

Conservative systems and two problems of Smale

by

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We wish to consider two problems which have been raised by S. Smale concerning the orbit structure of diffeomorphisms.

Problem 1. [13] Is it true that every C^r diffeomorphism f , $r \geq 1$, on the two-sphere S^2 (or the two-disk D^2) can be C^r approximated by a diffeomorphism g having a periodic sink?

Problem 2. [14] Are the stable sets $W^s(x, f)$ smooth manifolds generically for diffeomorphisms f of a compact manifold M where $x \in M$?

First recall that a periodic point p of a diffeomorphism f is a point p such that $f^n(p) = p$ for some $n > 0$. The point p is called a sink if all eigenvalues of the derivative, $T_p f^n$, of f^n at p have norm less than one.

The main motivation for problem 1 comes, of course, from the theory of ordinary differential equations. It is known that in Lienard's equation

$$(1) \quad \ddot{x} + h(x) \dot{x} + g(x) = F(t)$$

where h , g and F are smooth functions with F periodic of period 1, certain conditions on h , g , and F will insure that the time-one map f carries a disk in the (x, \dot{x}) -plane into itself. Periodic sinks of f correspond to asymptotically stable periodic solutions of (1). These are of direct physical interest in many instances. Equation (1) has been studied by many authors. An important problem is to determine necessary and sufficient conditions on h , g , and F for the map f to be structurally stable.

The general answer to problem 1 is no. There is an example due to Plikin [8] of a structurally stable diffeomorphism of S^2 with a one dimensional attractor and no periodic sinks. However, we do not know if an example of Plikin's type can occur in equation (1).

On the other hand, the analog of Problem 1 for conservative diffeomorphisms is true in the C^1 topology. A conservative diffeomorphism in this case is taken to mean an area preserving one. To be more precise about our analogy, observe that a sink p of period n of $f : S^2 \rightarrow S^2$ has arbitrarily small neighborhoods which are mapped into their interiors by f^n . By theorems of Kolmogorov, Arnold, Moser, and Rüssmann, if p is a generic elliptic periodic point of period n of a C^5 area preserving diffeomorphism $f : S^2 \rightarrow S^2$, then p has arbitrarily small neighborhoods which are mapped onto themselves by f^n . To say that p is a generic elliptic point of period n means that the eigenvalues of $T_p f^n$ have norm one, are not real, and a certain open dense condition on the 5-jet of f^n at p holds. Thus generic elliptic points in conservative diffeomorphisms are the analogs of sinks in non-conservative diffeomorphisms.¹

Theorem 1. Let f be a C^1 area preserving diffeomorphism of S^2 , and let U be any open set in S^2 . There is a C^5 area preserving diffeomorphism g arbitrarily C^1 close to f having a generic elliptic periodic point in U .

This result follows from a general C^1 approximation theorem for symplectic diffeomorphisms which we will now describe [5].

Let (M, ω) be a compact symplectic manifold (i.e. ω is a closed non-degenerate 2-form), and let $\text{Diff}_\omega^r(M)$ be the set of ω -symplectic diffeomorphisms (i.e. $f^*\omega = \omega$) with the uniform C^r topology. A periodic point p of f with period n is called 1-elliptic if $T_p f^n$ has one pair of non-real eigenvalues of norm one and all its other eigenvalues have norm different from one. The diffeomorphism $f : M \rightarrow M$ is called Anosov if there are a continuous splitting $TM = E^s \oplus E^u$ invariant by Tf so that, in some Riemann norm, $Tf|_{E^s}$ is contracting and $Tf|_{E^u}$ is expanding.

¹The conservative case arises in Lienard's equation (1) above from the situation when $h(x) \equiv 0$

Theorem 2. There is a residual subset $B \subset \text{Diff}_\omega^1(M)$ such that if f is in B , then either f is Anosov or the 1-elliptic periodic points of f are dense in M .

Theorem 1 follows from theorem 2 via an elementary smoothing technique since Anosov diffeomorphisms on two manifolds only exist on the two torus.

Another corollary of theorem 2 is

Corollary 3. A symplectic diffeomorphism is structurally stable if and only if it is Anosov.

Recall that f is structurally stable if for any $g \in C^1$ near f there is a homeomorphism $h : M \rightarrow M$ with $hf = gh$.

Now we turn to problem 2 above.

Theorem 4. On any manifold M with $\dim M > 1$ there is an open set $U \subset \text{Diff}^r M$, $r \geq 2$, such that each $f \in U$ has points x in M for which $W^S(x, f)$ is not a manifold.

The diffeomorphisms in theorem 4 are those which have been studied in [2] and [4] and which contain certain closed invariant sets which we will call wild homoclinic sets or wild horeshoes.

An f -invariant subset Λ will be called a wild homoclinic set (for f) if

- (2) Λ contains a hyperbolic subset Λ_0 such that $W^u(\Lambda_0, f)$ has a tangency with $W^s(\Lambda_0, f)$ and $\Lambda = \text{Cl}[W^u(\Lambda_0) \cap W^s(\Lambda_0)]$.
- (3) there is a C^2 neighborhood N of f so that each g in N has a hyperbolic set $\Lambda_0(g)$ near Λ_0 for which $W^u(\Lambda_0(g), g)$ has a tangency with $W^s(\Lambda_0(g), g)$.

It is to be hoped that wild homoclinic sets can be avoided in most applications.

A significant problem is to determine whether they can exist in the zones of instability [12] near an elliptic point of an area preserving map of the plane.

In view of theorem 4, one is led to ask if there is some understandable structure to stable sets. There are indications that in many cases they are actually locally the product of a Cantor set and a manifold. In fact, near the boundary of non-conservative Ω -stable systems, it seems that one will be able to obtain considerably more. The results we have in mind are best illustrated in the setting of bifurcation theory.

First recall Smale's notion of a fine filtration [10], [11]. A filtration for a diffeomorphism $f : M \rightarrow M$ is a decreasing sequence of submanifolds with boundary $M_{n-1} \supset \dots \supset M_1$ such that if $M_n = M$ and $M_0 = \emptyset$, then $f(M_i) \subset \text{int } M_i$ for $1 \leq i \leq n$. The filtration is fine if $\Omega(f) = \bigcup_{i=1}^n \left[\bigcap_{-\infty < n < \infty} f^n(M_i - M_{i-1}) \right]$. Here $\Omega(f)$ is the non-wandering set of f which is defined as the set of points $x \in M$ such that for each neighborhood U of x there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.

The sets $\Omega_i(f) = \bigcap_{-\infty < n < \infty} f^n(M_i - M_{i-1})$ will be called the basic sets for $\Omega(f)$ relative to the fine filtration $\{M_i\}$.

Let $\Phi^{k,r}$ be the set of C^k maps from $[0,1]$ into $\text{Diff}^r(M)$ with the uniform C^k topology. Let AS be the set of diffeomorphisms satisfying Axiom A and strong transversality. For $\xi \in \Phi^{k,r}$, set $b_0 = b_0(\xi) = \inf\{t \in [0,1] : \xi_t \notin \text{AS}\}$. For f in $\text{Diff}^r(M)$, let $L^-(f)$ be the closure of the α -limit points of f . Let $G(M)$ be the set of diffeomorphisms $f : M \rightarrow M$ such that

(4) f has a fine filtration with associated Ω -decomposition

$$\Omega(f) = \Omega_1 \cup \dots \cup \Omega_n$$

(5) $f|_{\Omega_i}$ is topologically transitive with periodic points dense

(6) for each $x \in M$, $W^s(x, f)$ is either an immersed open disk or locally the product of a Cantor set and a disk

(7) an open dense set of points of x have their ω -limit sets in finitely many hyperbolic attractors.

One should think of elements of $G(M)$ as being like Axiom A systems with a few non-hyperbolic saddle basic sets. In theorem 6 there will be one bad basic

set and it will be a wild homoclinic set.

Let $A^{k,r}$ be the set of curves $\xi \in \Phi^{k,r}$ such that

- (8) $\xi_o \in AS$ and $L^-(\xi_{b_o})$ is hyperbolic
 (9) $\Omega(\xi_{b_o})$ consists of $L^-(\xi_{b_o})$ and a single orbit $o(x)$ where x is a tangency between $W^u(p_1)$ and $W^s(p_2)$ with p_1 and p_2 are hyperbolic periodic points such that $\dim W^u(p_1) = \dim W^u(p_2)$.

An example of a curve in $A^{k,r}$ is obtained by modifying the horseshoe diffeomorphism of Smale as in Figure 4c, p. 143 of [3]. Further examples are studied in [6] and [7].

Theorem 5. For $k \geq 1$, $r \geq 2$, a residual set of ξ in $A^{k,r}$ satisfy the following. For $\epsilon > 0$, there are $\delta > 0$ and a set $B_\delta \subset [b_o, b_o + \delta)$ such that

- (1) $m(B_\delta) < \epsilon\delta$ (m is Lebesgue measure).
 (2) $\xi_t \in G(M)$ for $t \in [b_o, b_o + \delta) - B_\delta$.

Proofs of theorems 4 and 5 will appear elsewhere.

We remark that the structure of ξ_t on its wild basic sets, $t \in [b_o, b_o + \delta) - B_\delta$, can be described by a symbolic dynamics similar to that of Bowen's for Axiom A systems [1].

Recall that the examples in [4] showed that the set diffeomorphisms with fine filtrations is not residual. On the other hand, theorem 5 asserts that many diffeomorphisms near these examples do have fine filtrations. Thus one is led to ask if residuality may be replaced by another notion of largeness so that diffeomorphisms in that large set would have good structure. Toward this end, we say that a set $B \subset \text{Diff}^r(M)$ has full one-dimension measure or briefly, full measure, if there is a residual set $R \subset \Phi^{k,r}$ such that for $\xi \in R$, $m(\xi^{-1}(B)) = 1$. Of course, there are similar definitions of full k -dimensional measure, using maps ξ defined on the unit k -ball in \mathbb{R}^k , $k > 1$, instead of $[0,1]$.

Clearly, a countable intersection of sets of full measure again has full measure, and a set of full measure is dense. To begin to consider this concept, we want to understand its relationship with several well-known residual sets.

It follows from Sard's theorem that any residual sets defined by transversality have full measure. In particular, the Kupka-Smale diffeomorphisms (those having only hyperbolic periodic points whose stable and unstable manifolds meet transversely) have full measure. In fact, by a theorem of Sotomayor [6], [15], for a residual set a arcs ξ , ξ_t fails to be Kupka-Smale for at most countably many t 's in $[0,1]$. Also, given manifolds M, N, P with $N \subset P$, the set of C^r mappings from M to P which are transverse to N has full measure (with the obviously related definitions). On the other hand, the set of C^5 Morse-Smale diffeomorphisms of the circle does not have full measure even though it is dense and open. A similar statement holds for the C^5 diffeomorphisms whose periodic points are dense in their non-wandering sets.

To see why this occurs, consider a curve $\xi_t(\theta) = \theta + t$ where θ is an angular variable on the circle. The map $\hat{\xi}(t, \theta) = (t, \theta + t)$ is a twist map of the annulus $A : \frac{1}{2} \leq t \leq 1, 0 \leq \theta \leq 2\pi$. For any C^5 close curve of diffeomorphisms η_t , the map $\hat{\eta}(t, \theta) = (t, \eta_t(\theta))$ is such that any circle $t = t(\theta)$ in A meets its $\hat{\eta}$ -image. Thus, the twist theorem in [9], [12] guarantees that there is a subset $B \subset A$ of positive measure which is a union of invariant circles for $\hat{\eta}$ on each of which $\hat{\eta}$ is conjugate to an irrational rotation. Letting $p : A \rightarrow [\frac{1}{2}, 1]$ be the projection $(t, \theta) \rightarrow t$, it is easy to see that as soon as (t_0, θ_0) is in an $\hat{\eta}$ invariant circle, that circle must equal $p^{-1}(t_0)$. Thus, the map η_{t_0} is conjugate to an irrational rotation, and, hence, has no periodic points. Also, the set of t_0 's with this property has positive measure in $[\frac{1}{2}, 1]$.

In closing, we ask: Does the set of C^r diffeomorphisms of M possessing a fine filtration have full measure in M ?

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