ON CODIMENSION ONE ANOSOV DIFFEOMORPHISMS.

By S. E. Newhouse.

1. Throughout this paper, we let M be a compact connected C^∞ manifold without boundary. A C^r diffeomorphism $f: M \to M$, $1 \leqq r \leqq \infty$, is called an *Anosov diffeomorphism* if there is a continuous splitting of the tangent bundle $TM = E^s \oplus E^u$, a riemannian metric $\|\cdot\|$ on TM, and constants $c,\ c' > 0$, $0 < \lambda < 1$, such that

(i) $T_xf(E_x{}^s) = E_{f(x)}{}^s,\ T_xf(E_x{}^u) = E_{f(x)}{}^u$

(ii) For $v \in E^s$, $\| Tf^n(v) \| \leqq c\lambda^n \| v \|$, and for $v \in E^u$,
$$\| Tf^{-n}(v) \| \leqq c'\lambda^n \| v \|$$

where T_xf refers to the derivative of f at the point x. It can be shown that condition (ii) is independent of the riemannian metric on TM.

In the past few years, Anosov diffeomorphisms have been studied to a great extent. We refer the reader to [2] and [8] for background information, general references, and terms which are not defined here.

An Anosov diffeomorphism is said to be of *codimension* 1 if either $\dim E^u = 1$ or $\dim E^s = 1$ where E^u and E^s are as in the above definition.

Let A be an $n \times n$ matrix with integer entries such that $\det A = \pm 1$ and the eigenvalues of A are off the unit circle. Then A induces a diffeomorphism $\bar{A}$ of the n-dimensional torus T^n. The map $\bar{A}$ is called a *toral automorphism.* Two maps $f: M \to M$, $g: N \to N$ are called π_1-*conjugate* if there is an isomorphism $\phi: \pi_1(M) \to \pi_1(N)$ such that $\phi f_* = g_* \phi$ where $\pi_1(M)$, $\pi_1(N)$ are the fundamental groups and f_*, g_* are the induced maps. The maps f, g are called topologically conjugate if there is a homeomorphism $h: M \to N$ such that $hf = gh$.

For a diffeomorphism $f: M \to M$, we let $NW(f)$ denote the set of non-wandering points of f which is defined by $NW(f) = \{x \in M:$ for any neighborhood U of x, there is a positive integer $n(U)$ such that $f^{n(U)}(U) \cap U \neq \phi\}$. In [2], Franks proves the following theorem.

(1.1) Theorem. *Let $f: M \to M$ be a codimension 1 Anosov diffeomorphism such that $NW(f) = M$. Then f is topologically conjugate to a toral*

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automorphism. Any two codimension 1 *Anosov diffeomorphisms* $f: M \to M$, $g: N \to N$, *such that* $NW(f) = M$, $NW(g) = N$ *are topologically conjugate if and only if they are* π_1-*conjugate.*

Under the assumption that the stable and unstable foliations were of class C^2, partial results in the direction of Theorem (1.1) were obtained independently by H. Rosenberg.

In this paper, we prove

(1.2) THEOREM. *Let* $f: M \to M$ *be any codimension* 1 *Anosov diffeomorphism. Then* $NW(f) = M$.

Theorem (1.2) was proved earlier by Smale in the case where $\dim M = 2$ (see [2, (7.2)]).

In view of (1.1) and (1.2), we obtain

(1.3) COROLLARY. *Any codimension* 1 *Anosov diffeomorphism is topologically conjugate to a toral automorphism. Any two codimension one Anosov diffeomorphisms are topologically conjugate if and only if they are* π_1-*conjugate.*

Applying well-known facts we obtain

(1.4) COROLLARY. *If* $f: M \to M$ *is any codimension* 1 *Anosov diffeomorphism, then*

(1) *the periodic points of* f *are dense in* M [8], *and*

(2) f *has an invariant Lebesgue measure and* f *is ergodic* ([7] or [1]).

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2. In this section, we prove Theorem (1.2). It is well-known that an Anosov diffeomorphism satisfies Smale's Axioms A and B [8; (6.1) and (6.4)], and we wish to make use of this fact and several of its consequences. Thus we assume familiarity with §§ I. 3, I. 6, and I. 7 of [8].

To begin the proof of (1.2), we observe that since there are no Anosov diffeomorphisms of S^1, we may assume $\dim M \geqq 2$. Since $NW(f) = NW(f^{-1})$, we may assume $\dim E^u = 1$ and $\dim E^s = n - 1$ where $\dim M = n \geqq 2$. Further, by taking two two to one coverings if necessary, we may assume M is orientable, TM is oriented, and the line bundle E^u is oriented. Thus the unstable manifolds of f are oriented arcs. By replacing f by f^2 or f^4, assume Tf preserves the orientations of E^u and TM.

Let $NW(f) = \Omega_1 \cup \cdots \cup \Omega_n$ be the spectral decomposition of $NW(f)$. We let $W^s(x)\,(W^u(x))$ denote the stable (unstable) manifold of f at the point $x \in M$. For a subset Λ of M let

$$W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x), \text{ and let } W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x).$$

A source is a basic set Ω_i such that $W^s(\Omega_i) = \Omega_i$. A sink is a basic set Ω_i such that $W^u(\Omega_i) = \Omega_i$. It is easy to see that if Ω_i is a source, then $W^u(\Omega_i)$ is an open subset of M, and, if Ω_i is a sink, then $W^s(\Omega_i)$ is an open subset of M. If we show some source is a sink, then it follows that $NW(f) = M$. For, if Ω_i is a source and a sink, then $W^s(\Omega_i) = \Omega_i = W^u(\Omega_i)$ is an open and closed subset of M. Thus $\Omega_i = M$. We proceed to show, in fact, that any source must also be a sink.

Let Ω_i be a source. We will show

(2.1) $\quad W^u(\Omega_1) = \Omega_1.$

Given $y_1, y_2 \in W^u(x)$, we say $y_1 < y_2$ if $y_1 \neq y_2$ and the subarc of $W^u(x)$ from y_1 to y_2 has the same orientation as $W^u(x)$. For $y_1 < y_2$, let $[y_1, y_2]$ denote the compact subarc of $W^u(x)$ from y_1 to y_2. Let $l[y_1, y_2]$ denote the length of $[y_1, y_2]$. Given $0 < \alpha < \infty$, and $y_1 \in W^u(x)$, define

$$B_\alpha{}^+(y_1) = \{y \in W^u(x) : y_1 < y \text{ and } l[y_1, y] < \alpha\},$$

and

$$B^+(y_1) = \{y \in W^u(x) : y_1 < y\}.$$

Similarly, define

$$B_\alpha{}^-(y_1) = \{y \in W^u(x) : y < y_1 \text{ and } l[y, y_1] < \alpha\},$$

and

$$B^-(y_1) = \{y \in W^u(x) : y < y_1\}.$$

(2.2) LEMMA. (1) *For each* $x \in \Omega_1$, $B^+(x) \cap \Omega_1 \neq \emptyset$

(2) *For each* $x \in \Omega_1$, $B^-(x) \cap \Omega_1 \neq \emptyset$.

Before proving (2.2), we show how (2.1) follows from (2.2). Comments by M. Hirsch and J. Franks were useful in simplifying the original proof.

Let $x \in \Omega_1$. We prove that $W^u(x) \subset \Omega_1$. Choose an increasing sequence of integers $n_1 < n_2 < n_3 \cdots$ and a point $y \in M$ such that $f^{n_i}(x) \to y$ as $i \to \infty$.

Since Ω_1 is closed and invariant, $y \in \Omega_1$. By (2.2), we may choose $\alpha > 0$ such that $B_\alpha^\pm(y) \cap \Omega_1 \neq \emptyset$. But then there is an integer $N > 0$ such that for $i \geqq N$, $B_\alpha^\pm(f^{n_i}(x)) \cap \Omega_1 \neq \emptyset$. Since f^{-n_i} contracts the unstable manifolds, we see that x is an accumulation point of $B^\pm(x) \cap \Omega_1$. That is, x is an accumulation point of $W^u(x) \cap \Omega_1$ from both sides in $W^u(x)$. The same argument shows that any point $v \in W^u(x) \cap \Omega_1$ is an accumulation point of $W^u(x) \cap \Omega_1$ from both sides in $W^u(x)$. Now, since $W^u(x) \cap \Omega_1$ is closed in $W^u(x)$, if y were a point of $W^u(x) - \Omega_1$, then one could find an arc $[x_1, x_2]$ in $W^u(x)$ such that $y \in [x_1, x_2]$, $(x_1, x_2) \cap \Omega_1 = \emptyset$, and $x_1, x_2 \in \Omega_1$. But Ω_1 must accumulate on x_1 and x_2 from both sides in $W^u(x)$. This would contradict the fact that $(x_1, x_2) \cap \Omega_1 = \emptyset$. Thus no such y exists, so $W^u(x) \subset \Omega_1$ and (2.1) is proved.

Now we recall some definitions and results we will need for the proof of (2.2).

For each x, we let $W_\epsilon^u(x)$ be the intrinsic closed ϵ-ball about x in $W^u(x)$. This is obtained as follows. The Riemannian metric on TM induces one on $TW^u(x)$. This in turn induces a topological metric on $W^u(x)$ which makes $W^u(x)$ homeomorphic to the real line. Then $W_\epsilon^u(x)$ is to be the ball of radius ϵ about x in this topological metric. Similarly, let $W_\epsilon^s(x)$ be the intrinsic closed ϵ-ball about x in $W^s(x)$. Thus $W_\epsilon^u(x)$ is diffeomorphic to a closed $(n-1)$-disk where $\dim M = n$. For any subset Λ of M, and $y \in \Lambda$, let $W^s(y, \Lambda)$ be the connected component of $W^s(y) \cap \Lambda$ which contains y. Similarly, let $W^u(y, \Lambda)$ be the connected component of $W^u(y) \cap \Lambda$ which contains y. The following fact is an easy consequence of Theorem (7.3) of [8]. This theorem is proved in [4].

(2.3) There is an $\epsilon > 0$ such that for each $x \in M$, there is a neighborhood $V_\epsilon(x)$ of x satisfying

(2.3.1) $V_\epsilon(x)$ is homeomorphic to $W_\epsilon^u(x) \times W_\epsilon^s(x)$.

(2.3.2) for $y_1, y_2 \in V_\epsilon(x)$, if $W^u(y_1, V_\epsilon(x)) \cap W^u(y_2, V_\epsilon(x)) \neq \emptyset$, then $W^u(y_1, V_\epsilon(x)) = W^u(y_2, V_\epsilon(x))$; similarly, if

$$W^s(y_1, V_\epsilon(x)) \cap W^s(y_2, V_\epsilon(x)) \neq \emptyset,$$

then $W^s(y_1, V_\epsilon(x)) = W^s(y_2, V_\epsilon(x))$.

(2.3.3) for $y_1, y_2 \in V_\epsilon(x)$, $W^u(y_1, V_\epsilon(x)) \cap W^s(y_2, V_\epsilon(x))$ is a single point.

The interior of $V_\epsilon(x)$ is usually referred to as a *local product neighborhood*, and (2.3) is referred to as the *local product theorem.*

For our purposes it is convenient to use the notion of a product set which is a sort of elongated closed local product neighborhood. Let $x \in M$, and $\epsilon > 0$. A *stable product set relative to* $W_\epsilon{}^s(x)$ is a set, denoted by N or $N(W_\epsilon{}^s(x))$, satisfying the following conditions.

(2.4) $N = \cup \{W^u(y, N) : y \in W_\epsilon{}^s(x)\}$.

(2.5) For $y_1, y_2 \in N$, either $W^u(y_1, N) \cap W^u(y_2, N) = \emptyset$ or $W^u(y_1, N) = W^u(y_2, N)$; similarly, either $W^s(y_1, N) \cap W^s(y_2, N) = \emptyset$ or $W^s(y_1, N) = W^s(y_2, N)$.

(2.6) For $y_1 \in N$, $W^s(y_1, N)$ $(W^u(y_1, N)$ is homeomorphic to a closed ball in $W^s(y_1)$ $((W^u(y_1))$.

(2.7) There exists $\epsilon_1 > 0$ such that $W_{\epsilon_1}{}^u(y_1) \subset W^u(y_1, N)$ for all $y_1 \in W_\epsilon{}^s(x)$.

(2.8) If $y_1, y_2 \in N$, then $W^u(y_1, N) \cap W^s(y_2, N)$ is a single point.

For $K \subset W_\epsilon{}^s(x)$, a *stable product set N or $N(K)$ relative to K* is defined to be $\bigcup_{y \in K} W^u(y, N(W_\epsilon{}^s(x)))$ where $N(W_\epsilon{}^s(x))$ is some stable product set relative to $W_\epsilon{}^s(x)$.

Similarly, for $x \in M$, $\epsilon > 0$, we may define an unstable product set relative to $W_\epsilon{}^u(x)$ or K where $K \subset W_\epsilon{}^u(x)$.

2.9) *Remark.* 1. Using (2.3), and the compactness of $W_\epsilon{}^u(x)$ and $W_\epsilon{}^s(x)$, it is easy to check that for any $x \in M$, and any $\epsilon > 0$, product sets relative to $W_\epsilon{}^u(x)$ and $W_\epsilon{}^s(x)$ exist.

2. If N is a stable product set relative to $W_\epsilon{}^s(x)$, $K \subset W_\epsilon{}^s(x)$, and $y \in N$, then $N(K) \cap W^s(y, N)$ is homeomorphic to K. A similar statement holds for unstable product sets.

Now we prove Lemma (2.2). We prove (2.2.1) since the same methods yield (2.2.2). Let $A = \{x \in \Omega_1 : B^+(x) \cap \Omega_1 \neq \emptyset\}$. We proceed to show $A = \Omega_1$.

Step 1. A is an f-invariant subset of Ω_1, i.e. $f(A) = A$.

Proof. This follows easily from the fact that f preserves the orientation of E^u and the facts that for any $x \in M$, $f(W^s(x)) = W^s(f(x))$ and $f(W^u(x)) = W^u(f(x))$.

Step 2. If $x \in \Omega_1$ is not periodic, then $x \in A$.

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Proof. Choose $\epsilon > 0$ as in (2.3). Since x is not periodic, the orbit of x, $o(x)$, is infinite. We claim

(2.10) there is an infinite sequence $\{x_i\}_{i \geqq 1} \subset o(x)$ such that if $x_i \neq x_j$, then $W_\epsilon^s(x_i) \cap W_\epsilon^s(x_j) = \emptyset$.

If for each pair of integers $n_1 > n_2$, $W^s(f^{n_1}(x)) \cap W^s(f^{n_2}(x)) = \emptyset$, (2.10) is true. If there are integers $n_1 > n_2$ such that

$$W^s(f^{n_1}(x)) \cap W^s(f^{n_2}(x)) \neq \emptyset,$$

then $W^s(f^{n_1}(x)) = W^s(f^{n_2}(x))$, and so $f^{n_1-n_2}(W^s(x)) = W^s(x)$. Since $f^{-(n_1-n_2)}$ expands $W^s(x)$, there is an increasing sequence of integers $m_1, m_2, \cdots$ such that if $i \neq j$, the distance in $W^s(x)$ between $f^{-m_i(n_1-n_2)}(x)$ and $f^{-m_j(n_1-n_2)}(x)$ is larger than 2ϵ. Thus $W_\epsilon^s(f^{-m_i(n_1-n_2)}(x)) \cap W_\epsilon^s(f^{-m_j(n_1-n_2)}(x)) = \emptyset$, and we take $x_i = f^{-m_i(n_1-n_2)}(x)$ for (2.10).

Now let y be a limit point of $\{x_i\}$, and choose a subsequence $\{x_{i_j}\}$ such that $x_{ij} \to y$ as $j \to \infty$. Let $V_\epsilon(y)$ be a neighborhood of y, as described in (2.3). By (2.3) and the fact that the stable manifolds are $(n-1)$-dimensional, it is clear that if $x_{i_j} \neq x_{i_k}$ and $x_{i_j}, x_{i_k} \in V_\epsilon(y)$, then either $B^+(x_{i_j}) \cap W_\epsilon^s(x_{i_k}) \neq \emptyset$ or $B^+(x_{i_k}) \cap W_\epsilon^s(x_{i_j}) \neq \emptyset$, so either $x_{i_j} \in A$ or $x_{i_k} \in A$. In either case, since $\{x_{i_j}, x_{i_k}\} \subset o(x)$ and A is f-invariant (step 1), we obtain that $x \in A$.

It remains to show if $p \in \Omega_1$ is periodic, then $p \in A$. The arguments for this fact for the case where $\dim M = 2$ are different from those for the case where $\dim M > 2$.

First assume $\dim M = 2$, and $p \in \Omega_1$ is periodic of period m, i.e., $f^m(p) = p$. Since M has a nowhere vanishing line field and is orientable, we may assume M is the two torus T^2. Suppose $p \notin A$. Then $B^+(p) \cap W^s(p) = \emptyset$. Since Ω_1 is a source, $W^s(p) \subset \Omega_1 \subset NW(f) = NW(f^m)$. Thus there is a source Ω_1' for f^m such that $p \in \Omega_1'$ and p is a fixed point of f^m. Since $W^s(p)$ is dense in Ω_1', it recurs on itself.

Let $V_\epsilon(p)$ be as in (2.3). Since the unstable bundle E^u on M is oriented, so is the stable bundle E^s. Let q be a point of $W^s(p) \cap B_\epsilon^-(p)$, and consider the loop γ consisting of the arc in $W^s(p)$ from p to q followed by the arc in $B_\epsilon^-(p)$ from q to p. If γ is null-homotopic, then, since it is a topological circle embedded in a topological torus, it bounds a topological 2-disk. But then if we follow $W^s(p)$ in the direction from p to q and go beyond q to the first point q_1 where $W^s(p)$ meets $B_\epsilon^-(p)$ again, it is easy to see that the arc from p to q must cross $B_\epsilon^-(p)$ in the direction opposite to that of the arc from q to q_1. This contradicts the fact that the stable bundle E^s is oriented. Thus γ is *not* null-homotopic. Again since M is a torus, $M^2-\gamma$ is a topological cylinder which has an unstable foliation induced by $\{W^u(x)\}_{x\in M}$. Now it is easy to see how to define ω-limit sets and set up a Poincare-Bendixon theory (see [3] or [6]) for this continuous foliation on $M^2-\gamma$. (Alternatively, using the structurally stability of f, one could, by approximating, assume f is C^2 and use theorem (6.5) of [4] to get E^u is C^1 and apply the standard Poincare-Bendixon theory.) Since closure $(B^+(p)-B_\epsilon^+(p))\subset M^2-\gamma$, the ω-limit set of the leaf $B^+(p)$ is a non-empty subset of $M^2-\gamma$. Since the foliation $\{W^u(x)\}$ has no singularities, Poincare-Bendixon theory says the ω-limit set of $B^+(p)$ must be a circle in $M^2-\gamma$. But this is obviously impossible since all the unstable manifolds of f are injectively immersed cells. Thus the assumption that $p\notin A$ leads to a contradiction. This completes the proof of (2.2.1) for $\dim M=2$.

For the remainder of the proof of (2.2.1), we assume $\dim M\geqq 3$, and, as before, $p\in\Omega_1$ is periodic.

Step 3. By step 2, if $x\in W^s(p)-\{p\}$, then $x\in A$. For $x\in W^s(p)-\{p\}$, define $\tilde{\phi}(x)=\inf\{l[x,y]: y\in B^+(x)\cap\Omega_1\}$. If there is an x such that $\tilde{\phi}(x)=0$, then $p\in A$. Hence we may assume $\tilde{\phi}(x)>0$ for $x\in W^s(p)-\{p\}$.

Proof. This follows immediately from the continuous dependence of the stable manifolds on compact sets and the fact that if $x,y\in\Omega_1$, then $W^u(x)\cap W^s(y)\subset\Omega_1$.

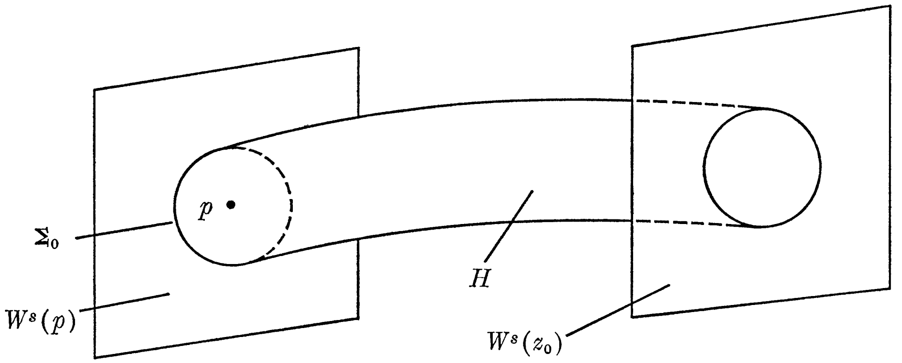
Step 4. For $x\in W^s(p)-\{p\}$, let $\phi(x)$ be the point in $B^+(x)$ such that $l[x,\phi(x)]=\tilde{\phi}(x)$. Then there is a $z_0\in M$ such that $\phi(x)\in W^s(z_0)$ for all $x\in W^s(p)-\{p\}$; i.e., all the $\phi(x)$ lie in the same stable manifold as x varies in $W^s(p)-\{p\}$.

Proof. For any $z\in M$, let $U_z=\{x\in W^s(p)-\{p\}: \phi(x)\in W^s(z)\}$. It is clear that U_z is open in $W^s(p)-\{p\}$ for all z. Clearly, if $W^s(z_1)\neq W^s(z_2)$, then $U_{z_1}\cap U_{z_2}=\emptyset$. Now since $\dim W^s(p)\geqq 2$, $W^s(p)-\{p\}$ is connected. Hence, if z_0 is such that $U_{z_0}\neq\emptyset$, then $U_{z_0}=W^s(p)-\{p\}$.

Step 5. The map $\phi: W^s(p)-\{p\}\to W^s(z_0)$ is continuous and injective.

Proof. Let $x\in W^s(p)-\{p\}$, and let γ be the arc $[x,\phi(x)]$. Let $\epsilon>0$. There is an unstable product set N relative to γ such that $W^s(\phi(x),N)\subset W_\epsilon^s(\phi(x))$ and $p\notin W^s(x,N)$. Then if $y\in W^s(x,N)$, $\phi(y)\in W^s(\phi(x),N)\subset W_\epsilon^s(\phi(x))$. So ϕ is continuous. Injectivity follows immediately from the definition of ϕ.

Thus if $K \subset W^s(p) - \{p\}$ is compact, $\phi\,|_K$ is a homeomorphism. Let E_0 be a closed $(n-1)$-ball in $W^s(p)$ containing p in its interior relative to $W^s(p)$. Let Σ_0 be the boundary of E_0 in $W^s(p)$ so that Σ_0 is an $(n-2)$-sphere in $W^s(p) - \{p\}$. Let $H = \bigcup_{x \in \Sigma_0} [x, \phi(x)]$. Then, clearly, H is homeomorphic to $\Sigma_0 \times I$ where I is the unit interval.



Step 6. For $x \in \Sigma_0$, let $\gamma_x = [x, \phi(x)]$. For $y \in \gamma_x$, letΣ_y be the path component of $W^s(y) \cap H$ which contains y. Then

(2.11) Σ_y is homeomorphic to an $(n-2)$-sphere; hence, by the Jordan-Brouwer separation theorem, [9, p. 198], Σ_y separates $W^s(y)$.

Proof. We first assert

(2.12) For each x, $x_1 \in \Sigma_0$, and each $y \in \gamma_x$, $\Sigma_y \cap \gamma_{x_1}$ is exactly one point.

If (2.12) is true, fix x, and define $\phi_y\colon \Sigma_0 \to \Sigma_y$ by $\{\phi_y(x_1)\} = \Sigma_y \cap \gamma_{x_1}$ for $x_1 \in \Sigma_0$. Then it is easy to check that ϕ_y is injective, surjective, and continuous, so it is a homeomorphism. Thus we need to prove (2.12).

We first prove

(2.13) if $y \in \gamma_x$, then $\Sigma_y \cap \gamma_{x_1} \neq \emptyset$ for all $x_1 \in \Sigma_0$.

Fix $y \in \gamma_x$. Let $\Sigma_0' = \{z \in \Sigma_0 : \Sigma_y \cap \gamma_z \neq \emptyset\}$. By the local product theorem (2.3), Σ_0' is open in Σ_0. We show $\Sigma_0 - \Sigma_0'$ is open in Σ_0.

Let $z \in \Sigma_0 - \Sigma_0'$. Then $\Sigma_y \cap \gamma_z = \emptyset$, so there is an unstable product set N relative to γ_z such that $N \cap \Sigma_y = \emptyset$. Now, by the version of (2.7) for the unstable product set N, $N \cap \Sigma_0$ is a neighborhood of z in Σ_0. But $N \cap \Sigma_0 \subset \Sigma_0 - \Sigma_0'$, so $\Sigma_0 - \Sigma_0'$ is open in Σ_0. Since $\dim \Sigma_0 \geqq 1$, and $\Sigma_0' \neq \phi$, $\Sigma_0' = \Sigma_0$. Thus (2.13) is proved.

Now for fixed $x \in \Sigma_0$, let $D_x = \{y \in \gamma_x : \Sigma_y \cap \gamma_{x_1}$ is one point for all

$x_1 \in \Sigma_0\}$. We show D_x and $\gamma_x - D_x$ are open in γ_x. By taking a stable product set relative to E_0, we see that $D_x \neq \emptyset$, so $D_x = \gamma_x$ which proves (2.12).

D_x open: Let $y \in D_x$. Then the map $\phi_y : \Sigma_0 \to \Sigma_y$ defined above is injective, surjective, and continuous, so it is a homeomorphism. Let N_y be a stable product set relative to Σ_y. Then $W^u(y, N_y) \cap \gamma_x$ is a neighborhood of y in γ_x which is contained in D_x.

$\gamma_x - D_x$ open: Let $y \in \gamma_x - D_x$. By (2.13), there exists an $x_1 \in \Sigma_0$ such that $\Sigma_y \cap \gamma_{x_1}$ has at least two points. Then using a stable product set relative to an arc in Σ_y which connects y and two points of $\Sigma_y \cap \gamma_{x_1}$, we see that y is an interior point of $\gamma_x - D_x$.

Thus (2.12), and hence (2.11) is proved.

The next step uses an argument similar to one frequently used by Haefliger.

Step 7. Fix $x \in \Sigma_0$. It is clear that the sets Σ_y vary continuously with $y \in \gamma_x$. That is, if y_1 is close to y_2, and $\{y_1, y_2\} \subset \gamma_x$, then there is a homeomorphism $\Psi : \Sigma_{y_2} \to \Sigma_{y_1}$ such that Ψ is C^0 close to the inclusion $i_{y_2} : \Sigma_{y_2} \to M$. Now each Σ_y, being a homeomorph of an $(n-2)$-sphere in $W^s(y)$, is the boundary of a bounded open path connected set V_y in $W^s(y)$. One can show the V_y vary continuously with y, but we do not need this. Let $\gamma = \gamma_x$. We claim

(2.14) $\quad B^+(p) \cap V_y \neq \emptyset$ for all $y \in \gamma$.

Notice that (2.14) implies that $B^+(p) \cap V_{\phi(x)} \neq \emptyset$. But since $\phi(x) \in \Omega_1$, $V_{\phi(x)} \subset \Omega_1$, so $B^+(p) \cap \Omega_1 \neq \emptyset$ which means that $p \in A$. Thus to complete the proof of (2.2.1) we need only prove (2.14).

Proof of (2.14). Let $D = \{y \in \gamma : B^+(p) \cap V_y \neq \emptyset\}$. Clearly, D is open in γ. Since D contains a neighborhood of x in γ, we are done if we prove $\gamma - D$ is open in γ, for then $D = \gamma$.

Let $y \in \gamma - D$. Take a stable product set N_y relative to V_y such that $B^+(p) \cap N_y = \emptyset$. By (2.7), continuous dependence of Σ_y on y, and the definition of V_y, we see that if y_1 is close to y, then $V_{y_1} \subset N_y$. Thus y is an interior point of $\gamma - D$.

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