Differential Equations on the two dimensional torus

We will now study some simple differential equations on a two-dimensional torus. These will provide interesting minimal sets.

The 2−torus $\mathbf{T}^2$ is the product $S^1 \times S^1$ of two circles.

Writing S^1 as

$$S^1 = \{z \in \mathbf{C} : \mid z \mid = 1\},$$

we get

$$S^1 \times S^1 = \{(z_1, z_2) \in \mathbf{C}^2 : \mid z_1 \mid = \mid z_2 \mid = 1.$$

There is an alternative viewpoint which is useful.

Consider the quotient group $\mathbf{R}/\mathbf{Z}$; i.e., the set of equivalence classes in $\mathbf{R}$ under the equivalence relation $x \sim y$ iff $x - y \in Z$. There is a natural map $\pi : \mathbf{R} \to \mathbf{Z}$ assigning to each real number x its equivalence class $[x]$.

Notice that the map $\phi : t \to e^{2\pi it}$ is a surjective map from $\mathbf{R}$ onto S^1 such that each pre-image $\phi^{-1}(z)$ is an equivalence class in $\mathbf{R}/\mathbf{Z}$. Thus, the map $\psi([x]) = \phi(x)$ gives a well-defined bijection from $\mathbf{R}/\mathbf{Z}$ to S^1.

We can use the map ψ to define a topology on $\mathbf{R}/\mathbf{Z}$, so we have notions of continuity, etc.

Note that we can also think of the circle as a closed interval $[a, b]$ with its endpoints identified.

We can now apply these ideas to the product $T^2 = S^1 \times S^1$. There is a bijection from $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ onto T^2. This defines a topology on T^2. There is another natural bijection between $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ and $\mathbf{R}^2/\mathbf{Z}^2$, and we can think of this latter quotient set as $\mathbf{T}^2$ as well. We will use all of these objects.

In particular, we think of T^2 as either $S^1 \times S^1$ or $\mathbf{R}^2/\mathbf{Z}^2$.

Now we wish to define differential equations on $\mathbf{T}^2$. We will use the representation $\mathbf{R}^2/\mathbf{Z}^2$.

For this purpose, let $f(x, y), g(x, y)$ be two real-valued functions of two real variables such that they are periodic of period 1 in both variables. That is,

$$f(x + 1, y + 1) = f(x, y), \ g(x + 1, y + 1) = g(x, y), \quad \forall x, y.$$

The planar vector field $X(x,y) = (f(x,y), g(x,y))$ is a map from $\mathbf{R}^2 \to \mathbf{R}^2$. The periodicity conditions imply that it induces a well-defined map from $\mathbf{R}^2/Z^2$ to R^2. We call this a vector field on the torus $\mathbf{T}^2$. The associated system of differential equations

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

is called a *differential equation on* $\mathbf{T}^2$.

The solutions (or orbits) are obtained by taking the orbits in $\mathbf{R}^2$ and projecting them down to $\mathbf{T}^2$.

Let us take a simple example.

Consider the constant vector field in $\mathbf{R}^2$

$$X(x,y) = \omega_1 \partial_x + \omega_2 \partial_y,$$

where ω_1, ω_2 are positive real numbers.

The associated system of differential equations is

$$\begin{aligned} \dot{x} &= \omega_1 \\ \dot{y} &= \omega_2 \end{aligned} \tag{1}$$

The periodicity conditions are obviously satisfied, so we get a differential equation on $\mathbf{T}^2$. This is called the *constant* or *linear* vector field on $\mathbf{T}^2$.

Let's us consider its orbits.

The solutions to (1) in $\mathbf{R}^2$ are the lines

$$x(t) = \omega_1 t + c_1, \;\; y(t) = \omega_2 t + c_2.$$

Each such line has slope $\frac{\omega_2}{\omega_1}$ and passes through the point (c_1, c_2).

On the torus $\mathbf{T}^2$ we simply take $(x(t), y(t))$ mod 1.

We have two main cases:

1. $\frac{\omega_2}{\omega_1}$ is rational. Write it as $\frac{\omega_2}{\omega_1} = \frac{p}{q}$ in lowest terms with p, q positive integers.

   Consider the orbit γ_0 in $\mathbf{R}^2$ through $(0,0)$ (i.e., $c_1 = c_2 = 0$).

   The solution in $\mathbf{R}^2$ has the form

$$x(t) = \omega_1 t, \quad y(t) = \left(\frac{p}{q}\right) \omega_1 t$$

Let $\pi : \mathbf{R}^2 \to \mathbf{T}^2$ denote the canonical projection.

When $t = q/\omega_1$, we have $x(t) = q,\ y(t) = p$, so the projection $\pi(\gamma_0) = \gamma$ into $\mathbf{T}^2$ is a closed curve (topological circle) in $\mathbf{T}^2$.

Since all the solutions are vertical tranlates of γ_0 we have that all solutions in $\mathbf{T}^2$ are periodic of the same period.

2. $\frac{\omega_2}{\omega_1}$ is irrational.

This case is more interesting. We will show that every orbit in $\mathbf{T}^2$ is dense in $\mathbf{T}^2$. Hence, all of $\mathbf{T}^2$ is a minimal set.

Let us show that the orbit through $(0, 0)$ is dense in $\mathbf{T}^2$. The analogous statement for other orbits is similar.

Consider the circle $S^1 = \pi(\{x = 0\})$.

There is a first return map $F : S^1 \to S^1$ (also called Poincaré map) obtained by taking a point $(0, y)$ and taking the point $(1, F(y))$ where the orbit through $(0, y)$ hits the line $\{x = 1\}$.

Let us compute F.

The orbit through $(0, y)$ is $\{(\omega_1 t, \omega_2 t + y),\ t \in \mathbf{R}\}$.

We get

$$\begin{aligned} \omega_1 t &= 1 \\ \omega_2 t + y &= F(y). \end{aligned}$$

Hence,

$$t = 1/\omega_1, \quad F(y) = y + \omega_2/\omega_1$$

So, letting $\alpha = \frac{\omega_2}{\omega_1}$ we get that $F(y) = y + \alpha\ (mod\ 1)$.

To show that the orbits of X are dense, it suffices to show that the iterates $\{F^n(y), n \geq 0\}$ are dense mod 1

This is the number theoretic statement: for α irrational, and

$$A = \{n\alpha + m : n \in \mathbf{Z}, m \in \mathbf{Z}\},$$

then

$$A \text{ is dense in } \mathbf{R}. \tag{2}$$

This elementary fact is proved as follows.

(a) A is clearly an additive subgroup of $\mathbf{R}$.

(b) To show that A is dense, it suffices to show that 0 is an accumulation point of A.

(c) For this latter fact, for each $n \in \mathbf{Z}$, let $m_n \in \mathbf{Z}$ be such that $x_n = n\alpha + m_n \in [0, 1)$. Since α is irrational, the numbers x_n are all distinct. Since $[0, 1]$ is compact, there is a subsequence x_{n_i} of (x_n) which converges. In particular, if $\epsilon > 0$ is arbitrary, there are points x_{n_1}, x_{n_2} such that

$$| \; x_{n_1} - x_{n_2} \; | < \epsilon.$$

But then this difference $x_{n_1} - x_{n_2}$ is in $A \bigcap (-\epsilon, \epsilon)$. Since, ϵ is arbitrary, we have that 0 is an accumulation point of A as required.

Let us now give an application of these ideas.

Consider the mass spring system whose equation is

$$m\ddot{x} + kx = 0$$

where m is the mass of an object suspended vertically by a spring, $k > 0$ is the spring constant, and $x = 0$ represents the equilibrium position.

Let $\omega = \sqrt{\frac{k}{m}}$. The associated first order planar system can be written

$$\begin{aligned} \dot{x} &= -\omega \; y \\ \dot{y} &= \omega \; x \end{aligned} \tag{3}$$

Let us write this system (3) in polar coordinates.

We have

$$x = rcos(\theta), \;\; y = rsin(\theta)$$

$$\begin{aligned} \dot{x} &= \dot{r}cos(\theta) - rsin(\theta)\dot{\theta} \\ \dot{y} &= \dot{r}sin(\theta) + rcos(\theta)\dot{\theta} \end{aligned} \tag{4}$$

So, we see that the simple polar coordinate system

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\theta} &= \omega \end{aligned} \tag{5}$$

is equivalent to (4).

The solutions off $(0,0)$ are

$$r = c > 0, \quad \theta(t) = \omega\ t + \theta_0.$$

Hence, the orbits in the plane off $(0,0)$ consist of the circles $r = c$ and the motion has constant rotational speed ω with period $\frac{2\pi}{\omega}$.

Now, consider two uncoupled springs whose equations in $\mathbf{R}^4$ become

$$\begin{aligned} \dot{x_1} &= -\omega_1\ y_1 \\ \dot{y_1} &= \omega\ x_1 \\ \dot{x_2} &= -\omega_2\ y_2 \\ \dot{y_2} &= \omega_2\ x_2 \end{aligned} \tag{6}$$

Using polar coordinates $(r_1, \theta_1, r_2, \theta_2)$ in $\mathbf{R}^4$ in the obvious way, we get the equivalent equations

$$\begin{aligned} \dot{r_1} &= 0 \\ \dot{\theta_1} &= \omega_1 \\ \dot{r_2} &= 0 \\ \dot{\theta_2} &= \omega_2 \end{aligned} \tag{7}$$

Hence all solutions in $\mathbf{R}^4$ off the union of the two subspaces $\mathbf{R}^2 \times \{(0,0\} \bigcup \{0,0\} \times \mathbf{R}^2$ lie on two dimensional tori $r_1 = c_1, r_2 = c_2$, and the motion on each is a constant vector field. If ω_2/ω_1 is rational, then all these orbits are periodic, and if ω_2/ω_1 is irrational, then all these tori are minimal sets.