Cone-fields, Domination, and Hyperbolicity

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1 Introduction

We obtain new conditions for dominated and hyperbolic splittings on compact invariant sets for a diffeomorphism in terms of its induced action on a cone field and its complement. The results are applied to give simple new proofs of well-known theorems for hyperbolicity of the set of bounded orbits in real and complex Henon mappings. In this latter case, much more complete information can be given making use of complex methods. See, for instance, [1]

Let M be a compact C^∞ Riemannian manifold, let $f \in \mathcal{D}^1(M)$, let Λ be a compact $f-$invariant set.

The notion of a dominated splitting arose in the work of Mane on the stability conjecture [4] and has been considered in several recent works. See for instance [9].

The usual definition is the following.

Definition. A *dominated splitting* on Λ is a splitting $T_xM = E_{1x} \oplus E_{2x}$ for each $x \in \Lambda$ such that there are constants $C > 0$ and $\lambda > 1$ such that

$$| Df^n_x \mid E_{2x} || Df^{-n}_{f^n x} \mid E_{1 f^n x} | \leq C\lambda^{-n} \text{ for } n \geq 0, \tag{1}$$

$$Df(E_{ix}) = E_{i,fx} \text{ for } i = 1, 2, \text{ and} \tag{2}$$

$$\text{the maps } x \to E_{ix} \text{ are continuous.} \tag{3}$$

Let us give an interpretation of these estimates.

For a linear map $L : \mathbf{R}^n \to \mathbf{R}^n$, and a subset $E \subset \mathbf{R}^n$ with $E \neq \{0\}$, set

$$|\ L \mid E\ | = \sup_{v \neq 0, v \in E} \frac{|\ Lv\ |}{|\ v\ |},$$

and

$$m(L \mid E) = \inf_{v \neq 0, v \in E} \frac{|\ Lv\ |}{|\ v\ |}.$$

If L is an isomorphism, and E is a subspace, then

$$m(L \mid E) = |\ L^{-1} \mid L(E)\ |^{-1}.$$

We call $m(L \mid E)$ the *minimal expansion* of L on E, and $m(L^{-1} \mid L(E))$ the *minimal co-expansion* of L on E.

We may write condition (1) in several equivalent ways. For $n \geq 0$,

$$|\ Df_x^n \mid E_{2x}\ | m(Df_x^n \mid E_{1x})^{-1} \leq C\lambda^{-n} \tag{4}$$

$$|\ Df_x^n \mid E_{2x}\ | \leq C\lambda^{-n} m(Df_x^n \mid E_{1x}) \tag{5}$$

$$m(Df_{f^nx}^{-n} \mid E_{2,f^nx})^{-1} m(Df_x^n \mid E_{1x})^{-1} \leq C\lambda^{-n} \tag{6}$$

$$m(Df_x^n \mid E_{1x}) m(Df_{f^nx}^{-n} \mid E_{2,f^nx}) \geq C^{-1}\lambda^n. \tag{7}$$

The last formulation (7) can be simply expressed as follows: the minimal expansion of Df_x^n on E_{1x} times the minimal co-expansion of Df_x^n on E_{2x} grows exponentially with n.

Remark:

1. Dominated splittings need not be unique. For example, consider a toral automorphism with a center and expanding subspace, or, more generally, any partially hyperbolic set.

2. We will show later that conditions (1) and (2) imply condition (3).

3. If $dim\ E_{1x}$ is constant, then it follows from Theorem 1.2 below that the dominated splitting is unique in the following sense. If $T_x = F_{1x} \oplus F_{2x}$ is another splitting of T_xM such that, for all $x \in \Lambda$,

$$m(Df_x^n \mid F_{1x}) m(Df_{f^n x}^{-n} \mid F_{2,f^n x}) \geq C^{-1}\lambda^n$$

and

$$dim\ E_{1x} = dim\ F_{1x},$$

then $E_{1x} = F_{1x}$ and $E_{2x} = F_{2x}$ for all $x \in \Lambda$.

In this note we are interested in establishing simple conditions which guarantee that dominated splittings exist. Our conditions will be in terms of the action of Df on certain cone fields over the set Λ. In many examples, these conditions are easy to verify. They also lead to an elegant and useful method for proving that invariant sets are uniformly hyperbolic. This has many applications in bifurcation theory and other areas. As a particular application, we give a simple new proof of hyperbolicity of the set of bounded orbits for certain Henon mappings in both the real and complex situations.

There are many known results containing su□cient conditions for hyperbolicity. See, for instance, the papers [3], [7], and the books [6], [8], and [10]. Our techniques here are analogous to some which appear in [8] and [10] for two dimensional systems involving invariant cone fields. We replace the invariance conditions in [8] and [10] by expansion conditions on the cones and their complements. This gives stronger results and fits in easily in the more general context of domination. The techniques in this section are simple, and, although we expect that they are known to experts, we have found no reference with analogous results in the generality presented here.

Let us begin be recasting the domination conditions in a way which gives uniqueness and continuity.

Let $E \subset \mathbf{R}^n$ be a proper subspace; i.e, $0 < dim\ E < n$. Let F be a complementary subspace; i.e., $\mathbf{R}^n = E \oplus F$.

The *standard unit cone* determined by the subspaces E and F is the set

$$K_1(E, F) = \{v = (v_1, v_2) : v_1 \in E, v_2 \in F, \text{ and } \mid v_2 \mid \leq \mid v_1 \mid\}.$$

A *cone* in $\mathbf{R}^n$ with core E, denoted $\mathcal{C}(E)$, is the image $T(K_1(E, F))$ where $T : \mathbf{R}^n \to \mathbf{R}^n$ is a linear automorphism such that $T(E) = E$. By a **cone** $\mathcal{C}$ in $\mathbf{R}^n$ we mean a set $\mathcal{C}(E)$ for some proper subspace E of $\mathbf{R}^n$.

Now consider a diffeomorphism f on the manifold M and compact invariant set Λ as above.

A *cone field* $\mathcal{C} = \{\mathcal{C}_x\}$ on Λ is a collection of cones $\mathcal{C}_x \subset T_xM$ for $x \in M$. We say that the cone field $\mathcal{C}_x$ has *constant orbit core dimension* on Λ if $dim\ E_x = dim\ E_{fx}$ for all $x \in \Lambda$ where E_x, E_{fx} are the cores of $\mathcal{C}_x, \mathcal{C}_{fx}$, respectively. Note that there is no a priori continuity assumption on $x \to \mathcal{C}_x$.

Given such a cone field $\mathcal{C} = \{\mathcal{C}_x\}_{x \in M}$, let

$$m_{\mathcal{C},x} = m_{\mathcal{C},x}(f) = \inf_{v \in \mathcal{C}_x \setminus \{0\}} \frac{\mid Df_x(v) \mid}{\mid v \mid},$$

and

$$m'_{\mathcal{C},x} = m'_{\mathcal{C},x}(f) = \inf_{v \notin \mathcal{C}_{fx}} \frac{\mid Df_{fx}^{-1}(v) \mid}{\mid v \mid}.$$

We call $m_{\mathcal{C},x}$ the *minimal expansion* of f on $\mathcal{C}_x$ or of Df on $\mathcal{C}_x$, and we call $m'_{\mathcal{C},x}$ the *minimal co-expansion* of f on $\mathcal{C}_x$ or of Df on $\mathcal{C}_x$.

We define the *domination coefficient* of f on $\mathcal{C}$ to be

$$m_d(\mathcal{C}) = m_d(\mathcal{C}, f) = \inf_{x \in \Lambda} m_{\mathcal{C},x} \cdot m'_{\mathcal{C},x}. \tag{8}$$

We say that f is *dominating* on $\mathcal{C}$ over Λ (or $\mathcal{C}$ is a *dominating cone field* on Λ) if $\mathcal{C}$ has constant orbit core dimension and $m_d(\mathcal{C}) > 1$.

Remark. This condition depends on the choice of Riemannian metric on M. An analogous weaker condition, which is independent of the Riemannian metric, is that $m_d(\mathcal{C}, f^{n_0}) > 1$ for some $n_0 \geq 1$. It is easy to see that the results below also hold under this weaker condition.

We have the following simple lemma.

Lemma 1.1 *Suppose that f is dominating on $\mathcal{C}$ over Λ. Then, $\mathcal{C}$ is an $f-$invariant cone field. That is, for $x \in \Lambda$, we have*

$$Df_x(\mathcal{C}_x) \subset \mathcal{C}_{fx}. \tag{9}$$

Proof.

Let $v \in \mathcal{C}_x \setminus \{0\}$, and let $w = Df_x(v)$.

Then,

$$\mid Df_x(v) \mid \geq m_{\mathcal{C},x}(f) \mid v \mid.$$

We want to show that $w \in \mathcal{C}_{fx}$. We actually show that $w \in interior(\mathcal{C}_{fx})$. Indeed, if this were not true, then

$$| Df_{fx}^{-1}(w) | \geq m'_{\mathcal{C},x} | w |.$$

This would give

$$\begin{aligned} | v | &= | Df_{fx}^{-1}(w) | \geq m'_{\mathcal{C},x} | w | \\ &\geq m'_{\mathcal{C},x} \cdot m_{\mathcal{C},x} | v | \\ &> | v | \end{aligned}$$

which is a contradiction. QED.

We say that f is *strongly dominating* on $\mathcal{C}$ over Λ if $\mathcal{C}$ has constant orbit core dimension and

$$\left(\inf_{x \in \Lambda} m_{\mathcal{C},x} \right) \cdot \left(\inf_{x \in \Lambda} m'_{\mathcal{C},x} \right) > 1. \tag{10}$$

Theorem 1.2 *Suppose that f is dominating on $\mathcal{C}$ over Λ. Then there is a unique $Df-$invariant splitting $T_\Lambda M = E_1 \oplus E_2$ such that for all $x \in \Lambda$, we have $E_{1x} \subset \mathcal{C}_x$ and $E_{2x} \subset T_x M \setminus \mathcal{C}_x$.*

Further, if f is strongly dominating on $\mathcal{C}$ over Λ, then the functions $x \to E_{1x},\ x \to E_{2x}$ are continuous in x.

Proof.
Existence:
Let f be dominating on $\mathcal{C}$.
Let $\mathcal{C}_x^c = T_x M \setminus \mathcal{C}_x$.
Set

$$\mathcal{C}_x^+ = \bigcap_{n \geq 0} Df_{f^{-n}x}^n(\mathcal{C}_{f^{-n}x}) \subset \mathcal{C}_x,$$

$$\mathcal{C}_x^- = \bigcap_{n \geq 0} Df_{f^n x}^{-n}(\mathcal{C}_{f^n x}^c) = \bigcap_{n \geq 0} Df_{f^n x}^{-n}(Closure(\mathcal{C}_{f^n x}^c)).$$

Let $E_{f^{-n}x}$ be the core of $\mathcal{C}_{f^{-n}x}$.

Then, $E'_{n,x} = Df^n_{f^{-n}x}(E_{f^{-n}x})$ is a linear subspace which is contained in $\mathcal{C}_x$, and they all have the same dimension.

Let

$$E_x = \lim_{n_i \to \infty} E'_{n_i,x}$$

in the Grassmann sense for some sequence $n_i \to \infty$.

Similarly, let $F_{f^n x}$ be a subspace contained in $\mathcal{C}^c_{f^n x}$ complementary to the core of $\mathcal{C}_{f^n x}$, and let $F'_{n,x} = Df^{-n}_{f^n x}(F_{f^n x})$.

We may assume that the sequence n_i is chosen such that $F_x = \lim_{n_i \to \infty} F'_{n_i,x}$.

Then, $E_x \subset \mathcal{C}_x,\ F_x \subset interior(\mathcal{C}^c_x)$, and $dim\ E_x + dim\ F_x = dim\ T_x M$. Also, since $E_x \bigcap F_x = \{0\}$, we have $T_x M = E_x \oplus F_x$.

We claim:

$$\mathcal{C}^-_x = F_x, \tag{11}$$

and

$$\mathcal{C}^+_x = E_x. \tag{12}$$

For $x \in \Lambda$, let $x_i = f^i x$, and set

$$m_{x_i} = m_{\mathcal{C},x_i}(f),\ m'_{s_i} = m'_{\mathcal{C},x_i}(f).$$

For $n \geq 1$ set

$$M_n = \prod_{j=0}^{n-1} m_{x_j},\ M'_n = \left(\prod_{j=0}^{n-1} m'_{x_j} \right)^{-1}.$$

From condition (8) we may choose $\rho > 1$ so that

$$m_{x_j} \cdot m'_{x_j} > \rho\ \forall j, \quad \text{which of course gives } \frac{M_n}{M'_n} > \rho^n. \tag{13}$$

Observe that for $v \in \mathcal{C}^-_x \setminus \{0\}$, we have

$$Df^n_x(v) \in \mathcal{C}^c_{x_n} \setminus \{0\}, \tag{14}$$

and

$$\mid Df^n_x(v) \mid \leq M'_n \mid v \mid, \tag{15}$$

while, for $v \in \mathcal{C}_x^+ \setminus \{0\}$, we have

$$Df_x^n(v) \in \mathcal{C}_{f^n x} \setminus \{0\}, \tag{16}$$

and

$$| Df_x^n(v) | \geq M_n | v |. \tag{17}$$

Let us now prove (11) and (12).

We know that $F_x \subset \mathcal{C}_x^-$ and $E_x \subset \mathcal{C}_x^+$. Suppose there is a $v \in \mathcal{C}_x^- \setminus F_x$. Write $v = v_1 + v_2$ with $v_1 \in E_x, \ v_2 \in F_x$. Of course, $v_1 \neq 0$.

From (15) and (17), we have

$$\begin{aligned} M_n' | v | &\geq | Df_x^n(v) | \\ &= | Df_x^n(v_1) + Df_x^n(v_2) | \\ &\geq M_n | v_1 | - M_n' | v_2 | \end{aligned}$$

or, by (13),

$$\frac{| v | + | v_2 |}{| v_1 |} \geq \frac{M_n}{M_n'} \geq \rho^n.$$

Since the left side of this equation is bounded, this is a contradiction, proving (11).

Replacing f by f^{-1} gives $\mathcal{C}_x^+ = E_x$, so (12) is also proved.

Now, we set $E_{1x} = E_x, \ E_{2x} = F_x$ to give the splitting required in Theorem 1.2. It is clearly Df-invariant since both $\mathcal{C}_x^+$ and $\mathcal{C}_x^-$ are.

Uniqueness:

If $\bar{E}_{1x} \subset \mathcal{C}_x, \bar{E}_{2x} \subset \mathcal{C}_x^c$ for all $x \in \Lambda$, and the $\bar{E}_{ix}$ are $Df-$invariant, then we clearly have $\bar{E}_{1x} \subset \mathcal{C}_x^+$ and $\bar{E}_{2x} \subseteq \mathcal{C}_x^-$. Hence, $\bar{E}_{1x} \subset E_x$ and $\bar{E}_{2x} \subset F_x$. If $dim \ \bar{E}_{1x} < dim \ E_x$, then $dim \ \bar{E}_{1x} \oplus \bar{E}_{2x} < dim \ M$, contradicting the assumption that we have a splitting of T_xM. Hence, $E_x = \bar{E}_{1x}$. Similarly, $F_x = \bar{E}_{2x}$.

Continuity:

Assume that f is strongly dominating over $\mathcal{C}$, and let $E \oplus F = \{E_x \oplus F_x\}$ be the induced $Df-$ invariant splitting over Λ with $E_x \subset \mathcal{C}_x, \ F_x \subset \mathcal{C}_x^c$.

Suppose that $z \to F_z$ is not continuous at some x. We we may choose a sequence y_i converging to x and subspaces $\tilde{F}_x, \tilde{E}_x$ of T_xM such that

$$dim\ F_{y_i} \text{ is constant,} \tag{18}$$

$$F_{y_i} \to \tilde{F}_x \text{ and } E_{y_i} \to \tilde{E}_x \text{ in the Grassmann sense ,} \tag{19}$$

and

$$\tilde{F}_x \neq F_x. \tag{20}$$

Case 1: There is a unit vector $v \in \tilde{F}_x \setminus F_x$.

Choose unit vectors $v_i \in F_{y_i}$ such that $v_i \to v$ as $i \to \infty$, and write $v = v_1 + v_2$ and $v_1 \in E_x,\ v_2 \in F_x$. Since v is not in F_x we have $v_1 \neq 0$.

For $n \geq 0$, let

$$M_n = \prod_{j=0}^{n-1} m_{f^j x},\ M'_n = \prod_{j=0}^{n-1} m'_{f^j x},\ M'_{n,i} = \prod_{j=0}^{n-1} m'_{f^j y_i}.$$

Now, for each $n > 0$, we have

$$\begin{aligned} \mid Df_x^n(v) \mid &= \lim_i \mid Df_{y_i}^n(v_i) \mid \\ &\leq \liminf_{i\to\infty} \frac{\mid v_i \mid}{M'_{n,i}} \\ &= \liminf_{i\to\infty} \frac{1}{M'_{n,i}}, \end{aligned}$$

and

$$\begin{aligned} \mid Df_x^n(v) \mid &= \mid Df_x^n(v_1) + Df_x^n(v_2) \mid \\ &\geq \mid Df_x^n(v_1) \mid - \mid Df_x^n(v_2) \mid \\ &\geq M_n \mid v_1 \mid - \frac{1}{\mid M'_n \mid} \mid v_2 \mid. \end{aligned}$$

Letting

$$m_0 = \inf_{x\in\Lambda} m_{\mathcal{C}}(f),\ m'_0 = \inf_{x\in\Lambda} m'_{\mathcal{C}}(f),$$

we have

$$M'_{n,i} \geq (m'_0)^n, \; M_n \geq m_0^n, \text{ and } M'_n \geq (m'_0)^n.$$

Then, the above inequalities give

$$\liminf_{i\to\infty} \frac{1}{M'_{n,i}} \geq M_n \mid v_1 \mid - \frac{\mid v_2 \mid}{M'_n},$$

or,

$$\frac{1}{(m'_0)^n} + \frac{\mid v_2 \mid}{(m'_0)^n} \geq m_0^n \mid v_1 \mid,$$

or,

$$\frac{1 + \mid v_2 \mid}{\mid v_1 \mid} \geq (m'_0 m_0)^n.$$

For n large this is a contradiction.

Case 2: There is a unit vector $v \in F_x \setminus \tilde{F}_x$.

Write the unit vector v as $v = v_{1i} + v_{2i}$ with $v_{1i} \in E_{y_{n_i}}$ and $v_{2i} \in F_{y_{n_i}}$. The angles between the subspaces E_{y_i} and F_{y_i} are bounded below, so, for large i, the norms $\mid v_{1i} \mid$ are bounded below. Hence, the quantities

$$\frac{1 + \mid v_{2i} \mid}{\mid v_{1i} \mid}$$

are bounded above. On the other hand, as in Case 1, we can show that they are no smaller than $(m'_0 m_0)^n$ for each n. This contradiction proves that $z \to F_z$ is continuous.

Similarly, we obtain that $z \to E_x$ is also continuous.

This proves Theorem 1.2. QED.

Proposition 1.3 *A sufficient condition for f to have a dominated splitting over Λ is that there is an integer $n_0 > 0$ such that f^{n_0} has a strongly dominated cone field $\mathcal{C}$ over Λ.*

Proof.

By Theorem 1.2, we get a continuous $Df^{n_0}-$invariant splitting $T_xM = E_1 \oplus E_2$ and a $\tau > 1$ such that, for all $x \in \Lambda$,

$$m(Df_x^{n_0} \mid E_{1x}) \cdot m(Df_{f^{n_0}x}^{-n_0} \mid E_{2,f_0^n x}) > \tau.$$

This implies that for each $k \geq 0$,

$$m(Df_x^{kn_0} \mid E_{1x}) \cdot m(Df_{f^{kn_0}x}^{-kn_0} \mid E_{2,f^{kn_0}x}) > \tau^k.$$

Let $n \geq 0$ and write $n = kn_0 + r$ with $0 \leq r < n_0$.
Then,

$$m(Df_x^n \mid E_{1x}) \geq m(Df^{kn_0} \mid E_{1x}) \cdot \inf_{0 \leq j < n_0} m(Df_{f^{kn_0}x}^j \mid E_{1,f^{kn_0}x}) \tag{21}$$

and

$$m(Df_{f^n x}^{-n} \mid E_{2,f^n x}) \geq m(Df_{f^{kn_0}x}^{-kn_0} \mid E_{2,f^{kn_0}x}) \cdot \inf_{0 \leq j < n_0} m(Df_{f^j x}^{-j} \mid E_{f^j x}), \tag{22}$$

so, we get

$$m(Df_x^n \mid E_{1x}) \cdot m(Df_{f^n x}^{-n} \mid E_{2,f^n x}) \geq C\tau^k \geq C_1 \lambda^{n_0 k + r} = C_1 \lambda^n$$

for some constants $C, C_1 > 0$ and $\lambda > 1$. This proves the Proposition. QED.

Let us now apply these results to give a useful condition for uniform hyperbolicity involving cone fields.

One standard definition of uniform hyperbolicity is the following.

Let $f : M \to M$ be a C^1 diffeomorphism of a Riemannian manifold M, and let Λ be a compact invariant subset of M; i.e, $f(\Lambda) = \Lambda$. We say that Λ is a *uniformly hyperbolic* set for f, if there are constants $C > 0, \lambda > 1$ and a continuous splitting $T_x M = E_x^u \oplus E_x^s$ for each $x \in \Lambda$ such that

1. $Df_x(E_x^s) = E_{fx}^s$ and $Df_x(E_x^u) = E_{fx}^u$,
2. for $v \in E_x^u$ and $n \geq 0$, we have

$$\mid Df_x^n(v) \mid \geq C\lambda^n \mid v \mid,$$

3. for $v \in E^s_x$ and $n \geq 0$, we have

$$| \, Df_x^{-n}(v) \, | \geq C\lambda^n | \, v \, |.$$

We now use our previous results and definitions to get

Theorem 1.4 *A necessary and sufficient condition for Λ to be a uniformly hyperbolic set for f is that there are an integer $N > 0$ and a cone field $\mathcal{C}$ with constant orbit core dimension over Λ such that f^N is both expanding and co-expanding on $\mathcal{C}$.*

Proof.
Sufficiency:
If $\mathcal{C}$ and N are as in the statement of the theorem, then, $\mathcal{C}$ is obviously strongly dominating for f^N over Λ. From Theorem 1.2 and Proposiiton 1.3 we get a dominated splitting $T_xM = E_x \oplus F_x$ for f over Λ with $E_x \subset \mathcal{C}_x$ and $F_x \subset \mathcal{C}^c_x$ for every $x \in \Lambda$. Thus, in fact, $E_x \oplus F_x$ is a uniformly hyperbolic splitting for f on Λ.
Necessity:
Let Λ be uniformly hyperbolic with splitting $T_x = E^u_x \oplus E^s_x$ for $x \in \Lambda$. It is known [3] that there is an adapted Finsler norm $| \cdot |_1$ on TM. This is a norm induced by a Finsler metric such that there is a $\tau > 1$ such that for $x \in \Lambda$,

1. if $v \in E^u_x$, then $| \, Df_x(v) \, |_1 \geq \tau | \, v \, |_1$, and
2. if $v \in E^s_x$, then $| \, Df_x^{-1}(v) \, |_1 \geq \tau | \, v \, |_1$,
3. if $v = v_1 + v_2$ with $v_1 \in E^u_x, \; v_2 \in E^s_x$, then $| \, v \, |_1 = \max(| \, v_1 \, |_1, | \, v_2 \, |_1)$.

For $x \in \Lambda$, set

$$\mathcal{C}_x = \{v = (v_1, v_2) \in E^u_x \oplus E^s_x : | \, v_1 \, |_1 \leq | \, v_2 \, |_1\}.$$

The core of $\mathcal{C}_x$ may be taken to be E^u_x. This is clearly an expanding and co-expanding cone field of constant core dimension for f relative to the adapted norm $| \cdot |_1$. But, since any two norms have uniformly bounded ratios on non-zero vectors, it is clear that in any norm there is a positive integer N such that $\mathcal{C}$ is expanding and co-expanding for f^N. QED.

2 Application to the Henon Family

In this section we apply the preceding results to two dimensional systems, both real and complex. This gives a simple proof of hyperbolicity on the set of bounded orbits in the Henon family $H(x, y) = (A - x^2 - By, x)$ for $| A |$ sufficiently large depending on $| B |$. In the real case this was originally proved by Devaney and Nitecki [2]. Their proof is elementary but heavily dependent on real geometry. In fact, it is not clear that their methods apply to the complex case. Before we began the work reported here, we asked several experts about the complex case and were told that R. Oberste-Vorth has a proof which, however, was never published. We do not know what estimates he obtained for the relative sizes of $| A |$ and $| B |$.

Devaney and Nitecki proved that hyperbolicity holds provided that

$$A > \left(\frac{5 + 2\sqrt{5}}{4}\right)(1 + | B |)^2. \tag{23}$$

At the end of our calculations below, we were, in fact, quite surprised to find that essentially the same estimate holds in the complex case. In retrospect, it seems that our arguments are very similar to those in [2], so our main contribution here, perhaps, is that we have made the arguments cleaner and removed the dependence on real geometry. After this work was done, John Smillie informed us that better estimates can be obtained using complex methods. In fact, Proposition 7.4.6 in [5] gives that the bounded orbits form a topological horseshoe for $A > 2(1 + | B |)^2$, and then complex methods can be used to show actual hyperbolicity. Nevertheless, our estimates are sufficiently elementary that we feel it is desirable to present them here.

As is well-known, all degree two polynomial diffeomorphisms are affinely conjugate to members of the Henon family. Hence, in studying their dynamical properties, there is no loss of generality in assuming that $0 < | B | \leq 1$

Theorem 2.1 *Consider the real or complex Henon family $H_{A,B}(x, y)$ with $0 < | B | \leq 1$. Let $\Lambda_{A,B}$ denote the set of points with bounded orbits. Assume that*

$$| A | > \left(\frac{5 + 2\sqrt{5}}{4}\right)(1 + | B |)^2. \tag{24}$$

In the complex case or the real case with $A > 0$, we have that $\Lambda_{A,B}$ is a non-empty compact invariant uniformly hyperbolic set. In the real case with

$A < 0$, the set $\Lambda_{A,B}$ is empty.

Let us proceed to the proof.

Consider $V = \mathbf{R}$ or $V = \mathbf{C}$ and the family of maps $H(x, y) = H_{A,B}(x, y)$ for $x, y \in V$. If $V = \mathbf{R}$, we let A, B be real, but we allow them to be complex if $V = \mathbf{C}$.

We begin with simple criteria for expansion and co-expansion of a linear map using the standard unit cone. We use the max norm

$$| v | = \max(| v_1 |, | v_2 |)$$

for a vector $v = (v_1, v_2) \in V^2$.

Define the standard unit cone, $\mathcal{K}$ in $V^2 = V \oplus V$.

$$\mathcal{K} = \{v = (v_1, v_2) : | v_2 | \leq | v_1 |\}.$$

The complementary cone is

$$\mathcal{K}^c = \{v = (v_1, v_2) : | v_1 | < | v_2 |\}.$$

For a linear map $L : V^2 \to V^2$ with matrix

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

and determinant $det(L) = L_{11}L_{22} - L_{12}L_{21}$, we have the inverse matrix

$$L^{-1} = \frac{1}{det(L)} \begin{pmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{pmatrix} .$$

Lemma 2.2 *A sufficient condition for L to be expanding on $\mathcal{K}$ is that*

$$| L_{11} | - | L_{12} | > 1. \tag{25}$$

Analogously, a sufficient condition for L to be co-expanding on $\mathcal{K}$ is that

$$| L_{11} | - | L_{21} | > | det(L) |. \tag{26}$$

Proof.
Let $\lambda > 1$ be such that

$$| L_{11} | - | L_{12} | > \lambda.$$

For $v = (v_1, v_2) \in \mathcal{K}$, we have $| v | = | v_1 |$, and

$$\begin{aligned} | Lv | &\geq | L_{11}v_1 + L_{12}v_2 | \\ &\geq | L_{11}| v_1 | - | L_{12} || v_2 | | \\ &= | v_1 |(| L_{11} | - | L_{12} |\frac{| v_2 |}{| v_1 |}) \\ &\geq \lambda| v_1 | \\ &= \lambda| v |. \end{aligned}$$

Similary, let $\lambda > 1$ be such that

$$| L_{11} | - | L_{21} | > \lambda| det(L) |.$$

For $v \in \mathcal{K}^c$, we have $| v | = | v_2 |$, and

$$\begin{aligned} | L^{-1}v | &\geq \frac{1}{| det(L) |}(| -L_{21}v_1 + L_{11}v_2 |)) \\ &\geq \frac{1}{| det(L) |}(| L_{11}v_2) | - | L_{21}v_1 |) \\ &\geq \frac{1}{| det(L) |}| v_2 |(| L_{11} | - | L_{21} |\frac{| v_1 |}{| v_2 |}) \\ &\geq \lambda| v_2 | \\ &= \lambda| v |. \end{aligned}$$

QED.

Now, we return to the Henon family $H = H_{A,B}(x, y) = (A - x^2 - By, x)$ with $0 < | B | \leq 1$.

For $C > 0$ let $Q_C = \{(x, y) \in V^2 : \max(| x |, | y |) \leq C\}$.

In the real case, the next result is contained in [2].

Proposition 2.3 *Let C_0 be the largest root of the quadratic polynomial*

$$p(r) = r^2 - (1 + \mid B \mid)r - \mid A \mid,$$

and, let $Q_0 = Q_{C_0}$. *Then,*

$$\Lambda_{A,B} \subset Q_0.$$

Proof.

Write $(x_1, y_1) = (A - x^2 - By, x)$.

Take any $C > C_0$. We show that $\Lambda_{A,B} \subset Q_C$. The result then follows taking the intersection

$$\bigcap_{C > C_0} Q_C.$$

Since C is greater than the largest root of the polynomial

$$r^2 - (1 + \mid B \mid)r - \mid A \mid,$$

we can choose $\lambda > 1$ close enough to 1 so that C is greater than the largest roots of the polynomials

$$p_1(r) = r^2 - (\lambda + \mid B \mid)r - \mid A \mid,$$

and

$$p_2(r) = r^2 - (1 + \lambda \mid B \mid)r - \mid A \mid.$$

Consider the sets $\mathcal{K}^+ = \mathcal{K} \setminus Q_C$, and $\mathcal{K}^- = \mathcal{K}^c \setminus Q_C$.

Let $\pi_1(x, y) = x,\ \pi_2(x, y) = y$ be the natural projections on $\mathbf{R}^2$, and write $H = H_{A,B}$.

We show

(a) $H(\mathcal{K}^+) \subset \mathcal{K}^+$ and $z \in \mathcal{K}^+$ implies that $\mid \pi_1 H(z) \mid > \lambda \mid z \mid$, and

(b) $H^{-1}(\mathcal{K}^-) \subseteq \mathcal{K}^-$ and $z \in \mathcal{K}^-$ implies that $\mid \pi_2 H^{-1}(z) \mid > \lambda \mid z \mid$.

These statements give that the norms of the forward iterates of points in $\mathcal{K}^+$ approach infinity while the norms of the backward iterates of points in $\mathcal{K}^-$ approach infinity. Hence, bounded orbits must be in Q_C.

If $(x, y) \in \mathcal{K}^+$, then $|\, y \,| \leq |\, x \,|$ and $|\, x \,| > C$, so

$$\begin{aligned} |\, x_1 \,| &= |\, A - x^2 - By \,| \\ &\geq |\, x \,|^2 - |\, A \,| - |\, B \,||\, y \,| \\ &\geq |\, x \,|^2 - |\, A \,| - |\, B \,||\, x \,| \end{aligned}$$

For $|\, x_1 \,| > \lambda |\, x \,|$, we need

$$|\, x \,|^2 - |\, A \,| - |\, B \,||\, x \,| > \lambda |\, x \,|,$$

or, $|\, x \,|$ must be greater than the largest root of $p_1(r)$. But, this follows from our choice of λ since $|\, x \,| > C$.

Further, since, $|\, y_1 \,| = |\, x \,| < |\, x_1 \,|$, we have $H(x, y) \in \mathcal{K}^+$.

This is statement (a).

Statement (b) is similar.

Suppose $(x_1, y_1) \in \mathcal{K}^-$.

Then, $|\, x_1 \,| < |\, y_1 \,|$ and $|\, y_1 \,| > C$ and

$$(x, y) = H^{-1}(x_1, y_1) = (y_1, \frac{A - y_1^2 - x_1}{B})$$

give

$$\begin{aligned} |\, y \,| &= \frac{1}{|\, B \,|} |\, A - y_1^2 - x_1 \,| \\ &\geq \frac{1}{|\, B \,|} (|\, y_1 \,|^2 - |\, x_1 \,| - |\, A \,|) \\ &\geq \frac{1}{|\, B \,|} (|\, y_1 \,|^2 - |\, y_1 \,| - |\, A \,|). \end{aligned}$$

To get $|\, y \,| > \lambda |\, y_1 \,|$ we need that $|\, y_1 \,|$ be larger than the largest root of $p_2(r)$.

Again, this follows from the choice of λ since $|\, y_1 \,| > C$. In addition, we have $|\, x \,| = |\, y_1 \,| < |\, y \,|$, so $(x, y) \in \mathcal{K}^-$. QED.

It remains to prove the hyperbolicity of $\Lambda_{A,B}$ under the assumption (24). Let

$$\alpha = 1 + | B |,$$

and, let

$$C_0 = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4| A |}\right)$$

be as in Proposition 2.3.

An elementary algebraic calculation shows that the inequality in (24) is equivalent to that in (27). So, Theorem 2.1 follows from

Proposition 2.4 *In the complex Henon map, suppose*

$$| A | > \frac{\alpha^2}{4} + \alpha C_0, \tag{27}$$

Then, $\Lambda_{A,B}$ *is a non-empty hyperbolic set for* $H_{A,B}$*.*

In the real Henon map, if (27) holds, then $\Lambda_{A,B}$ *is hyperbolic if* $A > 0$ *and empty if* $A < 0$*.*

Proof.

For notational ease, let us suppress the subscripts and write $H = H_{A,B}$, and $\Lambda = \Lambda_{A,B}$. .

We will prove that

($\star$) *If (27) holds, then* Λ *is hyperbolic provided that it is not empty.*

This takes care of the complex case. (In that case, $\Lambda_{A,B}$ is clearly non-empty since H has periodic points.)

For the real case, let U be the set of parameters $(A, B) \in \mathbf{R}^2$ satisfying (24), and let $G \subset U$ be the subset of those $(A, B) \in U$ such that $\Lambda_{A,B} \neq \emptyset$.

Letting

$$\beta = \left(\frac{5 + 2\sqrt{5}}{4}\right)\alpha^2$$

be the number on the right side of the inequality (24), we have that the set U is the disjoint union of the two open connected sets

$$U_1 = \{(A, B) : A > \beta\},$$

and

$$U_2 = \{(A, B) : -A < -\beta\}.$$

Statement ($\star$) and the local stability of hyperbolic isolated invariant sets (Theorem 3, page 157 in [8]) imply that G is an open subset of U. We claim $U \setminus G$ is also open.

Indeed, we have

$$\Lambda = \bigcap_n H^n(Q_0)$$

is the decreasing intersection of the compact sets

$$\bigcap_{-k \leq n \leq k} H^n(Q_0).$$

If Λ were empty, then for some finite $k_0 > 0$, we would have

$$\bigcap_{-k_0 \leq n \leq k_0} H^n(Q_0) = \emptyset,$$

and this property persists under small changes in (A, B).

It follows that $G \bigcap U_i$ is either empty or equal to U_i for $i = 1, 2$.

Fix $0 \leq |\ B\ | \leq 1$. Then,

$$|\ C_0\ | \leq 1 + \sqrt{1 + |\ A\ |}$$

Using this, it is easy to show that if A is very large and negative, then for any $(x, y) \in Q_0$, we have

$$\pi_1 H^n(x, y) \to -\infty$$

as $n \to \infty$. Thus, if $|\ A\ |$ very large and negative, we have $\Lambda = \emptyset$. Hence, Λ must be empty for all A in $(-\infty, -\beta)$. On the other hand, if A is very large and positive, then Λ is non-empty since $H_{A,B}$ has fixed points. So, Λ is non-empty for all A in (β, ∞).

It remains to prove ($\star$).
Now,

$$DH = \begin{pmatrix} -2x & B \\ 1 & 0 \end{pmatrix} \text{ and } det(DH) = B,$$

so, (25) and (26) are both implied by

$$\mid 2x \mid > 1 + \mid B \mid,$$

or,

$$\mid x \mid > \frac{1 + \mid B \mid}{2} = \frac{\alpha}{2}. \tag{28}$$

We want to consider $z = (x, y) \in Q_0$ such that

$$\mid x \mid \leq \frac{\alpha}{2}. \tag{29}$$

We show that (29) together with (27) implies that the image $H(z)$ is not in Q_0. This will imply that $\mathcal{K}$ is expanded and co-expanded by DH on $Q_0 \bigcap H^{-1}(Q_0)$. Since Λ is contained in this latter set, this will prove hyperbolicity.

To get $H(z) \notin Q_0$, it su□ces to have

$$\mid A - x^2 - By \mid > C_0.$$

Since $\mid x \mid \leq \frac{\alpha}{2}$ and $\mid y \mid \leq C_0$, this is implied by

$$\mid A \mid - \frac{\alpha^2}{4} - \mid B \mid C_0 > C_0,$$

or

$$\mid A \mid - \frac{\alpha^2}{4} > \alpha C_0. \tag{30}$$

But this is the same as (27). QED.

Remark. In addition to the hyperbolicity, Devaney and Nitecki prove that, in the real case with $\frac{A}{\mid B \mid}$ large, the set of bounded orbits of $H_{A,B}$ equals its non-wandering set, and the map $H_{A,B}$ restricted to this set is topologically conjugate to the full two-sided shift on two symbols. We easily obtain this in our situation even in the complex case. First note that these facts are fairly easy to prove for a real Henon map H_{A_1,B_1} with A_1 large and $| B_1 |$ suciently small. In the general (even complex) case, for arbitrary (A, B) satisfying (27), we connect $H_{A,B}$ to a real Henon map H_{A_1,B_1} through a curve $\{H_t = H_{A_t,B_t}, t \in [0, 1]\}$ of Henon maps keeping (27) and $\Lambda_t \neq \emptyset$ for all t. Then, each Λ_t is a hyperbolic isolated invariant set. Hence, the local stability of isolated invariant hyperbolic sets gives that all the pairs (H_t, Λ_t) are topologically conjugate.

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