp\Big(\sum_{i=1} S_{n_i-p}\phi^u(z_i)\Big)$$

$$\leq D_\varepsilon^r\, Z_{n_1+\cdots+n_r}$$

and (c) is established.

Now, we claim $Z_{n+P} \leq D_\varepsilon Z_n$ for all $n \geq 1$. If not, let n be such that $Z_{n+P} > (1+\alpha)D_\varepsilon Z_n$ with suitable small α.

Then, for $k \geq 1$, $Z_{k(n+2P)} \geq D_\varepsilon^{-k} Z_{n+P}^k$

$$\geq D_\varepsilon^{-k}(1+\alpha)^k D_\varepsilon^k Z_n^k .$$

So

$$\frac{1}{k(n+2P)} \log Z_{k(n+2P)} > \frac{1}{n+2P}\,[\log(1+\alpha) + \log Z_n].$$

By (a), the term on the left approaches zero as $k \to \infty$ which gives a contradiction. Similarly, we get $Z_{n-P} \geq D_\varepsilon^{-1} Z_n$ for all $n > P$. Then (b) follows setting $C_1 = D_\varepsilon^2$.

8. Diffeomorphisms with infinitely many attractors.

We have seen that hyperbolic attractors have a rich structure. Given a diffeomorphism f, one would like to know what kinds of attractors f possesses. In particular, are there finitely many? Do almost all points, say in the sense of Lebesgue measure, approach attractors? For even simple diffeomorphisms which arise in practice, these questions are very difficult.

On the other hand, if p is a hyperbolic periodic point of f, and $C\ell\, H_p(f)$ is hyperbolic, then $C\ell\, H_p(f)$ has a dense orbit and a local product structure. By theorem (3.8), $f|C\ell\, H_p(f)$ is a finite-to-one quotient of a subshift of finite type. Also, by theorem (3.7) there are neighborhoods U of $C\ell\, H_p(f)$ in M and N of f in $\mathrm{Diff}^1 M$ such that for $g \in N$, $g|\bigcap_n g^n(U)$ is topologically conjugate to $f|C\ell\, H_p(f)$. Thus, we understand the structure of $f|C\ell\, H_p(f)$ very well and this structure persists when f is perturbed.

Now there are many diffeomorphisms f for which $C\ell\, H_p(f)$ is not hyperbolic for some hyperbolic periodic point p, and we would like to understand these. At present our knowledge of these diffeomorphisms is quite incomplete, and here we shall merely focus on a few typical examples and some of their properties.

It follows from proposition (4.2) that a diffeomorphism with a hyperbolic limit set has only finitely many attractors. We will see below that on surfaces $C\ell\, H_p(f)$ not being hyperbolic frequently leads to the existence of infinitely many periodic attractors.

To begin with, let us try to imagine the simplest way we might have $C\ell\, H_p$ not hyperbolic. Clearly, we should try to find a non-transverse homoclinic point q of $o(p)$. However, the Kupka-Smale theorem (1.3) tells us that generically all homoclinic points are transverse, so even if we found such a q we could perturb it away. If we want a persistent lack of transversality, it is natural to replace $o(p)$ by an infinite hyperbolic set Λ and try to arrange for $W^u(\Lambda)$ and $W^s(\Lambda)$ to be non-transverse in a persistent way. This turns out to be easy to do in dimension larger than 2.

For instance, return to Plykin's example in section 5. This is a diffeomorphism f of a subset $D \subset \mathbb{R}^2$ into its interior such that $\bigcap_{n\geq 0} f^n D = \Lambda_1(f)$ is a one-dimensional hyperbolic attractor. Let p be a fixed point of f in Λ_1, and consider $W^u_\varepsilon(p)$ for some small ε. For each $x \in W^u_\varepsilon(p)$, $W^s_\varepsilon(x)$ is an interval, and $\{W^s_\varepsilon(x)\}_{x\in W^u_\varepsilon(p)}$ gives a 1-dimensional foliation of a neighborhood of p in D. Let $\lambda > 1$, and let $g(x) = \lambda x$ be a linear expansion on the line $\mathbb{R}$. Consider the mapping $f \times g : D \times \mathbb{R} \to D \times \mathbb{R}$. Clearly, (p,o) is a hyperbolic saddle point for $f \times g$ and $\dim W^u((p,o), f \times g) = 2$. Let γ be a curve joining two points $z_1 \in W^u_\varepsilon((p,o), f \times g) - \{(p,o)\}$ and $z_2 \in W^s_\varepsilon((p,o), f \times g) - \{(p,o)\}$ as in figure 8.1.

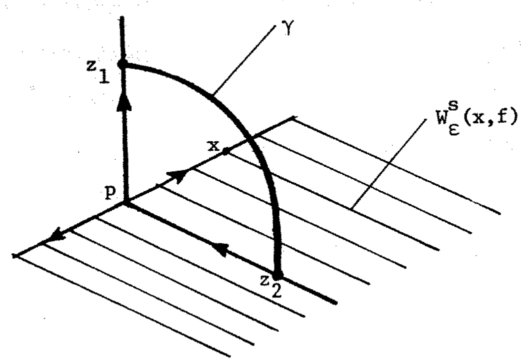


Figure 8.1

Let N be a small tubular neighborhood of γ, and let ϕ be a diffeomorphism such that $\phi(z) = z$ for $z \notin N$ and ϕ maps a disk Δ in $N \cap W^u_\varepsilon((p,o), f \times g)$ to a curved disk Δ' which meets $N \cap W^s_\varepsilon((p,o), f \times g)$ in a circle as in figure 8.2

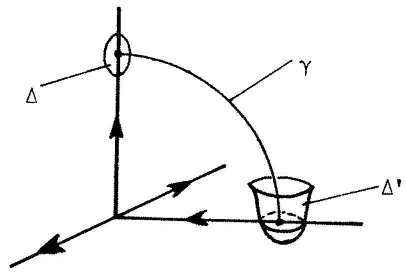


Figure 8.2

Let $\bar{f} = \phi \circ (f \times g)$. If N is small, one may check that $\Lambda' \equiv \Lambda_1 \times \{0\}$ is hyperbolic for $\bar{f}$, and for any f_1 C^1 near $\bar{f}$, $W^u(\Lambda'(f_1), f_1)$ has tangencies with $W^s(\Lambda'(f_1), f_1)$ near Δ'. That is, there are points $x, y \in \Lambda'(f_1)$ such that $W^u(x)$ is tangent to $W^s(y)$ near Δ'. Moreover, in this case, if f_1 is C^1 near f and p_1 is the hyperbolic fixed point of f_1 near (p,o) then $C\ell\, H_{p_1}(f_1)$ is not hyperbolic.

This shows that the Kupka-Smale theorem fails if we try to replace periodic points by hyperbolic sets. It also shows that Axiom A diffeomorphisms, or more generally diffeomorphisms with hyperbolic limit sets, are not dense in $\mathrm{Diff}^1 M$ for any M with $\dim M > 2$. It is less obvious that hyperbolic sets with persistently tangent stable and unstable manifolds exist in dimension two. We will see that they do in the C^r topology with $r \geq 2$, but it is still not known if $r = 1$.

Now let us assume that M is a compact two-dimensional manifold and $f \in \mathrm{Diff}^r M$ with $r \geq 2$. A hyperbolic basic set Λ for f is called *wild* if there is a neighborhood N of f in $\mathrm{Diff}^r M$ such that for any $g \in N$, there are points x and $y \in \Lambda(g)$ such that $W^u(x)$ and $W^s(y)$ are tangent somewhere.

We will omit the word basic and call such sets Λ, *wild hyperbolic sets.*

A periodic point p of period n of a diffeomorphism will be called *dissipative* if $\det T_p f^n < 1$. Let $S(f)$ denote the set of periodic sinks of f. That is, if $p \in S(f)$ and $f^n(p) = p$, then all eigenvalues of $T_p f^n$ have norm less than one.

Theorem (8.1). Suppose $f \in \mathrm{Diff}^r M^2$, $r \geq 2$, *has a wild hyperbolic set* Λ, *and* Λ *contains a dissipative periodic point* p. *There are a neighborhood* N *of* f *in* $\mathrm{Diff}^r M^2$ *and a residual subset* $B \subset N$ *such that if* $g \in B$, *then* $\mathcal{C}\ell\, H_p(g) \subset \mathcal{C}\ell\, S(g)$.

Thus under the conditions of theorem (8.1) each point of the h-closure $\mathcal{C}\ell\, H_p(g)$ is a limit of infinitely many periodic sinks. If Λ is a hyperbolic basic set for f and p is a periodic point in Λ, then $\Lambda \subset \mathcal{C}\ell\, H_p(f)$, so generically, each wild hyperbolic set with a dissipative periodic point is in the closure of the periodic sinks. This gives an infinite number of invariant open sets which accumulate on Λ, and is the reason why we call such Λ's wild.

The first part of the proof of theorem (8.1) is the next lemma.

Lemma 8.2. Suppose p *is a dissipative periodic point for* f *and* $W^u(o(p))$ *is tangent to* $W^s(o(p))$ *at a point* x. *If* U *is any neighborhood of* x *in* M *and* N *is any neighborhood of* f *in*

$\mathrm{Diff}^r M$, *then there is a* $g \in N$ *which has a periodic sink in* U.

<u>Proof</u>. Let $p_1, p_2 \in o(p)$ be such that $W^s(p_1)$ is tangent to $W^u(p_2)$ at x. Let γ_s and γ_u be small curves in $W^s(p_1)$ and $W^u(p_2)$ respectively such that γ_s and γ_u are tangent at s, and x is not in the boundary of γ_s or γ_u. Since x cannot be a fixed point of f, choose a small neighborhood U of x such that $f^{-1}U \cap U = \emptyset$ and $\gamma_s \cup \gamma_u \subset U$. Our perturbations of f will be of the form $g = \phi \circ f$ where ϕ is C^r near the identity and $\phi(z) = z$ for $z \notin U$. For U small enough, $f^n \gamma_s \cap U = \emptyset$ and $f^{-n}\gamma_u \cap U = \emptyset$ for $n > 0$. Thus, p is periodic for any such g and $\gamma_s \subset W^s(p_1, g)$ while $\phi(\gamma_u) \subset W^u(p_2, g)$. Let us write $\gamma_u(g) = \phi(\gamma_u)$.

We first choose ϕ_1 so that the curves γ_s and $\gamma_u(g_1)$ have nondegenerate second order contact at x.

This gives us figure 8.3.

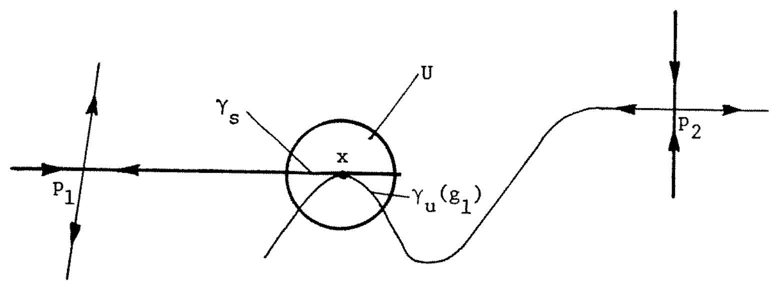


Figure 8.3

Assume we have x and U in a coordinate system (u,v) so that x corresponds to $(0,0)$, $\gamma_s \subset (v = 0)$, and $\gamma_u(g_1)$ corresponds to *periodic sink in* U.

$W^s(p_1)$ is tangent to

curves in $W^s(p_1)$ and

are tangent at s, and

nce x cannot be a fixed

of x such that

tions of f will be of the

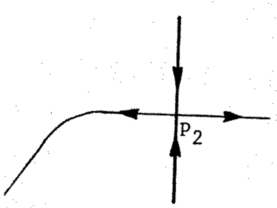
ntity and $\phi(z) = z$ for

nd $f^{-n}\gamma_u \cap U = \emptyset$ for

nd $\gamma_s \subset W^s(p_1,g)$ while

.

γ_s and $\gamma_u(g_1)$ have



system (u,v) so that

$g_1)$ corresponds to

$\{(u,v) : v = -au^2 + r(u)\}$ where $a > 0$ and $\lim_{u\to 0} \frac{|r(u)|}{|u|^2} = 0$. We may choose v_o near 0, a suitable small disk $D(v_o) = \{(u,v) : |u| \le \varepsilon_1, |v - v_o| \le \varepsilon_2\}$ where $\frac{\varepsilon_2}{\varepsilon_1}$ is small and an integer $n > 0$ so that $D(v_o)$ and $g_1^n D(v_o)$ look as in Figure 8.4.

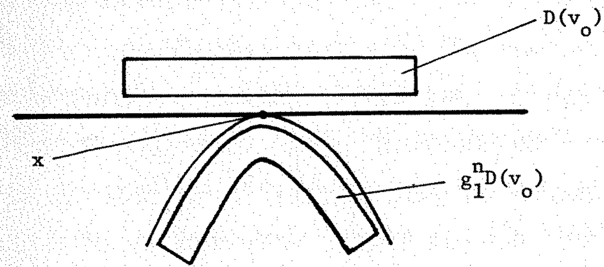


Figure 8.4

Of course $D(v_o)$ may be below γ_s, and $g_1^n D(v_o)$ may intersect $D(v_o)$. The important thing is that by translating $g_1^n D(v_o)$ up or down via a family $\{\phi_t\}$ with t in some interval $(-\delta,\delta)$, $\delta > 0$, we can find $s,t \in (-\delta,\delta)$ with $s \ne t$ such that

(1) $g_s^n D(v_o) \cap D(v_o) = \emptyset$

(2) $g_t^n D(v_o) \cap D(v_o) = A_1 \cup A_2$ and the index of g_t^n on $g_t^{-n} A_i$ is ± 1 for $i = 1,2$.

(3) g_ν^n has no fixed points on $\partial D(v_o)$ and the index of g_ν^n on $D(v_o)$ is 0 for $\nu \in (-\delta,\delta)$.

See figure 8.5

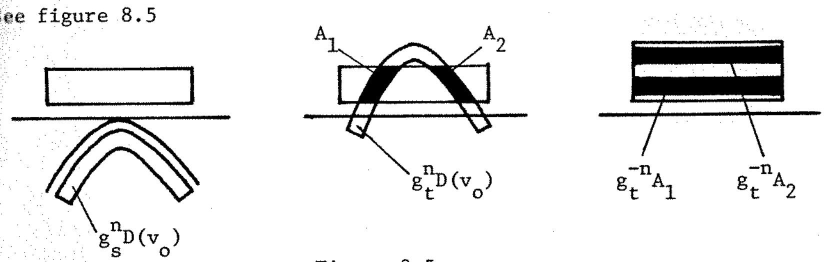


Figure 8.5

Now for some α between s and t, it follows that g_α^n has a fixed point z in $D(v_o)$ with eigenvalue 1. If $D(v_o)$ is very near x, then $\det T_w g_\alpha^n < 1$ for $w \in D(v_o)$ since most of the iterates $g^j D(v_o)$, $0 \le j \le n$, are near the dissipative orbit $o(p)$. Thus, the eigenvalues of $T_z g^n$ must be 1 and μ where $|\mu| < 1$.

Now there is a C^r g_α^n-invariant curve ζ through z $(r < \infty)$ tangent to the eigenspace of $T_z g_\alpha^n$ corresponding to 1. This uses the so-called center manifold theorem (see [18]). Then, $g_\alpha^n|\zeta$ is a diffeomorphism of a real interval with a fixed point of derivative 1, so we clearly can perturb $g_\alpha^n|\zeta$ to obtain a fixed point z_1 in ζ with derivative less than 1. If we realize this perturbation as $g^n|\zeta$ where $g = \phi \circ f$, then z_1 is a periodic sink for g.

Lemma (8.3). Let p *and* q *be* h*-related hyperbolic periodic points of a* C^r *diffeomorphism* f *with* $q \notin o(p)$. *Suppose* $W^u(o(p))$ *and* $W^s(o(p))$ *are tangent at some point. Then* f *may be* C^r *perturbed to* g *such that* $o(p)$ *and* $o(q)$ *are hyperbolic periodic orbits of* g, *and* $W^u(o(q),g)$ *and* $W^s(o(q),g)$ *have a tangency arbitrarily near* p.

<u>Proof</u>. Let x be the point at which $W^u(o(p))$ and $W^s(o(p))$ are tangent. Let $C(z,f)$ denote the connected component of a point z in a set F. For $\varepsilon > 0$ small, set $\gamma_\varepsilon^u = C(x, W^u(x) \cap B_\varepsilon(x))$ and $\gamma_\varepsilon^s = C(x, W^s(x) \cap B_\varepsilon(x))$ where $B_\varepsilon(x) = \{y \in M : d(x,y) \le \varepsilon\}$. Here, as usual, d is the distance function on M. If ε is small enough, then γ_ε^u is an interval about x in $W^u(x)$, γ_ε^s is an interval about x in $W^s(x)$, and, for $i \ne 0$, $f^i\gamma_\varepsilon^u \cap B_{2\varepsilon}(x) = \emptyset$ and $f^i\gamma_\varepsilon^s \cap B_{2\varepsilon}(x) = \emptyset$.

We will produce sequences of intervals $I_1, I_2, \ldots$ and $J_1, J_2, \ldots$ such that

(a) $I_i \subset W^u(o(q))$ and $J_i \subset W^s(o(q))$

(b) $I_i \to \gamma_\varepsilon^u$ and $J_i \to \gamma_\varepsilon^s$ in the C^r topologies

(c) for $n \geq 1$ and large i, $f^{-n}I_i \cap B_\varepsilon(x) = \emptyset$ and

$f^n J_i \cap B_\varepsilon(x) = \emptyset$.

Suppose these sequences have been found. Then for large i, (b) guarantees we may find a diffeomorphism ϕ C^r near the identity such that $\phi(I_i)$ is tangent to J_i near x and $\phi(\eta) = \eta$ for $\eta \notin B_\varepsilon(x)$. If we then set $g = \phi \circ f$, then (a) and (c) guarantee that $\phi(I_i) \subset W^u(o(q),g)$ and $J_i \subset W^s(o(q),g)$. Since the tangency of $\phi(I_i)$ and J_i is near x, its orbit under g will pass near p. Thus, to prove the lemma we only need to produce the sequences (I_i) and (J_i). We will produce the I_i's and leave the analogous construction of the J_i's to the reader.

Let τ be a common period of p and q. Let $p_1 \in o(p)$ and $p_2 \in o(p)$ be such that $W^u(p_1)$ and $W^s(p_2)$ are tangent at x. Let D be an interval in $W^u(p_1)$ such that $\{p_1,x\} \subset D - \partial D$ and $x \notin \bigcup_{1\leq j\leq \tau} f^{-j}D$, and let D' be an interval in $W^s(p_2)$ such that $p_2 \in D' - \partial D'$ and $x \notin D'$.

Since p and q are h-related, we may choose a point z of transverse intersection of $W^u(o(q))$ and $W^s(p_2)$ so that $z \in D' - \partial D'$. Because $o(q) \cap o(p) = \emptyset$, we have $W^u(o(q)) \cap W^u(o(p)) = \emptyset$, so we may choose a small $\varepsilon > 0$ and a small tubular neighborhood N of D such that $f^{-n}z \cap N = \emptyset$ for $n \geq 0$ and $\bigcup_{1\leq j\leq \tau} f^{-j}N \cap B_\varepsilon(x) = \emptyset$. Also, we may take a small tubular neighborhood N' of D' such that $\bigcup_{0\leq j\leq \tau} f^{j}N' \cap B_\varepsilon(x) = \emptyset$. Figure (8.6) gives a typical situation when $\tau = 1$. Observe that D may meet D' in several places

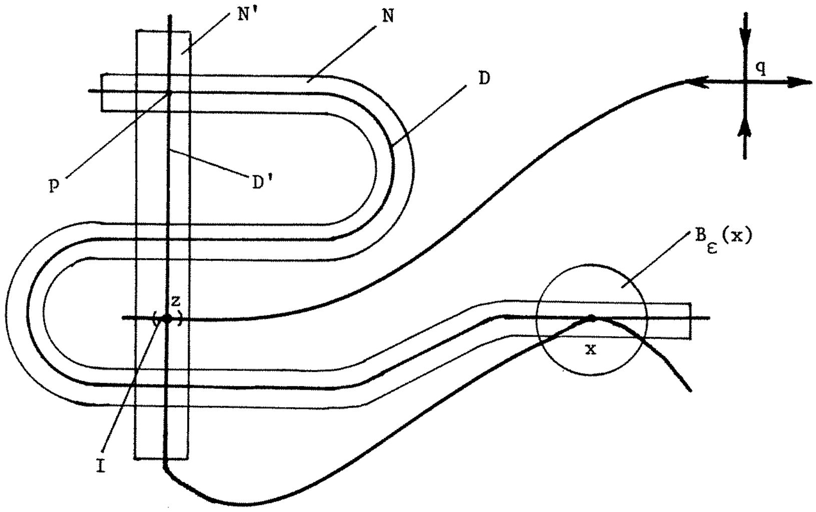


Figure 8.6

If I is a small interval about z in $W^u(o(q))$ and ε is small, then

(d) $f^{-n}I \cap B_\varepsilon(x) = \emptyset$ for $n \geq 0$

and

(e) If $k > 0$ is the smallest non-negative integer such that $f^k I \cap B_\varepsilon(x) \neq \emptyset$, then $f^i I \subset \bigcup_{1 \leq j \leq \tau} f^{-j}N \cup \bigcup_{0 \leq j < \tau} f^j N'$ for $0 \leq i < k$.

Now, for large i, a C^r version of (2.5) insures that $C(f^{\tau_i}I, N)$ is an interval C^r near D, so $C(f^{\tau_i}I, N) \cap B_\varepsilon(x)$ is an interval C^r near γ^u_ε. If we let $I_i = C(f^{\tau_i}I, N) \cap B_\varepsilon(x)$, then (d), (e), and the construction of N and N' guarantee that $f^{-n}I_i \cap B_\varepsilon(x) = \emptyset$ for $n \geq 1$, as required.

Lemma (8.4). Let Λ *be a hyperbolic basic set for* f *such that* $W^u(\Lambda)$ *and* $W^s(\Lambda)$ *are tangent at a point* x. *Let* p *be a periodic point in* Λ.

Then f *may be* C^r *perturbed to* g *so that* p *is periodic for* g *and* $W^u(o(p),g)$ *and* $W^s(o(p),g)$ *have a tangency arbitrarily near* x.

Proof. Using corollary (3.6), let $\varepsilon > 0$ be small enough so that if $U = \{y \in M : d(y,\Lambda) \leq \varepsilon\}$, then $\bigcap_n f^n U = \Lambda$ and $U \cap B_{2\varepsilon}(x) = \emptyset$. Let $z_1, z_2 \in \Lambda$ be such that $W^u(z_1)$ is tangent to $W^s(z_2)$ at x. Pick $n_o > 0$ so that $n \geq n_o$ implies $f^{-n}x \in W^u_\varepsilon(f^{-n}z_1) \subset U$ and $f^n x \in W^s_\varepsilon(f^n z_2) \subset U$. Then choose $\delta \in (0,\varepsilon)$ so that $f^j B_\delta(x) \cap B_\delta(x) = \emptyset$ for $|j| \leq n_o$. Let γ_u be a small interval in $f^{n_o} W^u_\varepsilon(f^{-n_o}z_1) \cap B_\delta(x)$ containing x. Then, $f^{-n}\gamma_u \cap B_\delta(x) = \emptyset$ for $n \geq 1$. Similarly, there is an interval $\gamma_s \subset W^s(z_2)$ about x such that $f^n\gamma_s \cap B_\delta(x) = \emptyset$ for $n \geq 1$. Since the orbits of the homoclinic points of p are dense in Λ, there are sequences $(r_i)_{i\geq 1}$, $(s_i)_{i\geq 1}$ of homoclinic points of $o(p)$ such that $r_i \to f^{-n_o}z_1$ and $s_i \to f^{n_o}z_2$ as $i \to \infty$. For large i, $W^u_{2\varepsilon}(r_i)$ is near $W^u_{2\varepsilon}(f^{-n_o}z_1)$, so there are intervals $I_i \subset W^u_{2\varepsilon}(r_i) \subset W^u(o(p))$ such that $f^{n_o}I_i \to \gamma_u$ in the C^1 (actually C^r) topology. For large i, one has $f^{-n+n_o}I_i \cap B_\delta(x) = \emptyset$ for $n \geq 1$. Similarly, there are intervals $J_i \subset W^s_{2\varepsilon}(s_i) \subset W^s(o(p))$ such that $f^{-n_o}J_i \to \gamma_s$ as $i \to \infty$, and, for large i and all $n \geq 1$, $f^{n-n_o}J_i \cap B_\delta(x) = \emptyset$. As in the proof of lemma (8.3), if we let ϕ be an appropriate function C^r near id with $\phi(w) = w$, for $w \notin B_\delta(x)$, and we set $g = \phi \circ f$, then we have, for some large i,

(a) $\phi(f^{n_o}I_i)$ is tangent to $f^{-n_o}J_i$ near x.

(b) p is a hyperbolic periodic point for g.

(c) $\phi(f^{n_o}I_i) \subset W^u(o(p),g)$ and $f^{-n_o}J_i \subset W^s(o(p),g)$.

This proves lemma (8.4).

Now we can prove theorem (8.1).

Let Λ be a wild hyperbolic set for f containing the dissipative periodic point p. Let N be a neighborhood of f so that if g is in N, then $W^u(\Lambda(g))$ is tangent somewhere to $W^s(\Lambda(g))$, and $p(g)$ is dissipative. For $n \geq 1$, let $\mathrm{Per}(n,g)$ be the set of hyperbolic periodic points of g of period less than or equal to n, and let $\mathrm{Per}(g)$ be the set of all hyperbolic periodic orbits of g. Finally, let H_n be the set of diffeomorphisms g in N such that for each $q \in \mathrm{Per}(n,g) \cap H_{p(g)}$, there is a periodic sink $s(q)$ of g such that $d(s(q),q) < \frac{1}{n}$. Clearly, H_n is an open subset of N. Using lemmas (8.2), (8.3), and (8.4), we can prove H_n is dense in N. To see this let $g \in N$, and let $q \in \mathrm{Per}(n,g) \cap H_{p(g)}$. Let $p = p(g)$. By lemma (8.4) we may perturb to g_1 so that $p \in \mathrm{Per}(g_1)$ and $W^u(o(p),g_1)$ is tangent somewhere to $W^s(o(p),g_1)$. From the proof of lemma (8.4), one sees that we may choose this g_1 so that $q \in \mathrm{Per}(g_1)$ and $p \sim q$. Now, lemma (8.3) says we may perturb to g_2 leaving p and q unaffected and introduce a tangency of $W^u(o(q),g_2)$ and $W^s(o(q),g_2)$ somewhere. Applying (8.3) again enables us to perturb g_3 so that $p \in \mathrm{Per}(g_3)$, $q \in \mathrm{Per}(g_3)$, and $W^u(o(p),g_3)$ has a tangency with $W^s(o(p),g_3)$ at some x with $d(x,q) < \frac{1}{2n}$. Now, lemma (8.2) gives us a g_4 near g which has a sink $s(q)$ within $\frac{1}{n}$ of q.

Thus, H_n is dense and open in N. So $B = \bigcap_{n\geq 1} H_n$ is residual in N. To complete the proof of theorem (8.1), we need only remark that if $g \in B$, and $q \in H_p(g)$, then q is a limit of a sequence $q_i \in \mathrm{Per}(n_i,g) \cap H_{p(g)}$ with $n_i \to \infty$. As we have already noted in section 3, this follows from the homoclinic theorem (2.3).

As an indication of the fact the wild hyperbolic sets occur frequently, we have

Theorem (8.5). *Let* p *be a dissipative hyperbolic saddle point of a* C^r *diffeomorphism* f *on* M^2, $r \geq 2$. *Suppose* $W^u(p)$ *and* $W^s(p)$ *be tangent at some point* x. *Then arbitrarily* C^r *near* f *there is a diffeomorphism* g *having a wild hyperbolic set near the orbit of* x.

For a proof of this theorem, see [32].

The condition that $W^u(p)$ and $W^s(p)$ be tangent somewhere occurs very naturally in one-parameter families of diffeomorphisms. For instance, it frequently occurs in arcs between two structurally stable diffeomorphisms of different types. The study of such arcs, or more generally, parametrized systems of diffeomorphisms and flows, is called bifurcation theory. This is a subject of wide scope with many interesting applications. We refer the reader to [24], [14], [13], [34], [35] for more information.

We now describe some specific diffeomorphisms of the two-disk D^2 with wild hyperbolic sets. Thinking of D^2 as a subset of any two manifold M^2, these diffeomorphisms provide examples on M^2. First, we need some preliminaries on Cantor sets.

A Cantor set F is a compact subset of the real line $\mathbb{R}$ such that each point of F is a limit point of F and the interior of F is empty. Given such a set we may write $\mathbb{R} - F = \bigcup_{i=-2}^{\infty} U_i$ where each U_i is an open interval and U_{-2} and U_{-1} are unbounded. We call the U_i's the gaps of F. Let F_o be the smallest closed interval containing F, and define, for $i \geq 1$, $F_i = F_o - \bigcup_{0 \leq j < i} U_j$. Then $F_o \supset F_1 \supset \ldots$ and $F = \bigcap_{i \geq 0} F_i$.

We call $\{F_i\}$ a defining sequence for F. It is obtained by specifying an ordering of the bounded gaps of F.

Each U_i, $i \geq 0$, lies in a single component $\overline{F_i}$ of F_i, and divides $\overline{F_i}$ into two components. Let $c_{i\ell}$ be the component of

$\overline{F_i} - U_i$ to the left of U_i, and let c_{ir} be the component of $\overline{F_i} - U_i$ to the right of U_i as in figure 8.7.

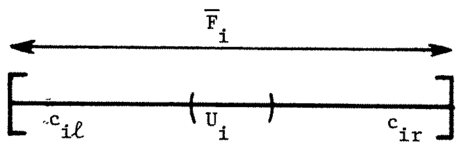


Figure 8.7

Let ℓJ be the length of an interval J and set

$$\tau(\{F_i\}) = \inf_{i\geq 0}\{\min(\frac{\ell c_{i\ell}}{\ell U_i}, \frac{\ell c_{ir}}{\ell U_i})\} \quad .$$

Finally, set $\tau(f) = \sup\{\tau(\{F_i\}) : \{F_i\}$ is a defining sequence for $F\}$. We call $\tau(F)$ the *thickness* of F. It measures the size of F in a certain sense.

Let us consider some examples. Let $0 < \beta < 1$. The middle β-set $F(\beta)$ is defined as follows.

Let F_o be a closed interval. Let U_o be the open interval of length $\beta\ell(F_o)$ centered at the midpoint of F_o. Then $F_1 = F_o - U_o$ has two components F_{11} and F_{12}. Form F_2 by removing open intervals U_{11}, U_{12} from F_{11} and F_{12} so that U_{1i} is centered in F_{1i} and $\ell(U_{1i}) = \beta\ell(F_{1i})$. Continue in this manner defining F_i by removing from each component c of F_{i-1} a centrally placed open interval of length $\beta\ell(c)$. Thus, if $\frac{1-\beta}{2} = \alpha$, then F_i is a union of 2^i components of length α^i. Finally, set $F(\beta) = \bigcap_{i\geq 0} F_i$. The F_i's in the example are not a defining sequence for $F(\beta)$, because we have taken out several gaps of F at each stage. It is easy to check that $\tau(F(\beta)) = \frac{\alpha}{\beta} = \frac{1-\beta}{2\beta}$. So $\tau(F(\beta)) \to \infty$ as $\beta \to 0$. Observe that the

Lebesgue measure of each $F(\beta)$ is 0 since the measure of $F_i = 2^i\alpha^i$.

The reason for thickness is the following.

Theorem 8.6. If F *and* G *are Cantor sets with* $\tau(F)\cdot\tau(G) > 1$ *and neither is contained in a gap of the other, then* $F \cap G \neq \emptyset$.

Proof. Let $\{F_i\}$ and $\{G_i\}$ be defining sequences of F and G respectively such that $\tau(\{F_i\})\cdot(\{G_i\}) > 1$. Since neither F nor G is in a gap of the other, $F_o \cap G_o \neq \emptyset$. We prove the following statement.

(*) If c is a component of F_i which meets G, then c contains a component of F_{i+1} which meets G.

Once (*) has been proved, we have that each F_i meets G, so F meets G. Let us prove (*).

Suppose c is a component of F_i such that $c \cap G \neq \emptyset$. If c is also a component of F_{i+1}, there is nothing to prove, so assume that F_{i+1} is obtained from F_i by removing the F-gap $U_i^F \subset c$. Let c_ℓ, c_r be the components of $c - U_i$ as in figure 8.8.

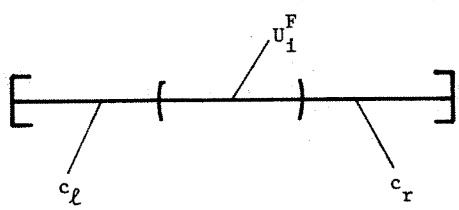


Figure 8.8

Assume by way of contradiction that c_ℓ and c_r do not meet G. Thus, both c_ℓ and c_r are in G-gaps. If both c_ℓ and c_r were in unbounded G-gaps, then we would have one of the following situations.

(a) G_o lies to the left of c_ℓ

(b) G_o lies to the right of c_r

(c) $G_o \subset U_i^F$.

Now (a) and (b) contradict the assumption that $c \cap G \neq \emptyset$, and (c) would put G in the F-gap U_i^F.

Hence, at least one of c_ℓ and c_r is in a bounded G-gap. Suppose c_ℓ is in the bounded G-gap U_j^G. The argument is similar if c_r is in a bounded G-gap.

Let c'_ℓ and c'_r be the components of G_{j+1} adjacent to U_j^G as in figure 8.9.

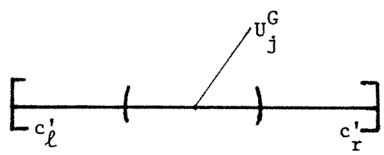


Figure 8.9

Now c_r cannot be in U_j^G for this would give $c \subset U_j^G$ contrary to hypothesis.

Then we have,

Case 1: $c'_r \subset U_i^F$

or

Case 2: $c_r \cap c'_r \neq \emptyset$.

In case 1, we have figure 8.10.

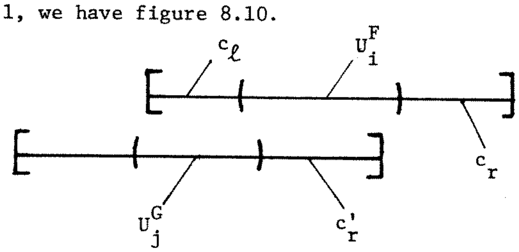


Figure 8.10

This gives $\dfrac{\ell c_\ell}{\ell(U_i^F)} \cdot \dfrac{\ell(c'_r)}{\ell(U_j^G)} \le 1$, a contradiction.

In case 2, we have c_r is contained in another G-gap U_k^G which is to c'_r. So we have figure 8.11.

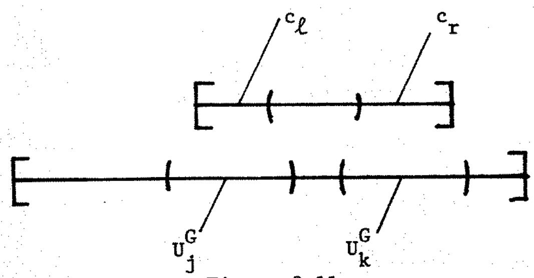


Figure 8.11

If c'' is the component of G_{k+1} adjacent to U_k^G and to its left, then $c''_\ell \subset U_i^F$, so $\dfrac{\ell c''_\ell}{\ell U_k^G} \cdot \dfrac{\ell c_r}{\ell U_i^F} \le 1$, a contradiction. This proves (*).

Now we describe our examples. All of our diffeomorphisms will be C^r with $r \ge 2$ fixed.

First return to the horseshoe diffeomorphism as defined in section 2. This is a diffeomorphism f of $\mathbb{R}^2$ to $\mathbb{R}^2$ for which there is a square Q such that $fQ \cap Q = A_1 \cup A_2$ with A_1 and A_2 disjoint rectangles such that $Tf|f^{-1}A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and $Tf|f^{-1}A_2 = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha^{-1} \end{pmatrix}$ with $0 < \alpha < \frac{1}{2}$. This time we adjust Q so that the left sides of Q and A_1 coincide and the right sides of Q and A_2 coincide, and we take $\alpha \in (\frac{1}{3}, \frac{1}{2})$. We also want the left lower corner of Q to be a hyperbolic fixed point p for f such that the boundary of Q is in $W^u(p) \cup W^s(p)$. See figure (8.12).

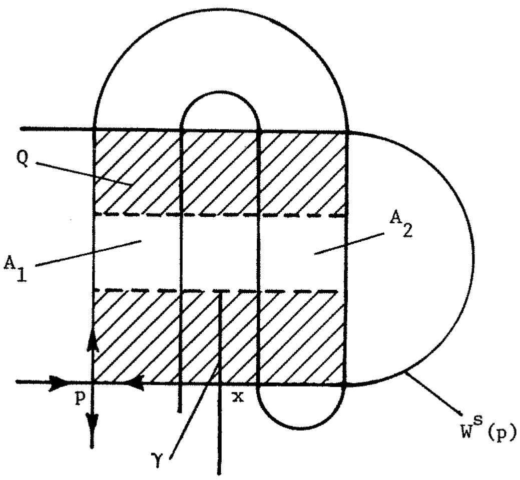


Figure 8.12

The shaded area is $f^{-1}A_1 \cup f^{-1}A_2$. Let x be the midpoint of the base of Q, and let γ be a vertical line segment through x dividing $f^{-1}A_1$ in half. Recall that $\bigcap_{n\leq 0} f^nQ$ has the form (interval) $\times$ (Cantor set). If we let $F^s(\gamma) = \gamma \cap \bigcap_{n\leq 0} f^nQ$, then $F^s(\gamma)$ is a middle β-set with $\alpha = \frac{1-\beta}{2}$. Since $\frac{1}{3} < \alpha < \frac{1}{2}$, we have $\beta < \frac{1}{3}$, and $\tau(F^s(\gamma)) = \frac{\alpha}{\beta} > 1$.

From the way we have defined f, there is an interval I in $W^u(p)$ below Q such that $\partial I \subset A_2$, $f(\partial I) \subset A_1$, and $f(I) \cap \text{int } Q = \emptyset$ as in figure 8.13a.

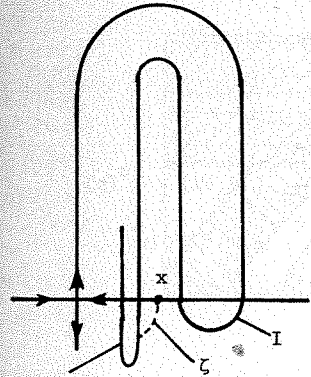


Figure 8.13a

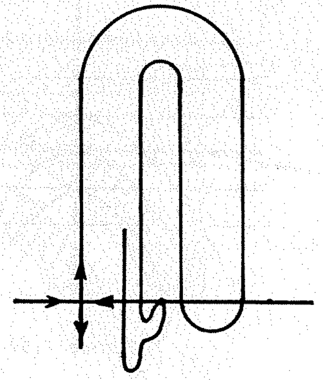


Figure 8.13b

We can modify f through a curve of diffeomorphisms to f_1 so that $f_1(z) = f(z)$ off a neighborhood of $A_1 \cup A_2$, $I \subset W^u(p,f_1)$, and $f_1(I)$ is tangent to $W^s(p,f_1)$ at x as in figure 8.13b.

This is most easily accomplished as follows.

Let ζ be a curve joining a point $f(I) - Q$ and x as in figure 8.13a. Let ϕ be a C^r diffeomorphism such that $\phi(z) = z$ outside a small neighborhood of ζ and ϕ maps a piece of $f(I)$ to a curve tangent at x to $W^s(p)$. Then take $f_1 = \phi \circ f$.

Let F^u be the vertical coordinate curves near the right boundary of Q, and let F^s be the horizontal coordinate curves near the bottom of Q.

Then $f_1 F^u$ is a foliation of a neighborhood of $f_1(I)$, and F^s is a foliation of a neighborhood of $W^s_\varepsilon(x)$ for small ε. Let us choose ϕ so that $f_1 F^u$ and F^s are tangent along a vertical curve which contains γ, and all of these tangencies are only first order contacts of the corresponding curves. This gives us Figure 8.14.

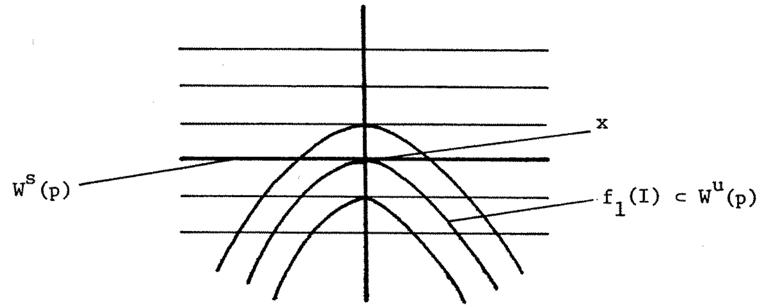


Figure 8.14

Thus, if U is a small neighborhood of x in M, and $\pi: U \to \gamma$ is the projection onto γ, then π restricted to any curve in f_1F^u has a unique non-degenerate maximum which lies on γ. Let $\bar{Q}$ be a small rectangular neighborhood of Q.

If we push $f_1(I)$ up slightly to give a new f_2, then we claim $\Lambda(f_2) = \bigcap_n f_2^n(\bar{Q}) = \bigcap_n f^n(\bar{Q})$ is a wild hyperbolic set for f_2.

The proof of this fact is as follows.

For g near f_2 and $0 \le n \le \infty$, let $\bar{W}_n^s(\Lambda(g)) = \bigcap_{-n \le j \le 0} g^j\bar{Q} \cap U$, and let $\bar{W}_n^u(\Lambda(g)) = g^2(\bigcap_{0 \le j \le n} g^j\bar{Q}) \cap U$. It is evident that $\bar{W}_\infty^s(\Lambda(f_2))$ and $\bar{W}_\infty^u(\Lambda(f_2))$ are two families of curves in $W^s(\Lambda(f_2))$ and $W^u(\Lambda(f_2))$, respectively, and $\bar{W}_\infty^s(\Lambda(f_2)) \cap \gamma$ and $\bar{W}_\infty^u(\Lambda(f_2)) \cap \gamma$ are Cantor sets. Set $F^s(f_2) = \bar{W}_\infty^s(\Lambda(f_2)) \cap \gamma$ and $F^u(f_2) = \bar{W}_\infty^u(\Lambda(f_2)) \cap \gamma$. Note that $F^s(f_2) = F^s(\gamma)$ since $f_2 = f$ off a small neighborhood of $f^{-1}\zeta$.

Figure 8.15 shows $\bar{W}_2^s(\Lambda(f_2))$ and $\bar{W}_2^u(\Lambda(f_2))$.

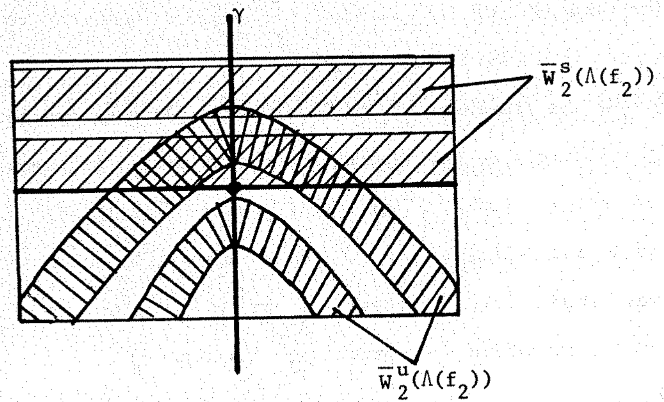


Figure 8.15

We assume f_2 is chosen so that $\tau(F^s(f_2)) = \frac{\alpha}{\beta} = \tau(F^u(f_2)) > 1$. From the construction, it is clear that the smallest closed intervals in γ containing $F^s(f_2)$ and $F^u(f_2)$ intersect. By lemma (8.6), $F^s(f_2)$ and $F^u(f_2)$ themselves intersect.

For g C^r near f_2, there are foliations $F^s(g)$ and $gF^u(g)$ which contain the sets $\overline{W}^s_\infty(\Lambda(g))$ and $\overline{W}^u_\infty(\Lambda(g))$, respectively. These foliations are C^1 near $F^s(f_2)$ and $f_2F^u(f_2)$, respectively, and have C^1 tangent line fields which are tangent along a C^1 curve $\gamma(g)$ which in turn is C^1 near γ. See lemma (9) in [32] for a proof of this. Now, $F^s(g) \equiv \overline{W}^s_\infty(\Lambda(g)) \cap \gamma(g)$, and $F^u(g) = \overline{W}^u_\infty(\Lambda(g)) \cap \gamma(g)$ are Cantor sets, and neither is in a gap of the other. If we show their thicknesses are near $\frac{\alpha}{\beta}$, then lemma (8.6) will show that $F^s(g) \cap F^u(g) \neq \emptyset$. Any point in this last set will be a tangency of $W^u(\Lambda(g)) \cap W^s(\Lambda(g))$, and we will have shown that $\Lambda(f_2)$ is a wild hyperbolic set.

Let us prove that $\tau(F^s(g))$ is near $\frac{\alpha}{\beta}$. The proof for $F^u(g)$ is similar.

For n large, let I be a component of $\overline{W}^s_n(\Lambda(g)) \cap \gamma(g)$, and let

J be the component of $[\bar{W}^s_{n-1}(\Lambda(g)) - \bar{W}^s_n(\Lambda(g))] \cap \gamma(g)$ which is adjacent to I. It is sufficient to prove that $\frac{\ell I}{\ell J}$ is near $\frac{\alpha}{\beta}$. Let $W^u_1(p(g))$ be the component of $p(g)$ in $W^u(p(g)) \cap \bar{Q}$, and let $\pi : \gamma(g) \to W^u_1(p(g))$ be the mapping induced by projecting along the curves of $F^s(g)$. Since $F^s(g)$ is C^1 near $F^s(f_2)$, $\gamma(g)$ is C^1 near $\gamma(f_2)$, and $W^u_1(p(g))$ is C^r near $W^u_1(p(f_2))$, one has the norm of the derivative $|T\pi|$ is near 1 on $\gamma(g)$. Thus, if $I_1 = \pi I$ and $J_1 = \pi J$, it suffices to prove $\frac{\ell I_1}{\ell J_1}$ is near $\frac{\alpha}{\beta}$.

Now

$$\frac{\ell(I_1)}{\ell(J_1)} = \frac{\alpha_1 \cdot \alpha_2 \cdots \alpha_{n-1} \cdot \ell g^{n-1} I_1}{\beta_1 \cdot \beta_2 \cdots \beta_{n-1} \cdot \ell g^{n-1} J_1}$$

where $\alpha_i = |T_{z_i}(g)^{-1} v_i|$, $\beta_i = |T_{z_i'}(g)^{-1} v_i'|$, $z_i \in g^i I_1$, $z_i' \in g^i J_1$, v_i is a unit vector in $T_{z_i} g^i I_1$, and v_i' is a unit vector in $T_{z_i'} g^i J_1$.

For g near f_2, there is a homeomorphism $h : \Lambda(g) \to \Lambda(f_2)$ near the inclusion $i_{\Lambda(g)} : \Lambda(g) \to \mathbb{R}^2$ such that $f_2 h = hg$. Also, $h(\partial I_1) = \partial(I_1(f_2))$ and $h(\partial J_1) = \partial(J_1(f_2))$ where $I_1(f_2)$ and $J_1(f_2)$ are intervals in $W^u(p(f_2))$, and

$$\frac{\ell(f_2^{n-1} I_1(f_2))}{\ell(f_2^{n-1} J_1(f_2))} = \frac{\alpha}{\beta} .$$

Since h is near $i_{\Lambda(g)}$, it follows that $\frac{\ell(g^{n-1} I_1)}{\ell(g^{n-1} J_1)}$ is near $\frac{\alpha}{\beta}$. Thus it suffices to show that $\prod_{i=1}^{n-1} \frac{\alpha_i}{\beta_i}$ is near 1. Since $\prod_{i=1}^{n-1} \frac{\alpha_i}{\beta_i} = \prod_{i=1}^{n-1} (1 - \frac{\beta_i - \alpha_i}{\beta_i})$, it suffices to show that $\sum \frac{|\beta_i - \alpha_i|}{|\beta_i|}$ is small. For g near f_2, one has $|\beta_i| > \frac{\alpha}{2}$ for all i, so we only need to show that $\sum |\beta_i - \alpha_i|$ is small. Let $K_1(g) = \sup_{z \in \bar{Q}} |T_z g^{-1}|$, let $K_2(g)$ be the maximum of the curvatures of $g^i I_1 \cup g^i I_2$ for $1 \le i \le n-1$, and let $K_1(g)$ be the maximum norm of the second derivatives of g^{-1} on $\bar{Q} \cap g^{-1}\bar{Q}$. Observe that $K_2(f_2) = K_3(f_2) = 0$ since f_2 is piecewise linear on $\bar{Q} \cap f_2^{-1}\bar{Q}$. Given $\varepsilon > 0$, choose $\varepsilon_1 > 0$ so that $4(2\varepsilon_1 K_1(f_2) + \varepsilon_1)\operatorname{diam}\bar{Q} < \varepsilon$, and then suppose g is close enough to f_2 so that $K_1(g) \le 2K_1(f_2)$, $\max\{K_2(g), K_3(g)\} < \varepsilon_1$ and $\ell(g^i I_1 \cup g^i J_1) \le (\frac{3}{4})^i \operatorname{diam}\bar{Q}$ for $1 \le i \le n-1$. Then,

$$\begin{aligned}
|\beta_i - \alpha_i| &= \big|\,|T_{z_i'}g^{-1}v_i'| - |T_{z_i}g^{-1}v_i|\,\big| \\
&\le \big|\,|T_{z_i'}g^{-1}v_i'| - |T_{z_i'}g^{-1}v_i|\,\big| + \big|\,|T_{z_i'}g^{-1}v_i| - |T_{z_i}g^{-1}v_i|\,\big| \\
&\le |T_{z_i'}g^{-1}(v_i' - v_i)| + |(T_{z_i'}g^{-1} - T_{z_i}g^{-1})v_i| \\
&\le (K_1(g)K_2(g) + K_3(g))\,\ell(g^i I_1 \cup g^i J_1) \\
&\le (2K_1(f_2)\varepsilon_1 + \varepsilon_1)(\tfrac{3}{4})^i \operatorname{diam}\bar{Q}
\end{aligned}$$

and, so, $\sum|\beta_i - \alpha_i| \le 4(2K_1(f_2)\varepsilon_1 + \varepsilon_1)\operatorname{diam}\bar{Q} < \varepsilon$.

Notes: The modification Plykin's example at the beginning of this section is a variation of examples first studied by Simon [52]. The remaining parts of this section are variations of results in [26], [29], and [32]. The first examples of open sets of diffeomorphisms with non-hyperbolic h-closures were given by Abraham and Smale [1], and variations of these were described by Shub [54], and studied in Hirsch, Pugh, and Shub [18].

Concluding Remarks.

We have tried in these lectures to present some of the methods and results which have been developed in recent years to describe non-trivial recurrence in ordinary differential equations. We understand hyperbolic systems quite well, and they provide a wide range of examples. On the other hand, it is clear that there are many examples of non-hyperbolic recurrence which have to be faced.

For future development, we wish to single out several problems.

1. Find examples of hyperbolic sets in models of specific physical systems. In particular, find non-trivial hyperbolic attractors.

2. Develop a structure theory for non-hyperbolic recurrence. For instance, is it true generically in $\mathrm{Diff}^r(S^2)$ that $L(f)$ has Lebesgue measure zero?

3. If $N \subset \mathrm{Diff}^r(M^2)$ is an open set of non-Ω-stable diffeomorphisms, is there an $f \in N$ which has a wild hyperbolic set?

4. If (σ, Σ_A) is a subshift of finite type, and $F \subset \Sigma_A$ is closed with $\sigma(F) = F$, we call (σ, F) a subshift on finitely many symbols. Is there a residual set $B \subset \mathrm{Diff}^r M$ such that for $f \in B$, each $\mathcal{C}\ell\, H_p(f)$ is a finite-to-one quotient of a subshift on finitely many symbols?

5. (Thom) Is there a residual set $B \subset \mathrm{Diff}^r M$ such that for $f \in B$ the union of the basins of the attractors of f is dense in M?

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