On some results of Hofbauer on maps of the interval

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March, 1991

Abstract

We present a new treatment of results of F. Hofbauer on piecewise monotone mappings of the interval with positive topological entropy. Countable state symbolic systems are obtained as models for the dynamics of the natural extensions of the interval mappings. It follows that the set of measures of maximal entropy is a finite dimensional simplex and that each ergodic measure of maximal entropy is a Markov measure.

1 Introduction

Symbolic dynamics plays a fundamental role in smooth Dynamical Systems. Frequently, deep geometric results can be obtained by passing to a symbolic model in which special combinatorial methods become available. The earliest use of symbolic dynamics goes back to Hadamard, Morse, Hedlund, and Birkhoff. In the $1960's$ Sinai established the existence of symbolic dynamics for Anosov diffeomorphisms [8]. Later this was done by Bowen for general hyperbolic basic sets [1]. In a series of papers, F. Hofbauer obtained remarkable results on the structure of piecewise monotone maps of an interval with positive topological entropy [5], [6], [7]. He showed that the natural extensions can be modeled by countable state Markov shifts after the exclusion of certain "small" sets. From this, he showed that the set of measures of maximal entropy formed a finite dimensional simplex and that the (natural extensions) of the ergodic measures of maximal entropy are isomorphic to Markov processes.

In this paper we will present new proofs of these results of Hofbauer.

2 Definitions and Statements of Results

We begin with a topological space X and its $\sigma-$algebra $\mathcal{B}$ of Borel sets. A map $f : X \to X$ is called Borel (or Borel measurable) if $f^{-1}(A) \in \mathcal{B}$ for every $A \in \mathcal{B}$.

*Partially supported by DARPA

A Borel automorphism of X is a bijective map $f: X \to X$ such that both f and f^{-1} are Borel. Let $\mathcal{M}(f)$ denote the set of $f-$invariant probability measures on X. We define the topological entropy $h(f)$ of f by

$$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$$

where $h_\mu(f)$ denotes the metric entropy of the measure μ with respect to f. If f has no invariant probability measures, we set $h(f) = 0$.

Given the Borel automorphism $f: X \to X$, a *zero entropy set* X_1 is a subset $X_1 \subseteq X$ such that

1. $f(X_1) = X_1$ (invariance).
2. If μ is an invariant probability measure for f on X such that $\mu(X_1) = 1$, then $h_\mu(f) = 0$.

By convention, if f has no invariant probability measures, then we say that all of X is a zero entropy set. We use the notation *ze* for zero entropy. Also, by convention, we agree that the empty set is a *ze* set.

There are the following facts.

1. Every invariant Borel subset of a zero entropy set is also a zero entropy set.
2. A countable union of zero entropy sets is also a zero entropy set.
3. The wandering set of a homeomorphism of a compact metric space is a zero entropy set.
4. The set of periodic points of a Borel automorphism is a zero entropy set.

There is a natural definition of isomorphism of Borel automorphisms modulo zero entropy sets. We say that (f, X) and (g, Y) are *isomorphic mod ze* iff there are sets $X_1 \subset X,\ Y_1 \subset Y$ such that

1. $f(X_1) = X_1$ and X_1 is a *ze* set.
2. $g(Y_1) = Y_1$ and Y_1 is a *ze* set.
3. There is a Borel isomorphism $\phi: X \setminus X_1 \to Y \setminus Y_1$ such that $g\phi = \phi f$.

We use the notation $(f, X) \underset{\text{ze}}{\simeq} (g, Y)$ to denote that (f, X) is isomorphic mod zero entropy to (g, Y).

Examples.

1. If $f : X \to X$ is any Borel automorphism and $X_1 \subset X$ is a zero entropy set, then $(f, X) \underset{\text{ze}}{\simeq} (f \mid X \setminus X_1, X \setminus X_1)$. That is, removing a ze−set yields a system isomorphic mod ze to the original system.

2. A particular case of the above is the following. Let f be a homeomorphism of the compact metric space X. Let $\Omega(f)$ be the non-wandering set of f. Then, $(f, X) \underset{\text{ze}}{\simeq} (f \mid \Omega(f), \Omega(f))$

We now consider a very useful class of models for the equivalence relation of isomorphism mod ze — the class of finite or countable Markov shifts.

Let $\mathcal{A} = \{v_1, v_2, \ldots\}$ be a finite or countable set. Let A be a $0-1$ valued matrix indexed by $\mathcal{A} \times \mathcal{A}$; i.e., a mapping $A : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$.

Let $\Sigma_A = \{\mathbf{a} \in \mathcal{A}^{\mathbf{Z}} : A_{a(i),a(i+1)} = 1 \ \forall i \ \in \mathbf{Z}\}$. Let d_0 denote the discrete metric on $\mathcal{A}$,

$$d_0(v_i, v_j) = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases}$$

Rescale the metric d_0 to d_1 defined by

$$d_1(v_i, v_j) = \frac{1}{min(i, j)} d_0(v_i, v_j)$$

The metric d_1 makes the completion of the space $(\mathcal{A}, d_1)$ topologically the same as the one point compactification of $(\mathcal{A}, d_0)$.

Define a metric d on Σ_A by

$$d(\mathbf{a}, \mathbf{b}) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{| i |}} d_1(a(i), b(i))$$

Let $\sigma : \Sigma_A \to \Sigma_A$ be defined by

$$\sigma(\mathbf{a})(i) = a(i+1) \quad \forall i \in \mathbf{Z}$$

Then, σ is a homeomorphism on Σ_A. We call the pair (σ, Σ_A) the Markov shift with alphabet $\mathcal{A}$ and incidence matrix A. We sometimes call Σ_A a Markov shift without explicitly mentioning σ.

We say $T : X \to X$ is Markov mod ze if it is isomorphic mod ze to some Markov shift (σ, Σ_A).

Let $(T, X, \mathcal{B}, \mu)$ be a measure preserving transformation. We say that μ is a *Markov measure* (for $(T, X, \mathcal{B})$) if there is a finite or countable partition $\alpha = \{A_1, A_2, \ldots\}$, a sequence $\pi = (\pi_1, \pi_2, \ldots)$ of non-negative real numbers, and a stochastic matrix p_{ij} such that

1. $\sum \pi_i = 1$.

2. $\sum_i \pi_i p_{ij} = \pi_j$.

3. For any finite sequence $(i_0, i_1, \ldots, i_{n-1})$ of positive integers, we have

$$\mu(A_{i_0} \bigcap T^{-1}A_{i_1} \bigcap \cdots \bigcap T^{-n+1}A_{i_{n-1}}) = \pi_{i_0} p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{n-2} i_{n-1}}$$

With these notations we can now state Hofbauer's results.

Let $I = [0,1]$ be the closed unit interval. A map $f : I \to I$ is called piecewise monotone if there is a partition $\{I_1, \ldots, I_s\}$ of I such that, for each $1 \leq j \leq s$,

1. each I_j is an interval.
2. $f \mid I_j$ is monotone and continuous.

Note that f is not assumed continuous on all of I: it may be discontinuous at boundary points of the $I_j's$.

Let $\hat{f} : \hat{I} \to \hat{I}$ denote the natural extension (inverse limit) of f. Denote by $\mathcal{M}_{max}(f)$ the set of measures of maximal entropy for f. As usual, we let $h(f)$ denote the topological entropy of f.

Theorem 1.(Hofbauer). *Suppose $f : I \to I$ is a piecewise monotone interval map with $h(f) > 0$. Then, $\hat{f}$ is Markov mod ze. Moreover, $\mathcal{M}_{max}(f)$ is a finite dimensional simplex and each ergodic maximal measure for $\hat{f}$ is a Markov measure.*

We proceed toward a new proof of Theorem 1. The main novelty in our proof of Hofbauer's theorem is the use of simple metric entropy methods to prove that a certain exceptional set in $\hat{I}$ is a $ze-$set. Hofbauer used a special coding technique in [4].

Denote by α the partition $\{I_1, \ldots, I_s\}$. We first use α to construct a natural one-sided symbolic system which carries all of the positive entropy ergodic measures of f.

Let $\Sigma^+(s)$ denote the one-sided full shift on s symbols, $\Sigma^+(s) = \{1, \ldots, s\}^{\mathbf{N}}$ where $\mathbf{N} = \{0, 1, 2, \ldots\}$. Let

$$\Sigma_1^+ = \{\mathbf{a} \in \Sigma^+(s) : \bigcap_{j \geq 0} f^{-j}(I_{a(j)}) \neq \emptyset\}.$$

with the shift $\sigma(\mathbf{a})(i) = a(i+1)\ \forall\, i$. Let $\zeta(x) = \mathbf{a}$ where $x \in \bigcap_{j\geq 0} f^{-j}(I_{a(j)})$ so that $\sigma\zeta = \zeta f$. Note that typically, Σ_1^+ is not a closed subset of $\Sigma^+(s)$.

Lemma 1.(Hofbauer [5]). *$Cl(\Sigma_1^+) \setminus \Sigma_1^+$ is at most countable.*

Proof. It is easy to see that

$$Cl(\Sigma_1^+) = \{\mathbf{a} \in \Sigma^+(s) : \text{for each } n > 0, \bigcap_{0 \leq j < n} f^{-j}(I_a(j)) \neq \emptyset\}.$$

Consider $\mathbf{a} \in Cl(\Sigma_1^+) \setminus \Sigma_1^+$. For $n > 0$, let

$$I_{a(0)a(1)\cdots a(n-1)} = \bigcap_{0 \leq j < n} f^{-j}(I_{a(j)}).$$

Note that, since $\mathbf{a} \notin \Sigma_{\mathbf{1}}^+$, the interval $I_{a(0)a(1)\cdots a(n-1)}$ is non-trivial; i.e., not a single point.

Let

$$int\ I_{a(0)a(1)\cdots a(n-1)} = (\alpha_n, \beta_n)$$

for each n. That is, α_n is the left boundary point of $int\ I_{a(0)a(1)\cdots a(n-1)}$ and β_n is the right boundary point of $int\ I_{a(0)a(1)\cdots a(n-1)}$.

Then, we have a decreasing sequence of non-trivial open intervals

$$(\alpha_1, \beta_1) \supseteq (\alpha_2, \beta_2) \supseteq \ldots$$

with empty intersection. The only way this can happen is if

1. $\beta_n - \alpha_n \to 0$ as $n \to \infty$.
2. there is a $k > 0$ such that for $n \geq k$, either
   (a) $\alpha_n = \alpha_k$
   or

   (b) $\beta_n = \beta_k$

   (Note that condition (1) and the fact that the intervals are non-trivial guarantees that exactly one of (2a) and (2b) holds).

Statement (2) means that the left or right boundary points of the (α_n, β_n)'s must be the same for $n \geq k$.

Let $\mathcal{K}$ be the set of boundary points of the intervals $int\ \bigcap_{0 \leq j < n} f^{-j}(I_{a(j)})$ as $\mathbf{a}$ runs through $Cl(\Sigma_1^+)$. Then, $\mathcal{K}$ is at most countable.

So to each $\mathbf{a} \in Cl(\Sigma_1^+) \setminus \Sigma_1^+$ we associate the eventually constant boundary sequence

$$\psi(\mathbf{a}) = (z_0, z_1, \ldots).$$

The set of eventually constant boundary sequences is in one-to-one correspondence with a subset of the set of finite sequences of elements of $\mathcal{K}$. This last set is at most countable, and the map ψ is at most two-to-one, so $Cl(\Sigma_1^+) \setminus \Sigma_1^+$ is at most countable. □

The next result basically says that mod ze we may assume ζ is one-to-one.

Let $\mathcal{M}_e(f)$ denote the set of ergodic $f-$invariant probabliity measures on I. Let $\mathcal{M}_e^+(f)$ denote those measures in $\mathcal{M}_e(f)$ with positive metric entropy.

Propostion 1. *The set* $H = \{x \in I : \zeta^{-1}(\zeta(x))$ properly contains $\{x\}\}$ *is a zero-entropy set.*

Proof. Suppose $x \in H$, and let $I_x = \zeta^{-1}(\zeta x)$. Then, I_x is a non-trivial interval containing x. Also, $f \mid I_x$ is monotone and continuous, and $f(I_x) \subset I_{fx}$. Thus, $f(H) \subset H$. We have H is the countable union of the intervals $I_x, \ \ x \in H$. Let $\mu \in \mathcal{M}_e(f)$ be such that $\mu(H) > 0$. Then, there is an $x \in H$ such that $\mu(I_x) > 0$. It follows that there is an $n > 0$ with $f^n I_x \bigcap I_x \neq \emptyset$. But then $f^n I_x \subset I_x$ and $f^n \mid I_x$ is monotone. Since a monotone self-map of an interval carries no invariant measure of positive entropy, it follows that $h_\mu(f^n \mid I_x) = 0$ whence $h_\mu(f) = 0$. □

Proposition 2. *The map ζ induces an entropy preserving map*

$$\zeta_\star : \mathcal{M}(f) \to \mathcal{M}(\sigma \mid \Sigma_1^+).$$

Moreover, the restriction of $\zeta_\star$ to $\mathcal{M}_e^+(f)$ is a one-to-one correspondence between $\mathcal{M}_e^+(f)$ and $\mathcal{M}_e^+(\sigma, \Sigma_1^+) = \mathcal{M}_e^+(\sigma, Cl(\Sigma_1^+))$.

Proof.

Since the fibers of the map ζ are intervals mapped monotonically into each other by f, it follows that for each $\mu \in \mathcal{M}(f)$, we have $h_{\zeta_\star \mu}(\sigma) = h_\mu(f)$. This is the first statement of Propostion 2. Let us proceed to the second statement.

Since H is a null set for each $\mu \in \mathcal{M}_e^+(f)$, and $\zeta \mid I \setminus H$ is injective, we have that $\zeta_\star$ maps $\mathcal{M}_e^+(f)$ injectively into $\mathcal{M}_e^+(\Sigma_1^+)$. Let $\mu_1 \in \mathcal{M}_e^+(\Sigma_1^+)$. Consider $\zeta(H)$. Each $\mathbf{a} \in \zeta(H)$ is such that $\zeta^{-1}(\mathbf{a}) = \bigcap_{i \geq 0} f^{-i}(I_{a(i)})$ is a non-trivial interval $J(\mathbf{a})$ in I. Moreover, $\mathbf{a} \neq \mathbf{b}$ in $\zeta(H)$ implies that $J(\mathbf{a}) \bigcap J(\mathbf{b}) = \emptyset$. Thus, $\zeta(H)$ is at most countable and hence $\mu_1(\zeta(H)) = 0$ and $\mu_1(\Sigma_1^+ \setminus \zeta(H)) = 1$. Since ζ maps $I \setminus H$ injectively onto $\Sigma_1^+ \setminus \zeta(H)$, the latter set is Borel measurable and, then, $\zeta_\star^{-1} \mu_1 \in \mathcal{M}_e^+(f)$ is mapped by $\zeta_\star$ onto μ_1. □

The next statement shows that $h(f)$ can be obtained as the growth rate of the number of elements of the partition $\bigvee_{i=0}^{n-1} f^{-i}\alpha$.

Proposition 3.

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log card \bigvee_{i=0}^{n-1} f^{-i}\alpha$$

Proof of proposition 3.

We have defined $h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$. We will show that

$$h(f) = h(\sigma, Cl(\Sigma_1^+)) \qquad \text{(a)}$$

From standard facts about subshifts on a finite alphabet, we get that

$$\lim_{n \to \infty} \frac{1}{n} \log card \bigvee_{i=0}^{n-1} f^{-i}\alpha$$

exists and equals $h(\sigma, Cl(\Sigma_1^+))$. Thus Proposition 3 follows from (a).

Since $Cl(\Sigma_1^+)$ is compact, we know

$$h(\sigma, Cl(\Sigma_1^+)) = \sup_{\mu \in \mathcal{M}(\sigma, Cl(\Sigma_1^+))} h_\mu(\sigma) = \sup_{\mu \in \mathcal{M}_e^+(\sigma, Cl(\Sigma_1^+))} h_\mu(\sigma).$$

Now, from Proposition 2, and the definitions, we get

$$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f) = \sup_{\mu \in \mathcal{M}(f)} h_{\zeta_\star \mu}(\sigma, Cl(\Sigma_1^+)) \leq h(\sigma, Cl(\Sigma_1^+)),$$

and,

$$h(f) \geq \sup_{\mu \in \mathcal{M}_e^+(f)} h_\mu(f) = \sup_{\mu \in \mathcal{M}_e^+(\sigma, Cl(\Sigma_1^+))} h_\mu(\sigma) = h(\sigma, Cl(\Sigma_1^+)).$$

This proves (a) $\square$.

Proposition 4. *The map f possesses ergodic invariant measures of maximal entropy.*

Proof.
Since $\mu \to h_\mu(\sigma)$ is uppersemicontinuous on $Cl(\Sigma_1^+)$, it follows that $(\sigma, Cl(\Sigma_1^+))$ has measures of maximal entropy. Further, since entropy integrates over ergodic decompositions, it follows that σ has an ergodic measure μ of maximal entropy. Then, $\zeta_\star^{-1}\mu$ is an ergodic measure of maximal entropy for f. $\square$

We proceed to a proof of the first statement of Theorem 1. Let H be the set defined in Proposition 1. Let H_1 be the total $f-$orbit of the set of boundary points of the intervals $I_1, \ldots, I_s$. Let $L = I \setminus (H \bigcup H_1)$, and let $\Sigma_2^+ = \Sigma_1^+ \setminus \zeta(H \bigcup H_1)$. The transformation ζ maps L injectively onto Σ_2^+ and H, H_1, and L are $f-$invariant. Moreover, for each finite word $w = a(k) \ldots a(j)$ in Σ_2^+, the set $I_w = \bigcap_{i=0}^{j-k} f^{-i} I_{a(k+i)}$ is an open interval in I.

The natural extension $\hat{f} : \hat{L} \to \hat{L}$ factors isomorphically onto the two sided shift

$$\Sigma_2 = \{\mathbf{a} \in \Sigma(s) : \forall k \in \mathbf{Z}, a(k)a(k+1) \cdots \in \Sigma_2^+\}$$

where $\Sigma(s)$ is the full 2-sided shift on the symbols $\{1, \ldots, s\}$.

Let $\mathcal{B}$ be the set of finite admissible words in Σ_2. Thus, if $b \in \mathcal{B}$, then there are infinite words b_1, b_2 such that $b_1 b b_2 \in \Sigma_2$. For each $b = b_0 b_1 \ldots b_r \in \mathcal{B}$, let $L(b_r)$ be its last element, and let $F(b) = \{w \in \Sigma_2^+ : bw \in \Sigma_2^+\}$ be its future set in Σ_2^+. Define the equivalence relation $\sim$ on $\mathcal{B}$ by $a \sim b$ iff $F(a) = F(b)$ and $L(a) = L(b)$. We say that an element $\mathbf{a} \in \Sigma_2$ *is past-stabilizing* if for every $k \in \mathbf{Z}$ there is an $l < k$ such that

$$m \leq l \Rightarrow a(m)a(m+1) \ldots a(k) \sim a(l) \ldots a(k)$$

Let Σ_3 be the set of past-stabilizing words in Σ_2. It is easily seen to be shift invariant. Moreover, the elements of Σ_3 are also past-stabilizing for Σ_3. Let $\hat{\zeta}$ be the natural extension of the map $\zeta : L \to \Sigma_2^+$. Thus, $\hat{\zeta}$ maps $\hat{L}$ injectively onto Σ_2.

Main Proposition. *The complement $\Sigma_2 \setminus \Sigma_3$ of Σ_3 in Σ_2 is a zero-entropy set. Consequently, so is its pull-back to $\hat{L}$, $\hat{\zeta}^{-1}(\Sigma_2 \setminus \Sigma_3)$.*

Proof.

For $j \in \mathbf{Z}$, let A_j be the set of all elements $\mathbf{a} \in \Sigma_2 \setminus \Sigma_3$ such that for every $n < j$ there is an $m < n$ such that $F(a(m) \dots a(j))$ is a proper subset of $F(a(n) \dots a(j))$. Then, since $\Sigma_2 \setminus \Sigma_3$ consists of non past-stabilizing elements, we have that

$$\Sigma_2 \setminus \Sigma_3 = \bigcup_{j \in \mathbf{Z}} A_j$$

Suppose $\mu \in \mathcal{M}_e(\sigma \mid \Sigma_2 \setminus \Sigma_3)$. We must show that $h_\mu(\sigma) = 0$.

Now, there is a $j \in \mathbf{Z}$ such that $\mu(A_j) > 0$. Let $\tilde{\alpha}$ be the partition of $\Sigma_2 \setminus \Sigma_3$ induced by the partition α of I. This is obtained as follows. Let $\pi_1 : \Sigma_2 \to \Sigma_2^+$, $\pi_2 : \hat{L} \to L$ be the natural projections from the inverse limits, and let α_L be the restriction of the partition α to L. So, $\tilde{\alpha}$ is the partition $\hat{\zeta}\pi_2^{-1}\alpha_L$ restricted to $\Sigma_2 \setminus \Sigma_3$.

Then, $\tilde{\alpha}$ generates μ, so $h_\mu(\tilde{\alpha}, \sigma) = h_\mu(\sigma)$.

Also,

$$\begin{aligned} h_\mu(\tilde{\alpha}, \sigma) &= h_\mu(\sigma) && (1) \\ &= H_\mu(\sigma^j \tilde{\alpha} \mid \bigvee_{i=j+1}^{\infty} \sigma^i \tilde{\alpha}) && (2) \end{aligned}$$

We prove the last quantity is 0.

This is equivalent to the statement that

> For $\mu-$ almost every word $\mathbf{a} \in \Sigma_2 \setminus \Sigma_3$, if one knows the symbols $a(k)$ for $k < j$, then the symbol $a(j)$ is uniquely determined. $(\star)$

Now we know that each $\mathbf{a}$ is not past-stabilizing. So, for arbitrarily large negative k,

$$F(a(k-1) \dots a(j)) \text{ is a proper subset of } F(a(k) \dots a(j)) \quad (3)$$

But each finite word $w \in \Sigma_2$ corresponds to an interval $I_w \subset I$. Also, (3) means

$$f(I_{a(k-1)}) \text{ does not contain } I_{a(k) \dots a(j)} \quad (4)$$

If we let f_j be the continuous extension of $f \mid I_j$ to its closure, then (4) gives $f_{a(k-1)}(\partial I_{a(k-1)}) \bigcap I_{a(k)\dots a(j)} \neq \emptyset$ where ∂I_j denotes the boundary of the interval I_j.

Let l_j be the left boundary point of I_j, and let r_j be the right boundary point of I_j. We have for each $\mathbf{a} \in \Sigma_2 \setminus \Sigma_3$, there are infinitely many negative integers $k < j$ such that either

$$f_{a(k-1)}(l_{a(k-1)}) \in I_{a(k)\dots a(j)} \tag{5}$$

or

$$f_{a(k-1)}(r_{a(k-1)}) \in I_{a(k)\dots a(j)} \tag{6}$$

Let Σ_L be the set of elements $\mathbf{a} \in \Sigma_2 \setminus \Sigma_3$ such that there are infinitely many $k's < j$ such that (5) holds, and let Σ_R be the set of elements $\mathbf{a} \in \Sigma_2 \setminus \Sigma_3$ such that there are infinitely many $k's < j$ such that (6) holds.

Since Σ_L and Σ_R are $\sigma-$ invariant, μ is ergodic, and $\Sigma_2 \setminus \Sigma_3$ is the union of Σ_L and Σ_R, it follows that either $\mu(\Sigma_L) = 1$ or $\mu(\Sigma_R) = 1$. We suppose that $\mu(\Sigma_L) = 1$. The other case is similar.

Given $\mathbf{a} \in \Sigma_L$, if one knows $a(k)$ for $k < j$, and $m < j$ is such that $f_{a(m-1)}(l_{a(m-1)}) \in I_{a(m)\dots a(j)}$, then $f^{m-j} f_{a(m-1)}(l_{a(m-1)}) \in I_{a(j)}$. Thus, knowing $a(m-1)$ tells us what $l_{a(m-1)}$ is, and then this uniquely specifies $a(j)$. This gives $(\star)$ and proves the Main Proposition. $\square$

Next consider the shift invariant set Σ_3. Let $\mathcal{B}'$ be the set of finite admissible words in Σ_3.

The equivalence relation $\sim$ on $\mathcal{B}$ restricts to one on $\mathcal{B}'$ which we also denote by $\sim$.

Let Σ_3^+ be the set of infinite words $a_0 a_1 \dots$ such that there is an infinite word $b = (\dots b_{-2} b_{-1})$ with $b a_0 a_1 \dots \in \Sigma_3$.

For $b = b_0 b_1 \dots b_r \in \mathcal{B}'$, let $L(b) = b_r, F(b) = \{w \in \Sigma_3^+ : bw \in \Sigma_3^+\}$. Then $w \sim z$ if and only if $L(w) = L(z)$ and $F(w) = F(z)$.

Let $\mathcal{A}$ be the set of equivalence classes of $\sim$ on $\mathcal{B}'$, and define the incidence relation

$$[w] \succ [z] \text{ iff } wL(z) \sim z.$$

This defines a matrix A indexed by $\mathcal{A} \times \mathcal{A}$ with values in $\{0,1\}$ by

$$A_{[w],[z]} = 1 \text{ iff } [w] \succ [z].$$

Then, one gets a directed graph $(\mathcal{A}, \succ)$ which of course yields a corresponding two-sided shift (σ, Σ_A) (on the doubly infinite words). Hopefully the common use of the letter σ to denote various shift automorphisms will not confuse the reader.

We note in passing that a compact shift-invariant set $\tilde{\Sigma}$ is called *sofic* if the set of $\mathcal{A}(\tilde{\Sigma})$ equivalence classes is a finite set. In that case, one of the irreducible components of the doubly infinite sequences of $\mathcal{A}(\tilde{\Sigma})$ exhibits $\tilde{\Sigma}$ as a boundedly finite-to-one factor. Because of the analogy to sofic systems, we call the construction of the shift (σ, Σ_A) the *sofic* construction. It is this construction which yields the Markov symbolic dynamics for $(\hat{f}, \hat{I})$. Our goal now is to show that $(\sigma, \Sigma_A) \underset{\text{ze}}{\simeq} (\hat{f}, \hat{I})$. For this purpose, it suffices to show that $(\sigma, \Sigma_A) \underset{\text{ze}}{\simeq} (\sigma, \Sigma_3)$. This is contained in the following lemma.

Lemma 2. *The map* ζ_1 *defined by*

$$\zeta_1(\dots a_n a_{n+1} \dots) \to (\dots L(a_n) L(a_{n+1}) \dots)$$

is an isomorphism mod ze from Σ_A *to* Σ_3.

Proof. The proof consists of three steps.
Let $K = \zeta_1^{-1}(\Sigma_3)$.

Step 1: Σ_3 is contained in the image of ζ_1.

Let $\mathbf{a} \in \Sigma_3$. Since $\mathbf{a}$ is past stabilizing for Σ_2, it must also be past stabilizing for Σ_3. Thus, there is a $k < 0$ such that $a(k)a(k+1)\dots a(0) \sim a(l)\dots a(0)$ for all $l < k$. Let w_0 be the equivalence class of $a(k)a(k+1)\dots a(0)$ in $\mathcal{A}$. For $i > 0$, let w_i be the equivalence class of $a(k)a(k+1)\dots a(i)$. Then, $w = w_0 w_1 \dots$ maps to the one-sided word $a(0)a(1)\dots$. To prove Lemma 2, we need to extend w to a two sided word which maps to $\mathbf{a}$. There is an $m < k$ such that for $l < m$, $a(l)\dots a(-1) \sim a(m)\dots a(-1)$. Let w_{-1} be the equivalence class of $a(m)\dots a(-1)$. Since $m < k$, $a(m)\dots a(0) \sim w_0$ so $w_{-1}w_0\dots$ maps to $a_{-1}a_0\dots$. Proceeding in this way, we get an element $x \in \Sigma_A$ which ζ_1 maps to $(\dots L(a_n)L(a_{n+1})\dots)$.

Step 2: $\zeta_1 \mid K$ is injective.

Let $w \in K$, and let $a = \zeta_1(w)$. We show that w is determined by a. Since $a \in \Sigma_3$, we have that $\bigcap_{i\in\mathbf{Z}} \hat{f}^{-i}\pi_2^{-1} I_{a(i)}$ is a single point, say $x \in \hat{I}$. Let $i \in \mathbf{Z}$, and consider the symbol $w_i \in \mathcal{A}$. Let z be a finite admissible word in $\mathcal{B}'$ which represents w_i, say $z = b_\ell \dots b_k$. Then, $b_k = a(i)$, and $I_{b_{k-j}}$ is the unique interval in α such that $\pi_2 \hat{f}^{i-j} x \in I_{b_{k-j}}$. Thus, knowledge of x specifies w_i uniquely. Since i was arbitrary, w is uniquely specified by a and Step 2 is proved.

Step 3. $\Sigma_{\mathcal{A}} \setminus K$ is a ze set.

Let $\Sigma_{\mathcal{A}}^+$ be the one-sided shift on the alphabet $\mathcal{A}$. The set $\Sigma_{\mathcal{A}} \setminus K$ is the inverse limit of the set $\tilde{\Sigma}_{\mathcal{A}}^+$ of elements $w = (w_0, w_1, \dots) \in \Sigma_{\mathcal{A}}^+$ such that $(L(w_0), L(w_1), \dots) \in Cl\ \Sigma_3^+ \setminus \Sigma_3^+$. As in the proof of Lemma 1, one can show that $\tilde{\Sigma}_{\mathcal{A}}^{+}$ is at most countable. Hence, it is a ze set. Thus, $\Sigma_{\mathcal{A}} \setminus K$ is the inverse limit of a ze set, and, hence, must also be a ze set.

This completes the proof of Theorem 1.

We next move to a new proof of the following result which is a consequence of Lemma 13 in [5].

Theorem 2. *Let $0 < c < h(f)$. Consider the Markov shift (σ, Σ_A). There are only finitely many irreducible components of Σ_A with entropy greater than or equal to c.*

We first need the following lemma.

Lemma 3. *Let $\mu_1, \mu_2, \ldots$ be a sequence of $f-$invariant ergodic measures converging weakly to the measure μ such that $h_{\mu_i}(f) > c$ for each $i = 1, 2, \ldots$. Then, $\mu(\partial\alpha) = 0$ where $\partial\alpha$ denotes the set of boundary points of the partition $\alpha = \{I_1, I_2, \ldots I_s\}$.*

Proof.
We assume the hypotheses of Lemma 3 and that $\mu(\partial\alpha) > 0$. We proceed to obtain a contradiction. Since $\partial\alpha$ is a finite set and $\mu(\partial\alpha) > 0$, it follows that $\partial\alpha$ contains at least one periodic point p which is a μ atom; i.e. $\mu(\{p\}) > 0$. Let AT be the set of these atoms.

Now, lift the measures $\mu_1, \mu_2, \ldots$ to measures $\tilde{\mu}_1, \tilde{\mu}_2, \ldots$ on Σ_3^+. Taking subsequences if necessary, we may assume that these measures converge to a measure $\tilde{\mu}$ on $Cl(\Sigma_3^+)$.

We have two cases:

1. $\tilde{\mu}(Cl(\Sigma_3^+) \setminus \Sigma_3^+) > 0$.
2. $\tilde{\mu}(Cl(\Sigma_3^+) \setminus \Sigma_3^+) = 0$.

In case 1, since $Cl(\Sigma_3^+) \setminus \Sigma_3^+$ is at most countable and invariant (same proof as for Σ_2^+), there must be a set of shift invariant (hence periodic) $\tilde{\mu}$ atoms in this set. Those atoms project to intervals in I, and at least one of them must contain a point of AT in its boundary. Pick one such interval, say J. It follows that some iterate f^n maps J monotonically into an interval which contains J. But then J cannot meet the support of an ergodic $f-$invariant measure of positive entropy. But convergence of the μ_i's to μ implies that the supports of the $\mu_i's$ must meet J giving a contradiction. Thus, case 1 is taken care of.

In case 2, the measure $\tilde{\mu}$ projects to μ, and we again get an interval J mapped monotonically into an interval containing J giving another contradiction as in case 1.

This proves Lemma 3. □

Before proceeding to the proof of theorem 2, we need to examine the incidence relation $\succ$ a bit further.

Since each representative of an equivalence class w has the same last element, it makes sense to define $L(w)$ by setting $L(w) = L(b_1 \dots b_k)$ for any word $b_1 \dots b_k$ representing w.

To each finite word $b = b_0 b_1 \dots b_{k-1}$ in $\mathcal{B}'$ we can associate two open intervals in I which we will call the *initial interval* I_b of b and the *final interval* F_b of b. The intervals I_b and F_b are defined by

$$I_b = \bigcap_{i=0}^{k-1} f^{-i}(int\ I_{b_i})$$

and

$$F_b = f^{k-1} I_b$$

Note that if $a \sim b \in \mathcal{B}'$, then $F_a \bigcap \pi_2 \hat{\zeta}^{-1}(\Sigma_3) = F_b \bigcap \pi_2 \hat{\zeta}^{-1}(\Sigma_3)$.

Recall that $w \succ z$ means that if $b_1 b_2 \dots b_k$ represents w, then $b_1 b_2 \dots b_k L(z)$ represents z. We write $w \succ\succ z$ if there is a finite sequence $w = w_1 \succ w_2 \succ \dots \succ w_k = z$. Given a shift invariant subset Σ of Σ_A, we write $\mathcal{A}(\Sigma)$ for the set of elements of the alphabet $\mathcal{A}$ which appear in doubly infinite words in Σ. An irreducible component Σ of Σ_A is a maximal shift invariant subset of Σ_A such that for any two symbols w, z in $\mathcal{A}(\Sigma)$, we have $w \succ\succ z$, and $z \succ\succ w$.

Lemma 4. *Let $z \in \mathcal{A}$ be an element of the alphabet of some irreducible component of Σ_A, let Σ be a possibly different irreducible component of Σ_A, and let $\mathbf{w} \in \Sigma$. Let $\mathbf{b} \in \Sigma_3$ and $k < 0$ be chosen so that, for $\ell \le k$, $b_\ell b_{\ell+1} \dots b_0$ represents z. Suppose that, for arbitrarily large $i > 0$, we have $\pi_2 \hat{\zeta}^{-1} \zeta_1(\sigma^{-i}\mathbf{w}) \in F_{b_k \dots b_0}$. Then, $z \succ\succ \gamma$ for every $\gamma \in \mathcal{A}(\Sigma)$.*

Proof. Assume that $\mathbf{w} = (\dots w_j w_{j+1} w_{j+2} \dots)$ and $a_j = L(w_j)$ for $j \in \mathbf{Z}$, so that $\mathbf{a} = (\dots a_j a_{j+1} \dots) = \zeta_1(\mathbf{w})$. Choose $k_1 < k$ so that for $l \le k_1$, $a_l \dots a_0$ represents w_0. Then, for $n \ge 0$, $\ell \le k_1$, we have that $a_\ell \dots a_n$ represents w_n. Pick $i > \mid k_1 \mid$ such that $x = \pi_2 \hat{\zeta}^{-1} \zeta_1(\sigma^{-i}\mathbf{w}) \in F_{b_k \dots b_0}$. Since $\mathbf{a} \in \Sigma_3$, we have $\bigcap_{j \ge -i} f^{-j} I_{a_i} = \{x\}$. This implies that, for large $n > 0$, $I_{a_{-i} \dots a_n} \subset F_{b_k \dots b_0}$. Then, $b_0 = a_{-i}$, and,

$$F(b_k \dots b_0 a_{-i+1} \dots a_n) = F(b_k \dots b_{-1} a_{-i} \dots a_n) = F(a_{-i} \dots a_n)$$

(future sets in Σ_3^+), so $z \succ\succ w_n$. But, since Σ is irreducible, we have $w_n \succ\succ \gamma$ for any $\gamma \in \mathcal{A}(\Sigma)$. So, we also have $z \succ\succ \gamma$ for any γ. $\Box$

Now consider an irreducible component Σ of Σ_A with topological entropy $h(\Sigma) > c > 0$.

Let $\tilde{G}_\Sigma$ be a compact shift invariant topologically transitive subset of $Cl\ \zeta_1(\Sigma)$ with topological entropy greater than c. For instance, we could take an ergodic measure μ on Σ with $h_\mu(\sigma) > c$, and let $\tilde{G}_\Sigma$ be the closure of the $\zeta_{1\star}\mu$-generic points in the support of $\zeta_{1\star}\mu$. Alternatively, we could take a shift invariant subset corresponding to a finite irreducible subgraph of the graph of Σ with entropy greater than c.

Let $G_{\sum} = \pi_2 \hat{\zeta}^{-1} \tilde{G}_{\sum}$, so that $G_{\sum}$ is an uncountable subset of I which is invariant under forward and backward $f-$iterates. Let $\alpha(\Sigma)$ be the set of intervals $I_i \in \alpha$ such that $int\ I_i \bigcap G_{\sum} \neq \emptyset$. For $i \in \alpha(\Sigma)$, let $\tilde{I}_i$ be the interior of the smallest closed interval containing $int\ I_i \bigcap G_{\sum}$. Then,

$$\text{each boundary point of } \tilde{I}_i \text{ is in the closure of } G_{\sum} \tag{7}$$

Also, we can use the partition $\alpha_{\sum} = \{\tilde{I}_i : i \in \alpha(\Sigma)\}$ to give symbolic dynamics for $(\hat{f}, \tilde{G}_{\sum})$ which is isomorphic mod ze (in fact, up to orbits in the boundary of elements of $\alpha(\Sigma)$) to that of the partition α. The important point about using the intervals $\tilde{I}_i$ is (7). Next, construct the open intervals $\tilde{F}_b$ for representatives b of $w \in \mathcal{A}(\zeta_1^{-1}(\tilde{G}_{\sum}))$. Note that for two representatives a, b of $w \in \mathcal{A}(\zeta_1^{-1}(\tilde{G}_{\sum}))$, we have $\tilde{F}_a = \tilde{F}_b$ (since $F_a \bigcap \pi_2 \hat{\zeta}^{-1}(\Sigma_3) = F_b \bigcap \pi_2 \hat{\zeta}^{-1}(\Sigma_3)$). Thus, we may define $\tilde{F}_w$ to be $\tilde{F}_a$ for any representative a of $w \in \mathcal{A}(\zeta_1^{-1}(\tilde{G}_{\sum}))$.

From lemma 4 and (7) it follows that

$$\text{if } \Sigma_1, \Sigma_2 \text{ are distinct irreducible components of } \Sigma_A \text{ with } h(\Sigma_i) > c, \text{ and sets } G_{\sum_i}, \alpha_{\sum_i} \text{ as above, then for } w_i \in \mathcal{A}(\Sigma_i), \text{ we have } \tilde{F}_{w_1} \bigcap \tilde{F}_{w_2} = \emptyset. \tag{8}$$

Proof of the Theorem 2.

For each irreducible component Σ of Σ_A with entropy greater than c, we construct $\tilde{G}_{\sum}, \alpha_{\sum}$ as above. Let $\mathcal{A}_1(\Sigma) = \mathcal{A}(\zeta_1^{-1}(\tilde{G}_{\sum}))$, and let

$$t(\Sigma) = \max\{length(\tilde{F}_w) : w \in \mathcal{A}_1(\Sigma)\}.$$

Then, let $t(f)$ be the infimum of the set of $t(\Sigma)'s$ where Σ runs through the irreducible components of Σ_A of entropy greater than c.

We claim that

$$t(f) > 0 \tag{9}$$

Since any disjoint collection of open intervals in I with lengths bounded away from zero is finite, theorem 2 follows from (8) and (9).

To prove the claim we assume that $t(f) = 0$.

Let $\Sigma_1, \Sigma_2, \ldots$ be a sequence of irreducible components of Σ_A with entropy greater than c with $t(\Sigma_i) \to 0$ as $i \to \infty$. Let $\mu_1, \mu_2, \ldots$ be ergodic $\sigma-$invariant measures with entropy greater than c with μ_i supported on Σ_i for each i. Let $\mu_i \to \nu$ as $i \to \infty$. Let μ be the f-invariant measure $(\pi_2 \hat{\zeta}^{-1} \zeta_1)_\star \nu$. By Lemma 3, $\mu(\partial\alpha) = 0$. Let $0 < \epsilon$ be such that $\mu(B_\epsilon(\partial\alpha)) < \frac{c}{2}$. If x_i is a μ_i-generic point, then for large i, we have the orbit of $\pi_2\hat{\zeta}^{-1}\zeta_1(x_i)$ spends relatively little time in $B_\epsilon(\partial\alpha)$. But the only times that new symbols can be created are when the intervals corresponding to symbols have their images meeting elements of $\partial\alpha$. Since the intervals associated to Σ_i are small for large i, and the orbits of these intervals spend a small frequency of time near $\partial\alpha$, we see that for large i, the entropy of μ_i must be less than $\frac{3c}{4}$, which is a contradiction and (9) is proved. □

In [5], Hofbauer proves that each irreducible component of Σ has at most one measure of maximal entropy, and that if such a measure exists, its natural extension is a Markov measure. These results were also given by Gurevic [2], [3].

In view of Theorem 2, we thus have the following corollary.

Corollary. *The set to measures of maximal entropy for f is a finite dimensional simplex. Moreover, the natural extension of each ergodic maximal measure is Markov.*

This Corollary is the last statement of Theorem 1.

Problem. *Suppose that f has s branches on each of which it is monotone and continuous. Is $h(f) \leq \log s$?.*

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