A TWO-DIMENSIONAL VERSION OF THE FOLKLORE THEOREM

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Abstract. We formulate some sufficient conditions for the existence of Sinai-Ruelle-Bowen measures for piecewise C^2 diffeomorphisms with unbounded derivatives. The result can be viewed as a two-dimensional version of the well known one-dimensional Folklore Theorem on the existence of absolutely continuous invariant measures. Here we formulate the results and outline the main ideas and tools of our approach. The detailed version will appear elsewhere.

§1. The Folklore Theorem and SRB measures

The well-known Folklore Theorem in one-dimensional dynamics can be formulated as follows.

Folklore Theorem. *Let $I = [0,1]$ be the unit interval and $\{I_1, I_2, \dots\}$ a countable collection of disjoint open subintervals of I such that $\bigcup_i I_i$ has the full Lebesgue measure in I. Suppose there are constants $K_0 > 1$ and $K_1 > 0$ and mappings $f_i : I_i \to I$ satisfying the following conditions:*

1. *f_i extends to a C^2 diffeomorphism from* Closure(I_i) *onto* $[0,1]$, *and* $\inf_{z \in I_i} |Df_i(z)| > K_0$ *for all i.*
2. $\sup_{z\in I_i} \dfrac{|D^2 f_i(z)|}{|Df_i(z)|} |I_i| < K_1$ *for all i,*

where $|I_i|$ denotes the length of I_i. Then the mapping $F(z)$ defined by $F(z) = f_i(z)$ for $z \in I_i$ has a unique invariant ergodic probability measure μ equivalent to Lebesgue measure on I.

For a proof of the Folklore Theorem and the ergodic properties of μ, see, for example, [1] and [17].

Here we formulate a theorem that can be considered as a two-dimensional version of this Folklore Theorem.

Let $\tilde{Q}$ be a Borel subset of the unit square Q in the plane $\mathbf{R}^2$ with positive Lebesgue measure, and let $F : \tilde{Q} \to \tilde{Q}$ be a Borel measurable map. An F-invariant Borel probablility measure μ on Q is called a *Sinai-Ruelle-Bowen* measure (or SRB-measure) for F if μ is an ergodic measure with non-zero Lyapunov exponents and

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there is a set $A \subset \tilde{Q}$ of positive Lebesgue measure such that for $x \in A$ and any continuous real-valued function $\varphi : Q \to \mathbf{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(F^k x) = \int \varphi \, d\mu. \tag{1}$$

The set of all points x for which (1) holds is called the *basin* of μ.

We are interested in giving conditions under which certain two-dimensional maps F which piecewise coincide with hyperbolic diffeomorphisms f_i, have SRB measures. As in the one-dimensional situation, there is an essential difference between a finite and an infinite number of f_i. In the case of an infinite number of f_i, their derivatives grow with i and relations between first and second derivatives become crucial.

A different class of piecewise hyperbolic maps $F = f_i \mid E_i$, in which the number of domains E_i is finite but derivatives $Df_i(z)$ are allowed to grow when z approaches the boundary of E_i, was considered in [10] and [13]. In these works it was assumed that the derivatives $DF(z)$ grow at most exponentially depending on the distance between the point z and the singular set where F is not defined. In our work we allow the domains E_i to accumulate towards the limit set in an arbitrary way. Then the conditions of [10], [13] are typically violated even for linear maps f_i.

In [12] a method was developed that reduces the existence of an SRB measure for F to the existence of an absolutely continuous invariant measure for the one-dimensional map G obtained from F by factorizing along the stable manifolds. The technique of distortion estimates that we develop here might be used to check the conditions of [12]. In general, for the systems under consideration, checking the conditions of [12] looks similar to the straightforward generalization of [3].

§2. Hyperbolicity and geometric conditions

Consider a countable collection $\xi = \{E_1, E_2, \dots, \}$ of full height closed curvilinear rectangles in Q. Assume that each E_i lies inside a domain of definition $\mathcal{E}_i$ of a C^2-diffeomorphism f_i which maps E_i onto its image $S_i \subset Q$. We assume each E_i connects the top and the bottom of Q. Thus each E_i is bounded from above and from below by two subintervals of the line segments $\{(x,y) : y = 1, \ 0 \le x \le 1\}$ and $\{(x,y) : y = 0, \ 0 \le x \le 1\}$. We assume that the left and right boundaries of E_i are graphs of smooth functions $x^{(i)}(y)$ with $\left|\frac{dx^{(i)}}{dy}\right| < \alpha$. We further assume that the images $f_i(E_i) = S_i$ are strips connecting the left and right sides of Q and that they are bounded on the left and right by the two subintervals of the line segments $\{(x,y) : x = 0, \ 0 \le y \le 1\}$ and $\{(x,y) : x = 1, \ 0 \le y \le 1\}$ and above and below by the graphs of smooth functions $Y^i(X)$, $\left|\frac{dY^{(i)}}{dX}\right| < \beta$. The bounds on derivatives follow from hyperbolicity conditions H1, H2 that we formulate below.

The sets E_i are called ***posts*** , the sets S_i are called ***strips***. We say that the E_i's are of *full height* in Q, while the S_i's are of *full width* in Q.

We shall assume that the margins between the posts E_i and the extended domains $\mathcal{E}_i$ containing these posts are comparable to E_i in the following sense. For a point $z \in Q$, let l_z denote the horizontal line through z. If $E \subseteq Q$ is a subset of Q, let $\delta_z(E)$ denote the diameter of the horizontal section $l_z \bigcap E$. We call $\delta_z(E)$ the *z-width* of E.

For a given $k > 0$ and any $z \in \operatorname{int} E_i$, let $B_{(k)}(z)$ be the ball of radius $k\delta_z(E_i)$ centered at z. We assume there exists $k > 0$ independent of i and $z \in \operatorname{int} E_i$ such that $\mathcal{E}_i$ contains the set $\bigcup_{z \in E_i} B_{(k)}(z)$.

We assume the following *geometric conditions*:

G1 int $E_i \cap$ int $E_j = \emptyset$ for $i \neq j$;

G2 $\operatorname{mes}(Q \setminus \cup_i \text{ int } E_i) = 0$ where mes is the Lebesgue measure;

G3 $S_i \cap S_j = \emptyset$ for $i \neq j$ and S_i are disjoint from the top and from the bottom of Q;

and the condition G4 will be formulated below.

In the standard coordinate system for a map $f : (x, y) \to (f_1(x, y), f_2(x, y))$ we use $Df(x, y)$ to denote the differential of f at some point (x, y) and f_{jx}, f_{jy}, f_{jxx}, f_{jxy}, etc., for partial derivatives of f_j, $j = 1, 2$.

We use a version of hyperbolicity conditions introduced by Alekseev ([2]) who generalized the conditions of Smale ([14]). See [6] for another version of such conditions.

Hyperbolicity conditions. There exist positive constants $\mu_1, \mu_2, \varepsilon_{12}, \varepsilon_{21}$, such that for all i the map

$$F(z) = f_i(z) \quad \text{for } z \in \mathcal{E}_i$$

satisfies the conditions

H1. $|F_{2y} - F_{1y}F_{2x}F_{1x}^{-1}| \leq \mu_1$, $|F_{1x}^{-1}| \leq \mu_2$, $|\frac{F_{1y}}{F_{1x}}| \leq \varepsilon_{12}$, $|\frac{F_{2x}}{F_{1x}}| \leq \varepsilon_{21}$;

H2. $\mu_1\mu_2 < 1$;

H3. $\mu_1 + \mu_2 - \mu_1\mu_2 + \varepsilon_{12}\varepsilon_{21} \leq 1$.

For a positive real number $\alpha > 0$, we define the cones

$$K^u_\alpha = \{(v_1, v_2) : |v_2| \leq \alpha|v_1|\},$$
$$K^s_\alpha = \{(v_1, v_2) : |v_1| \leq \alpha|v_2|\}.$$

As is proved in [2], the hyperbolicity conditions H1–H3 imply that there exist two disjoint families of cones K^u_α and K^s_β independent of the point z such that DF maps K^u_α into the interior of K^u_α and DF^{-1} maps K^s_β into the interior of K^s_β. Also there exist a constant $K_0 > 1$ and a constant $c_0 > 0$ such that for any $v^{(u)} \in K^u_\alpha$ and for any $v^{(s)} \in K^s_\beta$ we have

$$|DF^n v^{(u)}| \geq c_0 K_0^n |v^{(u)}|, \tag{2}$$

$$|DF^{-n} v^{(s)}| \geq c_0 K_0^n |v^{(s)}|. \tag{3}$$

The cone conditions imply that any intersection $FE_i \cap E_j$ is full width in E_j. Also $E_{ij} = E_i \cap F^{-1}E_j$ is a full height subpost of E_i and F^2E_{ij} is a full width substrip of Q.

Proceeding with finite strings, we get that each set

$$E_{i_{-n}} \cap F^{-1}(E_{i_{-n+1}}) \dots F^{-n}(E_{i_0}) = P_{i_{-n}i_{-n+1}\dots i_0}$$

is a full height subpost of $E_{i_{-n}}$, and each set

$$FE_{i_0} \cap F^2 E_{i_{-1}} \dots F^{n+1} E_{i_{-n}} = F^{n+1}(P_{i_{-n} i_{-n+1} \dots i_0})$$

is a full width strip in Q. Then it follows from Theorem 4 in [2] that the corresponding infinite intersections are C^1 curves.

Namely, the following proposition holds.

Proposition 1. *Any C^1 map F satisfying the above geometric conditions* G1–G3 *and hyperbolicity conditions* H1–H3 *has a "topological attractor"*

$$\Lambda = \bigcup_{\dots i_{-n} \dots i_{-1} i_0} \bigcap_{k=0}^{\infty} F^{k+1} E_{i_{-k}}.$$

The infinite intersections $\bigcap_{k=0}^{\infty} F^{k+1} E_{i_{-k}}$ define C^1 curves $\gamma = y(x)$, that are the unstable manifolds for the points of the attractor. The infinite intersections $\bigcap_{n=0}^{\infty} P_{i_{-n} i_{-n+1} \dots i_0}$ define C^1 curves $x(y)$, that are the stable manifolds for the points of the attractor.

Remark 1. Let $\tilde{Q}$ be the set of points whose forward orbits always stay in $\bigcup_i \operatorname{int} E_i$. Then $\tilde{Q}$ has full Lebesgue measure in Q and F maps $\tilde{Q}$ into itself. The union of the stable manifolds contains $\tilde{Q}$, which has full measure in Q. The trajectories of all points in $\tilde{Q}$ converge to Λ. This is the reason to call Λ a topological attractor. However, the convergence of Birkhoff averages to the unique SRB measure is a much stronger property.

Remark 2. The distortion condition D1 and *distortion estimates* below imply that if our maps f_i are C^2 smooth, then the unstable manifolds are actually C^2 smooth. Similar conditions on the inverses of f_i imply that the stable manifolds are C^2 smooth, see [9].

§3. Distortion conditions and the main theorem

Since we have a countable number of domains, the derivatives of f_i grow. We will need certain assumptions on the second derivatives.

In a given coordinate system, we write $f_i(x,y) = (f_{i1}(x,y), f_{i2}(x,y))$. We use $f_{ijx}, f_{ijy}, f_{ijxx}, f_{ijxy}$, etc. for partial derivatives of f_{ij}, $j = 1,2$. Next we formulate distortion conditions. These will be used to control the fluctuation of the derivatives of iterates of F as in Lemma 1 below.

Suppose there is a constant $C_0 > 0$ such that the following *distortion conditions* hold:

D1. $\sup_{z \in \mathcal{E}_i, i \geq 1} \dfrac{|f_{ijkl}(z)|}{|f_{i1x}(z)|} \delta_z(E_i) < C_0$, where $j = 1,2$, $k,l = x,y$.

Remark 1. The widths $\delta_z(E_i)$ vary continuously as z varies in E_i, but as z moves from the top to the bottom of Q, the widths can become arbitrarily small.

The following geometric condition is sufficient to control the fluctuations of the widths of E_i.

Let $\delta_{i,\min} = \min_{z \in E_i} \delta_z(E_i)$, $\delta_{i,\max} = \max_{z \in E_i} \delta_z(E_i)$. Let K_0 be the expansion constant from (2).

G4. There exists $K_0 > K_1 > 1$ such that if we let $a_n = \sum_i \delta_{i,\max}$, the sum taken over those i that satisfy $\delta_{i,\min} \leq \frac{K_1^n}{K_0^n}$, then $\sum_n a_n < \infty$

Condition G4 implies that

$$-\sum_i \delta_{i,\max} \log \delta_{i,\min} < \infty. \tag{4}$$

On the other hand, (4) implies G4 for any $K_0 > K_1 > 1$. So, if G4 holds for some K_1 as above , then it also holds for all $K_0 > K_1 > 1$.

Remark 2. Condition G4 is satisfied if there are constants $0 < C_1, C_2,\ 0 < a < b < 1$ such for every i and for every $z \in E_i$, one has $C_1 a^i \leq \delta_z(E_i) \leq C_2 b^i$.

Theorem 1. *If F is a piecewise smooth mapping as above satisfying the geometric conditions* G1–G4*, the hyperbolicity conditions* H1–H3*, and the distortion conditions* D1*, then, F has an SRB measure whose basin has full Lebesgue measure in Q.*

§4. Distortions and ratios of derivatives

We proceed toward a sketch of the proof of Theorem 1.

We start with the following procedure of *width-reducing* .

Let ξ be a finite partition of Q into narrow full height rectangular posts $\bar{P}$, all of the same width χ, and let $\xi_1 = F^{-1}\xi$. Every element P_1 of ξ_1 is a full height curvilinear subpost of some orginal post E_i. Let $s_z \subset E_i$ be the horizontal crossection of E_i. The crossection s_z is mapped by F onto the full width curve γ in Q. Bounded distortions imply (see estimate (23) below) that there exists a constant C_{01} depending on C_0 such that for any vertical post $\bar{P}$ we have

$$\frac{|F^{-1}(\gamma \cap P)|}{|s_z|} \leq C_{01}|P \cap \gamma|. \tag{5}$$

It follows from (5) that for any $\varepsilon_1 > 0$ we can choose original partition ξ so that

$$\frac{|s_z(P_1)|}{|s_z(E_i)|} \leq \varepsilon_1. \tag{6}$$

Thus for an arbitrarily small $d_0 > 0$ we can choose ε_1 such that the distortion estimates D1 restricted to the posts P_1 become

D1′. $\sup_{z\in P_1, i\geq 1} \dfrac{|f_{ijkl}(z)|}{|f_{i1x}(z)|}\delta_z(P_1) < d_0$.

We must estimate the distortion of iterates of our map F. For this purpose it is convenient to introduce certain affine coordinate systems centered at various points z, Fz in such a way that, with respect to these coordinates, the Jacobian matrix of F at z is diagonal. We will formulate our distortion estimates so that they are uniform for all such affine coordinate representations of F. Our assumption that f_i extends to the neighborhood $\mathcal{E}_i$ guarantees that f_i will be defined and C^2 smooth on each *admissible parallelogram* defined below.

Fix a post $P_1 \subset E_i$. Let $z \in P_1$ and let v, w be a pair of unit vectors with $v \in K^u_\alpha$, $w \in K^s_\beta$. If E is a parallelogram with edges parallel to v, w, then the edges parallel to v are called the *top* and *bottom* of E, and the edges parallel to w the *sides* of E. We define the width of E to be the common length of the top and bottom, and the height of E to be the common length of the sides. Let $s_{z,v}$ be the crossection of P_1 through z by the line spanned by v and let $l_{z,w}$ be the line through z spanned by w.

We fix some $T_0 > 1$ depending on hyperbolicity conditions, as described below. Let $E_{z,v,w}$ be a parallelogram of the smallest width with edges parallel to v, w such that

(1) The left and right sides of $E_{z,v,w}$ do not meet the interior of P_1.
(2) $\text{height}(E_{z,v,w}) = T \, \text{width}(s_{z,v}), 0 < T \le T_0$.
(3) z divides the height of $E_{z,v,w}$ into equal parts.

Although z is not necessarily located in the center of $(s_{z,v})$, we say $E_{z,v,w}$ as indicated is an admissible parallelogram centered at z_0 inscribed in P_1 .

The hyperbolicity conditions imply that there exists n_0 such that for $v^{(u)} \in K^u_\alpha$, $v^{(s)} \in K^s_\beta$ we have

$$|DF^{n_0} v^{(u)}| \ge 2|v^{(u)}|, \tag{7}$$

$$|DF^{-n_0} v^{(s)}| \ge 2|v^{(s)}|. \tag{8}$$

Then there exists $T_0 \ge 1$ such that for $1 \le n \le n_0$ we have

$$|DF^{n} v^{(u)}| \ge 2T_0^{-1}|v^{(u)}|, \tag{9}$$

$$|DF^{-n} v^{(s)}| \ge 2T_0^{-1}|v^{(s)}|. \tag{10}$$

We use this T_0 in the above definition of the admissible parallelograms.

Since the tangent lines to the vertical boundaries of the posts belong to the stable cone and we choose v in the unstable cone, the admissible horizontal directions are transversal to the boundaries and $E_{z,v,w} \cap P_1 = E^{\text{dyn}}_{z,v,w}$ is a full width curvilinear subrectangle of P_1.

Given a point z and a pair of linearly independent vectors v, w, consider the associated unit vectors $\bar{v} = v/|v|$, $\bar{w} = w/|w|$. Let $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$ denote the standard unit vectors in $\mathbf{R}^2$. Consider the affine automorphism $A_{z,v,w}$ of $\mathbf{R}^2$ such that $A_{z,v,w}(z) = z$, $DA_{z,v,w}(e_1) = \bar{v}$, $DA_{z,v,w}(e_2) = \bar{w}$.

Some of these affine automorphisms will give us coordinate changes which will be useful to control distortions.

For $z \in P_1$, $v \in K^u_\alpha$, $w \in K^s_\beta$, write $f = f_i$ and consider the local coordinate representation

$$\tilde{f}_{z,v,w} = A^{-1}_{fz, Df_z(v), w} \circ f \circ A_{z,v,Df^{-1}_{fz} w}.$$

The map $\tilde{f}_{z,v,w}$ sends z to $f_i z$ and its Jacobian matrix at z is a diagonal matrix. We say that $\tilde{f}_{z,v,w}$ is an adapted representation of f_i or that we have a system of adapted coordinates for f_i. We also call $\tilde{f}_{z,v,w}$ the (z, v, w)-adapted representation of f_i.

We shall use admissible parallelograms $E_{z,v,Df_{fz}^{-1}w}$ with sides parallel to adapted axes.

For a local diffeomorphism f and a parallelogram E on which f is defined and C^2 smooth, we define the following quantities:

$$\varepsilon_{12}(f,E) = \sup_{z\in E} \frac{|f_{1y}(z)|}{|f_{1x}(z)|}, \tag{11}$$

$$\varepsilon_{21}(f,E) = \sup_{z\in E} \frac{|f_{2x}(z)|}{|f_{1x}(z)|}, \tag{12}$$

$$\varepsilon_{22}(f,E) = \sup_{z\in E} \frac{|f_{2y}(z)|}{|f_{1x}(z)|}, \tag{13}$$

$$\Phi_{11}(f,E) = \max\left(\sup_{z\in E} \frac{|f_{1xx}(z)|\Delta(E)}{|f_{1x}(z)|}, \sup_{z\in E} \frac{|f_{2xx}(z)|\Delta(E)}{|f_{1x}(z)|} \right), \tag{14}$$

$$\Phi_{12}(f,E) = \max\left(\sup_{z\in E} \frac{|f_{1xy}(z)|\Delta(E)}{|f_{1x}(z)|}, \sup_{z\in E} \frac{|f_{2xy}(z)|\Delta(E)}{|f_{1x}(z)|} \right), \tag{15}$$

$$\Phi_{22}(f,E) = \max\left(\sup_{z\in E} \frac{|f_{1yy}(z)|\Delta(E)}{|f_{1x}(z)|}, \sup_{z\in E} \frac{|f_{2yy}(z)|\Delta(E)}{|f_{1x}(z)|} \right), \tag{16}$$

where $\Delta(E) = \max\{\text{height}(E), \text{width}(E)\}$ is *the size of* E.

Finally, we set

$$\Theta_{jk}(f_i) = \sup_{z\in P_1\in E_i, v\in K_\alpha^u, w\in K_\beta^s} \Phi_{jk}(\tilde{f}_{z,v,w}, E_{z,v,Df_{fz}^{-1}w}) \tag{17}$$

for $(j,k) = (1,1), (1,2)$, or $(2,2)$.

An upper bound on $\Theta_{jk}(f_i)$ is a uniform upper bound on the distortions of the local coordinate representations of f_i, using those affine coordinate systems centered at $(z, f_i z)$ that diagonalize the Jacobian matrix of f_i at z.

We claim that $\Theta_{jk}(f_i)$ are uniformly bounded by a constant depending on distortion conditions and on hyperbolicity conditions . First we notice that second derivatives with respect to various adapted coordinates differ by a bounded factor. Next, hyperbolicity conditions imply that the vectors belonging to the unstable cone are expanded proportionally to $|f_{1x}|$. Then there exists $p_1 > 0$ such that for all i at the center z_0 of any adapted coordinate system we have

$$|f_{1x,\text{adapted}}(z_0)| > p_1 |f_{1x,\text{standard}}(z_0)|.$$

Finally, for an arbitrary point z of an admissible parallelogram we have a similar estimate because the difference of derivatives at z_0 and at z is estimated in terms of distortions (see below), which we made arbitrarily small. That gives

$$|f_{1x,\text{adapted}}(z)| > p_0 |f_{1x,\text{standard}}(z)|. \tag{18}$$

It remains to estimate the difference between the width of an admissible parallelogram and the width of the corresponding dynamically defined horizontal crossection s_z of P_1. That difference depends on the horizontal deviation of the boundary curves of P_1 within admissible height, which by definition is less than $(1/2)T_0|s_z|$. The horizontal deviation is a function of y given by

$$\delta x(y) = \int \left| \frac{f_{1y}}{f_{1x}} \right| dy.$$

In an adapted coordinate system, $f_{1y}(z_0) = 0$ and $\left| \frac{f_{1y}}{f_{1x}} \right|$ is estimated using distortion conditions D1 as in (21) below. Since we made the distortions arbitrarily small by reducing the width we get that the horizontal shift is arbitrarily small compared to the width of the horizontal section. Therefore

$$\text{width}(E_{z,v,Df_{f_z}^{-1}w}) \leq (1+\varepsilon)\delta_z(E^{\text{dyn}}_{z,v,Df_{f_z}^{-1}w}) \tag{19}$$

for any $z \in P_1$

This implies that D1 holds for any adapted coordinate system with another small constant depending on d_0. We shall keep the notation d_0 for that constant. So we get $\sup_{ijk}\ \Theta_{jk}(f_i) < d_0$.

§5. Fluctuation of derivatives

Let $z \in E_i$ and let $\hat{E}$ be an admissible parallelogram containing z (perhaps centered at a different point z_0 of E_i). Let $\hat{\Delta}$ be the size of $\hat{E}$. Write f for f_i. We can estimate $\varepsilon_{ij}(f)$ in terms of Φ_{ij} by the mean value theorem using the fact that in adapted coordinates $f_{1y}(z_0) = f_{2x}(z_0) = 0$. We get

$$\frac{|f_{1y}(z)|}{|f_{1x}(z)|} \leq \frac{|f_{1xy}(\tau)|}{|f_{1x}(z)|}\hat{\Delta} + \frac{|f_{1yy}(\tau)|}{|f_{1x}(z)|}\hat{\Delta}, \tag{21}$$

$$\log|f_{1x}(\tau) - \log f_{1x}(z)| \leq \frac{|f_{1xx}(\tau_1)|}{|f_{1x}(\tau_1)|}\hat{\Delta} + \frac{|f_{1xy}(\tau_1)|}{|f_{1x}(\tau_1)|}\hat{\Delta}. \tag{22}$$

So we get

$$\frac{|f_{1x}(\tau)|}{|f_{1x}(z)|} \leq \exp(\Phi_{11}(f) + \Phi_{12}(f)). \tag{23}$$

Using (21), (23) we get

$$\varepsilon_{12} \leq (\Phi_{12}(f) + \Phi_{22}(f))\exp(\Phi_{11}(f) + \Phi_{12}(f)). \tag{24}$$

Similarly from

$$\frac{|f_{2x}(z)|}{|f_{1x}(z)|} \leq \frac{|f_{2xx}(\tau)|}{|f_{1x}(z)|}\hat{\Delta} + \frac{|f_{2xy}(\tau)|}{|f_{1x}(z)|}\hat{\Delta} \tag{25}$$

we get

$$\varepsilon_{21} \leq (\Phi_{11}(f) + \Phi_{12}(f))\exp(\Phi_{11}(f) + \Phi_{12}(f)). \tag{26}$$

To estimate $\dfrac{|f_{2y}(z)|}{|f_{1x}(z)|}$, we use (9), (10) and get $\dfrac{|f_{2y}(z_0)|}{|f_{1x}(z_0)|} \leq \tau_0$, where $\tau_0 = T_0^2/4$.

For other points $z \in \hat{E}$ we get as above

$$\varepsilon_{22} \leq (\tau_0 + \Phi_{12}(f) + \Phi_{22}(f)) \exp(\Phi_{11}(f) + \Phi_{12}(f)) \tag{27}$$

Finally, we see that after restricting to small parallelograms E^1 obtained after width-reducing, we have

$$\frac{|f_{1x}(\tau)|}{|f_{1x}(z)|} \leq \exp(2d_0), \tag{28}$$

$$\varepsilon_{12}, \varepsilon_{21} \leq (2d_0)\exp(2d_0), \tag{29}$$

$$\varepsilon_{22} \leq (\tau_0 + 2d_0)\exp(2d_0). \tag{30}$$

So by width-reducing we make $\varepsilon_{12}, \varepsilon_{21}$ arbitrarily small, ratios of derivatives arbutrarily close to one, and ε_{22} arbitrarily close to τ_0.

Estimates for $\Phi_{ij}(f)$ allow us to get bounds for the fluctuation of derivatives of f. Namely, for two unit tangent vectors both within a small cone about an adapted x axis, $v^1 = (v_x^1, v_y^1)$ at z_1 , $v^2 = (v_x^2, v_y^2)$ at z_2 , $z_1, z_2 \in \hat{E}$ as above, we have the following result.

Lemma 1. *We have*

$$\frac{|Df_{z_1}(v^1)|}{|Df_{z_2}(v^2)|} \leq 1 + k_1\Phi_{11}\frac{|z_1 - z_2|}{\hat{\Delta}} + k_2(|v^1 - v^2|) \tag{31}$$

where $|v| = |v_x| + |v_y|$ and the distances are measured with respect to the adapted coordinate system under consideration and k_1, k_2 are positive constants.

To prove Lemma 1, we present the numerator of (31) as

$$Df_{z_2}(v^2) + Df_{z_2}(v^1 - v^2) + (Df_{z1} - Df_{z_2})(v^1)$$

and we take $|f_{1x}(z_2)(v_x^2)|$ out of the denominator. Then (28), (29), (30) give (31).

§6. Distortions of compositions

We estimate Φ_{kl} for compositions of hyperbolic maps. Let g be the map f_i and let f be the map f_j. The map $h = f_j \circ f_i = f \circ g$ is defined in $E_{ij} \equiv E_i \cap f_i^{-1}E_j$, which is a full height subpost of E_i. Let $z_0 \in E_i \cap f_i^{-1}E_j$. Earlier we estimated distortions on admissible parallelograms inscribed in the posts P_1 , but now we need to know the distortions on the preimages of such posts. As ξ is not a Markov partition, the posts P_1 can be in arbitrary positions relative to the posts P. To overcome that problem, we shall vary the initial partition ξ which we use for width reducing. Thus, instead of the fixed partition ξ we consider a family of partitions ξ_t obtained by translating each post by t along the x-axis. The distortion estimates obtained above do not depend on t.

We shall also use initial partitions with elements of two sizes. Let χ be the width of an element of ξ which is a standard rectangular post. Let ζ be a similar partition of Q into rectangular full height posts P_ζ that are so narrow that the width of any crossection spanned by $v \in K^u_\alpha$ of any preimage $F^{-k}(P_\zeta)$, $1 \le k \le n_0$, is less than $\chi/2$.

Any admissible parallelogram inscribed in $F^{-k}(P_\zeta)$ has width less than $(1+\varepsilon)(\chi/2)$ and can be put in the middle of an element P_t of an appropriate partition ξ_t. There we can use distortion estimates (20)–(30).

For $v \in K^u_\alpha,\ w \in K^s_\beta$, let us consider the (z_0, v, w)-adapted representation $\tilde{h}$ for h. Thus, $\tilde{h} = A^{-1}_{h(z_0),Dh(v),w} \circ h \circ A_{z_0,v,Dh^{-1}(w)}$. We can express $\tilde{h}$ as $\tilde{h} = \tilde{f} \circ \tilde{g}$, where $\tilde{g}$ is the $(z_0, v, Df^{-1}w)$-adapted representation of g, and $\tilde{f}$ is the $(gz_0, Dg(v), w)$-adapted representation of f. Let $P_1 = f_j^{-1}(P_\zeta)$, $P_2 = f_i^{-1}(P_1) = f_i^{-1} \circ f_j^{-1}(P_\zeta)$.

Let E_h be the minimal parallelogram centered at z_0 with edges parallel to $v, Dh^{-1}(w)$ such that

(1) the left and right sides of E_h do not meet the interior of P_2;
(2) $\text{height}(E_h) = \text{width}(s_{z_0}(P_2))$.

Then E_h is contained in an admissible parallelogram E_g inscribed in $f_i^{-1}P_t$ with the height of E_g equal to the width of $s_{z_0}(f_i^{-1}P_t)$ centered at z_0 with edges parallel to $v, Dh^{-1}w$.

Although the vectors in K^s_β can expand under a single iterate, we know that they cannot expand by more than $T_0/2$. Respectively, the image $g(E_h \cap P_2)$ is contained in an admissible parallelogram E_f inscribed in P_1, centered at $g(z_0)$, and having edges parallel to $Dg(v), Df^{-1}(w)$. Let $\delta_h = \text{width}(E_h)$, $\delta_f = \text{width}(E_f)$, $\delta_g = \text{width}(E_g)$.

As in (19), we get that lengths of horizontal crossections δ_z of E_h and the width of E_h are related by

$$\delta_z \le \delta_h \le (1+\varepsilon_h)\delta_z, \tag{32}$$

where ε_h is a small constant.

Since the map Dg restricted to E_h is close to a constant diagonal the horizontal crossections are mapped into curves close to horizontal crossections of E_f. Any horizontal crossection s_{E_f} of E_f lies in the middle of the corresponding crossection s_P of some rectangular post $P \in \xi_t$. Let us denote by $\delta_{f(\text{rltv})}$ the ratio $\frac{|s_{E_f}|}{|s_P|}$. By the choice of partitions ζ and ξ we have

$$\delta_{f(\text{rltv})} \le (1/2)(1+\varepsilon). \tag{33}$$

When we pull back, the ratios are multiplied by a factor due to distortion which is close to one because E_f is a small parallelogram. We obtain the following estimate:

$$\Phi_{jk}(g \mid E_h) \le (1+\bar{\varepsilon})\delta_{f(\text{rltv})}\Phi_{jk}(g \mid E_g). \tag{34}$$

Then we have

$$|\tilde{h}_{1x}| \ge |\tilde{f}_{1x}\tilde{g}_{1x}| \left(1 - \frac{|\tilde{g}_{2x}\tilde{f}_{1y}|}{|\tilde{g}_{1x}\tilde{f}_{1x}|}\right). \tag{35}$$

where the partial derivatives of g are computed on E_h. We define

$$\eta = \left(1 - \frac{|\tilde{g}_{2x}\tilde{f}_{1y}|}{|\tilde{g}_{1x}\tilde{f}_{1x}|}\right)^{-1} \tag{36}$$

and get, as in (25) and (26),

$$\eta < 1 + \varepsilon_{12}(\tilde{f})\delta_{f(\mathrm{rltv})}(1+\bar{\varepsilon})(\Phi_{11}(\tilde{g}) + \Phi_{12}(\tilde{g}))\exp(\Phi_{11}(\tilde{g}) + \Phi_{12}(\tilde{g})) \tag{37}$$

We also introduce

$$\gamma = \sup_{z_1, z_2 \in E_h} \frac{|\tilde{g}_{1x}(z_1)|}{|\tilde{g}_{1x}(z_2)|} \tag{38}$$

and get, as in (22) and (23),

$$\gamma \le \exp\left[(\Phi_{11}(\tilde{g}) + \Phi_{12}(\tilde{g}))\delta_{f(\mathrm{rltv})}(1+\bar{\varepsilon})\right]. \tag{39}$$

Using the chain rule for partial derivatives, we get

$$\begin{aligned}\Phi_{11}(\tilde{h}) \le \eta\big[&\Phi_{11}(\tilde{f})\gamma(1+\varepsilon_h) + 2\Phi_{12}(\tilde{f})\varepsilon_{21}(\tilde{g})\gamma + \Phi_{22}(\tilde{f})\varepsilon_{21}(\tilde{g})^2\gamma \\ &+ \Phi_{11}(\tilde{g})\delta_{f(\mathrm{rltv})}(1+\bar{\varepsilon})\big], \\ \Phi_{12}(\tilde{h}) \le \eta\big[&\Phi_{11}(\tilde{f})\varepsilon_{12}(\tilde{g})\delta_{f(\mathrm{rltv})}(1+\bar{\varepsilon})\gamma + \Phi_{12}(\tilde{f})[\varepsilon_{12}(\tilde{g})\varepsilon_{21}(\tilde{g})\gamma + \varepsilon_{22}(\tilde{g})\gamma] \\ &+ \Phi_{22}(\tilde{f})\varepsilon_{22}(\tilde{g})\varepsilon_{21}(\tilde{g})\gamma + \Phi_{12}(\tilde{g})\delta_{f(\mathrm{rltv})}(1+\bar{\varepsilon})\big].\end{aligned}$$

The quantity $\Phi_{22}(\tilde{h})$ can be expressed similarly.

Note that in the formula for $\Phi_{11}(\tilde{h})$ the value $\Phi_{11}(\tilde{f})$ is multiplied by factors that are of the form $(1 + \mathrm{const}\,\delta_{f(\mathrm{rltv})})$ and by $(1+\varepsilon_h)$ from (32).

So when the domains of $\tilde{f}$ and of $\tilde{h}$ are small, these factors are close to one. In the formula for $\Phi_{12}(\tilde{h})$, distortions of f are multiplied by small factors in each term except for ε_{22} estimated in (30) and the same is true for $\Phi_{22}(\tilde{h})$.

Taking the suprema over adapted coordinates, we get analogous estimates for $\Theta_{jk}(h)$.

In a similar way we estimate distortions for compositions $F^{n_0} = f_{i_1} \circ \ldots \circ f_{i_{n_0}}$ and taking into account that the widths of the initial rectangles P can be taken arbitrary small, we get for distortions of compositions F^{n_0}

$$\sup_{ijk} \Theta_{jk}(F^{n_0}) < d_1, \tag{40}$$

where $d_1 = d_1(d_0)$ and $\lim_{d_0 \to 0} d_1 = 0$.

By construction, we have in adapted coordinates

$$\left|\frac{F^{n_0}_{2y}}{F^{n_0}_{1x}}\right| \le \frac{1}{4}. \tag{41}$$

Now if we consider the iterates $F^{n_0 m}$ of the map F^{n_0}, we get the uniform decrease of $\Theta_{12}(F^{n_0 m}), \Theta_{22}(F^{n_0 m})$.

This implies, by induction, the following estimate similar to the one-dimensional estimate of distortion for the compositions of hyperbolic maps [5, Lemma 1].

Proposition 2. *For an arbitrary composition $F^k = f_{i_k} \circ \dots f_{i_1}$ restricted to the preimages $f_{i_k} \circ \dots f_{i_1}^{-1}(P)$ of the elements P of the partition ζ we have*

$$\Theta_{11}(F^k) \le c_1 d_1, \tag{42}$$

$$\max\left(\Theta_{12}(F^k), \Theta_{22}(F^k)\right) \le c_1 d_1 (q)^k, \tag{43}$$

where d_1 is from (40) *and $c_1 > 0$, $0 < q < 1$ are determined by the initial parameters from the hyperbolicity and distortion conditions.*

§7. Sinai local measures

According to Proposition 1, to any infinite sequence $(\dots i_{-n} \dots i_{-1} i_0)$ there corresponds an unstable manifold $W^u_{(\dots i_{-n} \dots i_{-1} i_0)}$ of full width in E_{i_0}. Moreover, $f_{i_0} W^u_{(\dots i_{-n} \dots i_{-1} i_0)}$ is a full width unstable manifold in Q. The curve $W^u_{(\dots i_{-n} \dots i_{-1} i_0)}$ as well as all its preimages have tangents inside $K^{(u)}_\alpha$.

Let us denote $\tilde{E}_{-n} = E_{i_{-n}} \cap F^{-1} E_{i_{-n+1}} \dots \cap F^{-n} E_{i_0}$, and let $\tilde{W}^u_{(-n)} = F^{-n} W^u_{(\dots i_{-n} \dots i_{-1} i_0)} \cap \tilde{E}_{-n}$. First we notice that the maps F^n from $\tilde{W}^u_{(-n)}$ onto the full width unstable manifolds in Q all have uniformly bounded distortions. For a given $k > 0$ we fix two points $z_1, z_2 \in \tilde{W}^u_{(-k)}$ and connect their images Z_1, Z_2 by a chain $Y_0 = Z_1, Y_1, \dots, Y_m = Z_2$ of $m \le N+1$ points, $N = \operatorname{card} \zeta$, such that Y_i, Y_{i+1} belong to the same elements of the partition ζ. Then z_1, z_2 are connected by the chain of preimages y_i, and for y_i, y_{i+1}, Proposition 2 and Lemma 1 give the uniform bound on the ratio of derivatives. Namely, applying Lemma 1 to the pairs $(y_0, v_1), (y_2, v_1), \dots, (y_{m-1}, v_1)$, (y_m, v_2) we get from (31)

$$\frac{|DF^k_{z_1}(v^1)|}{|DF^k_{z_2}(v^2)|} \le (1 + k_1 \Phi_{11})^{N+1} (1 + k_2(|v^1 - v^2|) = C_1. \tag{44}$$

The estimate (44) implies that for any iterate of F the ratios of distances between points are uniformly preserved up to some constant which depends on C_1. Then we fix a large k, take $\tilde{E}_{-k} = E_{i_{-k}} \cap F^{-1} E_{i_{-k+1}} \dots \cap F^{-k} E_{i_0}$, and denote by $\tilde{\delta}_{-k}$, $\delta_{i_{-k}}$ the widths of $\tilde{E}_{-k}, E_{i_{-k}}$ respectively. Let $\tilde{W}^u_{(-k)} = F^{-k} W^u_{\dots i_{-n} \dots i_{-k} \dots i_{-1} i_0} \cap \tilde{E}_{-k}$.

We want to estimate the ratios of derivatives at $z_1, z_2 \in \tilde{W}^u_{(-k)}$ and v_{z_1}, v_{z_2} tangent to $\tilde{W}^u_{(-k)}$ at z_1, z_2.

Using the hyperbolicity conditions and bounded distortion we get that the ratio of widths of $\tilde{E}_{-k}$ and $E_{i_{-k}}$ satisfies the estimate

$$\frac{\tilde{\delta}_{-k}}{\delta_{i_{-k}}} \le C_2 K_0^{-k}. \tag{45}$$

To compare unit tangent vectors v_{z_1}, v_{z_2}, we go backward to some high order preimage $\tilde{W}^{(u)}_{(-m)}$ of $\tilde{W}^{(u)}_{(-k)}$, take the preimages w_1, w_2 of z_1, z_2, and take unit horizontal vectors u_1, u_2 at w_1, w_2. Then v_{z_1}, v_{z_2} will be within the respective const $K_0^{-2(m-k)}$ cones around $DF^{m-k} u_1, DF^{m-k} u_2$. These last vectors have components $F^{m-k}_{1x}(w_j), F^{m-k}_{2x}(w_j)$, $j = 1, 2$. We can assume that the points w_1, w_2

belong to an admissible parallelogram $\hat{E}_{(m)}$ that is mapped by F^m onto a full width curvilinear subrectangle of an element of the initial partition ζ. Since the elements of ζ have fixed widths, by using again the bounded distortion property we get

$$\frac{|w_1 - w_2|}{|\hat{E}_{(m)}|} \leq C_3 K_0^{-k}. \tag{46}$$

We can take w_1 as the origin of an adapted coordinate system for $\hat{E}_{(m)}$ so that $F_{2x}^{m-k}(w_1) = 0$. Then

$$\left| \frac{F_{2x}^{m-k}(w_2)}{F_{1x}^{m-k}(w_2)} \right| \leq \frac{C_3}{K_0^k},$$

and, respectively,

$$|v_{z_1} - v_{z_2}| \leq \frac{C_3}{K_0^k}. \tag{47}$$

Using (45), (46), (47), and (31) we get

$$\frac{|DF_{z_1}(v^1)|}{|DF_{z_2}(v^2)|} \leq \exp\left(\frac{C_4}{K_0^k}\right) \tag{48}$$

Let us denote by $D^u F(z)$ the derivative of F along the unstable manifold $W^u(z)$ at z. Then (48) implies that for any two points $z_1, z_2 \in \tilde{W}^u_{(-n)}$ the following limit exists:

$$\lim_{n\to\infty} \frac{\prod_{s=1}^n D^u F(F^{-s} z_2)}{\prod_{s=1}^n D^u F(F^{-s} z_1)}. \tag{49}$$

Considering $\xi(z_1, z_2)$ obtained in the preceding limits as functions of z_1, we get, up to constants, densities of special measures on the unstable manifolds. The family of these measures defined on local unstable manifolds is invariant in the following sense. If we have two Lebesgue measurable subsets of a local unstable manifold $W^u(z)$, and their images are subsets of the local unstable manifold $W^u(F(z))$, then the ratios of local measures of these subsets are preserved, see for example [15, Lecture 16]. We call these measures *Sinai local measures* or just *local measures.* Let $\rho_{W^u(z)}$ denote the normalized Sinai local measure on $W^u(z)$.

§8. Construction of an SRB measure

A global SRB measure is obtained by averaging the iterates of a local measure on an arbitrary unstable manifold. Let $W_0 = W^u_{(\ldots i_{-n} \ldots i_{-1} i_0)}$ and let μ_0 be the local measure on W_0. Then the measures $\mu_n = F_*^n \mu_0$ are defined on $F^n(W_0)$ by $\mu_n(A) = \mu_0(F^{-n}(A))$. For any E_i we fix full height curve transversal to unstable manifolds, for example, some stable manifold γ_i. Every iterate of W_0 is a union of unstable manifolds that are full width in Q. Each of these manifolds intersects γ_i at a unique point z. The piece of that manifold cut by E_i is the local unstable manifold denoted by $W_i(z)$. Then on each E_i the sequence of discrete factor measures m_{in} is defined by assigning to $z \in \gamma_i$ the measure $\mu_n(W_i(z))$.

Then we consider sequences of measures

$$\lambda_n = \frac{1}{n} \sum_{k=0}^{n-1} \mu_k$$

and

$$\lambda_n^i = \frac{1}{n} \sum_{k=0}^{n-1} m_{ik}$$

for each i and choose a subsequence n_k such that the corresponding measures weakly converge to

$$\lambda = \lim_{k\to\infty} \lambda_{n_k}, \quad \lambda^i = \lim_{k\to\infty} \lambda^i_{n_k}$$

Then λ is an F-invariant measure. The following lemma is a modification of [15, Lecture 17, Theorem 5].

Lemma 2. *λ is a Gibbsian measure , i.e., the conditional measures that λ generates on the local unstable manifolds coincide with the local measures.*

Outline of the proof. Let

$$z_{(\dots i_{-n}\dots i_{-1} i_0)} = W^u_{(\dots i_{-n}\dots i_{-1} i_0)} \cap \gamma_{i_0}, \qquad X_{i_0} = \bigcup z_{(\dots i_{-n}\dots i_{-1} i_0)}$$

Let $\varphi(x)$ be a continuous function with the support inside E_{i_0}. The uniformly bounded distortions imply that

$$\psi(z) = \int \varphi(y)\, d\rho_{W_{i_0}(z)}(y)$$

is a continuous function of $z \in X_{i_0}$.

Since we are dealing with an infinite number of E_i, the union of the strips S_i, as well as the intersection of that union with γ_{i_0}, are not closed sets, and the same is true for $\bigcup_i F^n(E_i)$ for any n. Respectively, X_{i_0} is not a closed subset of γ_{i_0} and $\psi(z)$ is not continuous on the closure $\bar{X}_{i_0}$. However, since the measure of the union of E_i with $i > n$ tends to zero as n tends to ∞, the bounded distortion conditions imply that $Y_{i_0} = \bar{X}_{i_0} \setminus X_{i_0}$ has uniform measure zero with respect to all μ_k in the following sense: for every $\varepsilon > 0$ there exists an open cover U_{ε, i_0} of Y_{i_0} such that $\mu_k(U_\varepsilon) \le \varepsilon$ for all k.

With this modification the proof of Lemma 2 is similar to the proof of Theorem 5 in [15, Lecture 17].

Remark. When constructing λ we implicitly use the following result.

For any smooth curve $Y = y(x)$ transversal to the stable foliation with angles uniformly bounded away from zero, $Y \cap \left(\bigcup_i E_i\right)$ has full Lebesgue measure in Y.

In order to get this property we notice that G4 implies that the measure of $Y \cap \left(\bigcup_i E_i\right)$ depends continuously on the curve belonging to some smooth foliation. Then the property follows from G2 using the Fubini theorem.

§9. Absolute continuity and corollaries

The key property of SRB measures — absolute continuity of the projection along the stable foliation — is proved similarly to [3,9], but again requires a modification because of the infinite number of E_i in the initial partition.

Let us take two smooth curves W_1, W_2 with tangent lines within K^u_α in the same E_i and iterate them forward long enough. After n iterates the distance between the pieces of images W_{n1}, W_{n2} that had the same itinerary $[i_1, i_2, \dots i_n]$ will be less than $\text{const}\, K_0^{-n}$. When we prove the absolute continuity property we compare the Lebesgue measures of preimages $W_{1,[i_1,i_2,\dots i_n]} = F^{-n}(W_{n1}) \in W_1$ and $W_{2,[i_1,i_2,\dots i_n]} = F^{-n}(W_{n2}) \in W_2$. By the uniformly bounded distortion, we have

$$\text{mes}(W_{k,[i_1,i_2,\dots i_n]}) \in [c_1, c_2]\ \text{mes}(W_{nk})\ (\prod_{s=1}^{n} D^u F^s(x_k))^{-1}, \quad k = 1, 2 \tag{50}$$

where $x_1 \in W_{1,[i_1,i_2,\dots i_n]}$, $x_2 \in W_{2,[i_1,i_2,\dots i_n]}$ belong to the same stable manifold.

Formula (50) implies that the ratios of the measures of $W_{k,[i_1,i_2,\dots i_n]}$ are expressed up to a uniform constant by

$$\frac{\prod_{s=1}^{n} D^u F(F^s x_1)}{\prod_{s=1}^{n} D^u F(F^s x_2)} \frac{\text{mes}\, W_{n2}}{\text{mes}\, W_{n1}} \tag{51}$$

However, contrary to the classical case, the ratios of unstable Jacobians at $F^s(x_k)$ are close to 1 only if the distances between $F^s(x_k)$ are small compared to the widths of those E_{i_s} that contain $F^s(x_k)$. Similarly, the ratios of the measures of W_{nk} are close to one under the same condition.

Here we use the geometric condition G4. According to this condition, if we avoid at step s those E_i that have widths less than $K_1^s K_0^{-s}$, we delete at this step the subset of W_k of the relative measure less than the corresponding term of a converging series. So we get at the limit a Cantor set C^1_k of positive measure in W_k, $k = 1, 2$. On this set the ratios (51) are uniformly bounded between two positive constants r_{11}, r_{12}. If π is the projection along stable manifolds, then it follows from [3] that for any subset of positive Lebesgue measure $M_1 \subset C^1_1$ and for $M_2 = \pi M_1 \subset C^1_2$ the ratios of Lebesgue measures of M_1 and M_2 are uniformly bounded.

The remaining points in W_k belong to the preimages of E_i with large i. For these sets we take several extra forward iterates and repeat the previous arguments. We obtain Cantor sets C^2_k of positive measure, disjoint from C^1_k, with uniformly bounded ratios (51), but with different constants r_{21}, r_{22}. Repeating this construction, we get

$$W_k = \bigcap_{j=1}^{\infty} C^j_k \mod 0,$$

where C^j_k are disjoint Cantor subsets of W_k of positive measure with uniform estimates r_{j1}, r_{j2} of ratios (51) on C^j_k, $k = 1, 2$. That proves absolute continuity of π.

The ergodicity of λ follows from the absolute continuity of π and from the "Bernoulli" topological structure of the map F (see, for example, [3]). The same arguments show that any two measures constructed by the above averaging starting from different unstable manifolds, coincide. Finally, the absolute continuity of π implies that for Lebesgue a.e. point in Q and any continuous function φ, (1) holds with $\gamma = \mu$.

§10. Further ergodic properties of (F, λ)

10.1. Let us denote by α the original partition of the attractor Λ

$$\alpha = (\Lambda \cap E_i)_{i=1}^{\infty}.$$

Let

$$\eta = \bigvee_{-\infty}^{0} F^i \alpha.$$

Up to a set of measure zero, the elements of η coincide with the intersections of full height stable manifolds with the attractor. The partition η satisfies the following properties:

$$F\eta \succ \eta, \qquad \vee_n F^n \eta = \varepsilon, \qquad \bigwedge_n F^n \eta = \nu.$$

So we get from [16] that (F, λ) is a K-system.

Proposition 3. *The map (F, λ) is Bernoulli.*

The following *weak Markov property* was introduced in [11]. It was used to prove the Bernoulli property of Anosov flows (see [4,11]).

Let β be any partition,

$$\beta_k^l = \bigvee_{k \le i \le l} F^i \beta.$$

We say that β is weak Markov (WM) if for any $\varepsilon \geq 0$ there exists $N = N(\varepsilon)$, a set $P = P(\varepsilon)$ of atoms of β_0^{∞}, $\lambda(P) \geq 1 - \varepsilon$, and a set $M = M(\varepsilon)$ of atoms of $\beta_{-\infty}^N$, $\lambda(M) \geq 1 - \varepsilon$, such that if $\bar{x}, \bar{y} \in P \cap x_0^N$, $x_0^N \in \beta_0^N$, then for any set $A \subset M$ of atoms of $\beta_{-\infty}^N$ one has

$$\left| \frac{\lambda(A|\bar{x})}{\lambda(A|\bar{y})} - 1 \right| \leq \varepsilon. \tag{52}$$

We take for β our original partition α of the attractor. Then the corresponding α_0^{∞} is the partition into unstable manifolds $W^u_{\dots i_{-n} \dots i_0}$, elements of α_0^N are $S_{i_{-n} \dots i_0} = F^n E_{i_{-n}} \cap F^{n-1} E_{i_{-(n-1)}} \cap \dots \cap F E_{i_{-1}} \cap E_{i_0}$, and $\alpha_{-\infty}^N$ is the partition into pieces of stable manifolds within α_0^N.

In order to get P, we first delete an open cover U as in Lemma 2. Then $\bigcup_{i \leq i_0} E_i \setminus U$ is a closed set and we can cover it by a finite number of strips $S_{i_{-n} \dots i_0}$, $n \leq N$, such that for any two points belonging to the same stable manifold $z_1, z_2 \in W_0^s \cap S_{i_{-n} \dots i_0}$ the ratio of densities of local measures (49) differs from 1 by less that ε. The union of the unstable manifolds $W^u_{\dots i_{-n} \dots i_0}$ that belong to the above strips $S_{i_{-n} \dots i_0}$, $n \leq N$, constitutes P. For M we take the union of stable leaves outside of $\bigcup_{i \geq i_0} E_i$ and within $S_{i_{-n} \dots i_0}$. Then, the WM property is satisfied.

Using the WM property for α we get from [1, Proposition 2.2] that every partition $\zeta_k = \{E_1, E_2, \dots, E_k, \bigcup_{i\geq k} E_i\}$ is weak Bernoulli. As $k \to \infty$, the partitions ζ_k converge to the generating partition α and we get Bernoulli property for (F, λ), see [8].

10.2. Entropy formula. By construction, the measures of E_i satisfy

$$c_1\delta_{i,\min} < \lambda(E_i) < c_2\delta_{i,\max}. \tag{53}$$

Using (4), (56) we get that the entropy of the generating partition α is finite and

$$h_\lambda(F) \leq H(\alpha) < \infty. \tag{54}$$

Similarly,

$$(c_1\delta_{i,\min})^{-1} > |D^u F \mid E_i| > (c_2\delta_{i,\max})^{-1}$$

implies

$$\int \log |D^u F| d\lambda < \infty. \tag{55}$$

Let $\xi = \bigvee_{k=0}^{\infty} F^k\alpha$. The elements of ξ coincide with the local unstable manifolds $W_i(z)$. Respectively, ξ is an increasing partition with respect to F^{-1} and it has the same K-properties as the above η.

Since α is generating and $\alpha \prec \xi$, we get, using the properties of the entropy, that

$$h_\lambda(F) = H(F^{-1}\alpha|\xi) = H(F^{-1}\xi|\xi). \tag{56}$$

Then the arguments of Theorem 5.1 from [16] (proved for systems with smooth invariant measure) or similar arguments from Section 4 of [7] (where the smoothness of invariant measure is not assumed) give

$$H(F^{-1}\xi|\xi) = \int \log |D^u F| d\lambda, \tag{57}$$

which proves the entropy formula.

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References

1. R. Adler, *Afterword to R. Bowen, Invariant measures for Markov maps of the interval*, Comm. Math. Phys. **69** (1979), no. 1, 1–17.
2. V. M. Alekseev, *Quasi-random dynamical systems*, Math. of the USSR, Sbornik **5** (1968), no. 1, 73–128.
3. D. V. Anosov and Ya. G. Sinai, *Some smooth ergodic systems*, Russian Math. Surveys **22** (1967), 103–167.
4. L. Bunimovich, *On a class of special flows*, Math. USSR-Izv. **8** (1974), 219–232.
5. M. V. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. **81** (1981), 39–88.

6. M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets*, Proceedings of Symposia in Pure Mathematics **14** (1970), 133–164.
7. F. Ledrappier and J.-M. Strelcyn, *A proof of the estimation from below in Pesin's entropy formula*, Ergodic Theory Dynamical Systems **2** (1982), 203–219.
8. D. Ornstein, *Ergodic theory, randomness, and dynamical systems*, Yale University Press, New Haven, CT, 1975.
9. C. Pugh and M. Shub, *Ergodic attractors*, Trans. Amer. Math. Soc. **312** (1989), no. 1, 1–54.
10. Ya. B. Pesin, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergodic Theory Dynamical Systems **12** (1992), 123-151.
11. M. Ratner, *Anosov flows with gibbs measures are also bernoullian*, Israel J. Math. **17** (1974), 380–391.
12. M. Rychlik, *Mesures invariantes et principe variationel pour les applications de Lozi*, Compte Rend. Acad. Sci. Paris **296** (1983), 19–22.
13. E. A. Sataev, *Invariant measures for hyperbolic maps with singularities*, Russian Math. Surveys **47** (1992), 191–251.
14. S. Smale, *Diffeomorphisms with many periodic points*, Differential and combinatorial topology, Princeton University Press, Princeton, NJ, 1965, pp. 63-80.
15. Ya. G. Sinai, *Topics in ergodic theory*, Princeton University Press, Princeton, NJ, 1994.
16. ———, *Classical systems with countable Lebesgue spectrum.* II, Izv. Akad. Nauk. SSSR , Ser. Math. **30** (1966), 15 - 68; Amer. Math. Soc. Transl. **68** (1968), 34-88.
17. P. Walters, *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc. **236** (1978), 121–153.