We have proved that solutions to differential equations depend continuously on parameters.

Now we wish to investigate the smooth dependence of solutions in systems which depend smoothly on parameters.

Theorem. *Suppose that $f(t, x, \lambda)$ is a C^1 function of the variables (t, x, λ) in an open set $D \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$. For, $(t_0, x_0, \lambda_0) \in D$, let $\phi(t, t_0, x_0, \lambda)$ be the solution of the initial value problem*

$$\dot{x} = f(t, x, \lambda), \quad x(t_0) = x_0 \tag{1}$$

.

Then, the solution $\phi(t, u, x, \lambda)$ to the initial value problem

$$\dot{x} = f(t, x, \lambda), \quad x(u) = x \tag{2}$$

is a C^1 function of the variables (t, u, x, λ) for (u, x, λ) near (t_0, x_0, λ_0). Moreover, the $x-$partial derivative of the solution

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x}(t, u, x, \lambda) \equiv J(t)$$

with respect to the space variable x satisfies the matrix (operator) initial value problem

$$\dot{J} = \frac{\partial f}{\partial x}(t, \phi(t, u, x, \lambda), \lambda) \cdot J, \quad J(u) = id \tag{3}$$

Remark. The differential equation in (3) (without the λ parameter) is usually called the *variational equation* of (1).

We will leave the proof of this last theorem and some of its generalizations to the exercises.

Differential Inequalities

Let D_r denote the right hand derivative of a function; i.e.,

$$D_r f(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

If $\omega(t, u)$ is a scalar function for the scalar variables t, u in some open connected set Ω, we say a function $v(t)$ is a *solution of the differential inequality*

$$D_r v(t) \leq \omega(t, v(t)) \tag{4}$$

on $[a,b)$ if $v(t)$ is continuous on $[a,b)$, right differentiable on $[a,b)$ and satisfies (4) on $[a,b)$.

Lemma. *If $x(\cdot)$ is a continuously differentiable map from a closed interval $[a,b]$ in $\mathbf{R}$, then $D_r| x(t) |$ exists on $[a,b)$ and $| D_r(| x(t) |) | \leq | \dot{x}(t) |$ for all $t \in [a,b)$.*

Proof.

For any two $n-$ vectors, x, u, and $0 < \tau \leq 1, 0 < h$, we have

$$| x + \tau hu | - | \tau x + \tau hu | \leq (1-\tau)| x |$$

or

$$| x + \tau hu | - | x | \leq \tau(| x + hu | - | x |)$$

or

$$\frac{| x + \tau hu | - | x |}{\tau h} \leq \frac{| x + hu | - | x |}{h}$$

This implies that the difference quotient

$$\phi(h) = \frac{| x + hu | - | x |}{h}$$

is a non-decreasing function of h.

Also,

$$\begin{aligned} \phi(h) &= \frac{| x + hu | - | x |}{h} \\ &= \frac{| x - (-hu) | - | x |}{h} \\ &\geq \frac{| x | - | -hu | - | x |}{h} \\ &\geq -| u |, \end{aligned}$$

so, $\phi(h)$ is bounded below.

Thus,

$$\lim_{h \to 0+} \phi(h)$$

exists.

Now, suppose $x(t)$ is C^1.

then,

$$\begin{aligned} \Big| \; (\mid x(t+h) \mid - \mid x(t) \mid) \quad &- \quad (\mid x(t) + h\dot{x}(t) \mid - \mid x(t) \mid) \; \Big| \\ &= \Big| \; \mid x(t+h) \mid - \mid x(t) + h\dot{x}(t) \mid \; \Big| \\ &\leq \; \mid x(t+h) - x(t) - h\dot{x}(t) \mid = o(h) \end{aligned}$$

as $h \to 0+$.

Thus, $D_r(\mid x(t) \mid)$ exists and equals

$$\lim_{h \to 0+} \frac{\mid x(t) + h\dot{x}(t) \mid - \mid x(t) \mid}{h}$$

Taking the norm of both sides gives $\mid D_r(\mid x(t) \mid) \mid \leq \mid \dot{x}(t) \mid$ which proves the Lemma. QED.

Theorem. *Let $\omega(t,u)$ be continuous and locally Lipschitz on an open connected set $\Lambda \subset \mathbf{R}^2$. Suppose $u(t)$ is a solution to $\dot{u} = \omega(t,u)$ on the interval $[a,b]$ and $v(t)$ is a solution of the differential inequality (4) on $[a,b)$ with $v(a) \leq u(a)$. Then, $v(t) \leq u(t)$ for $t \in [a,b)$.*

Proof.

Consider the sequence of equations

$$\dot{u} = \omega(t,u) + \frac{1}{n} \tag{5}$$

for $n = 1, 2, \ldots$.

Let $u_n(t)$ denote the solution to the equation $\dot{u} = \omega(t,u) + \frac{1}{n}$ with $u_n(a) = u(a)$. By the global continuity theorem, there is an $n_0 > 0$ such that for $n \geq n_0$, the solution $u_n(t)$ is defined on all of $[a,b]$ and converges uniformly to $u(t)$ as $n \to \infty$.

We show that

(*) $v(t) \leq u_n(t)$ for all $t \in [a,b]$ and $n \geq n_0$

Once this is done, since u_n converges uniformly to u on $[a,b]$, it follows that $v(t) \leq u(t)$ for all t.

If (*) fails for some $n \geq n_0$, then then there exist $t_2 < t_1$ in (a,b) such that $v(t) > u_n(t)$ on $(t_2, t_1]$ and $v(t_2) = u_n(t_2)$. Therefore, $v(t) - v(t_2) > u_n(t) - u_n(t_2)$ on $(t_2, t_1]$.

This implies that

$$\begin{aligned} D_r v(t_2) &\geq \dot{u}_n(t_2) = \omega(t_2, u_n(t_2)) + \frac{1}{n} \\ &= \omega(t_2, v(t_2)) + \frac{1}{n} \\ &> \omega(t_2, v(t_2)) \end{aligned}$$

which contradicts the assumption that v is a solution to (4). QED

Corollary. *Suppose $w(t,u)$ is continuous and locally Lipschitz in u in an open set $D \subset \mathbf{R}^2$. Let $u(t)$ be a solution of $\dot{u} = \omega(t,u)$ defined on the closed interval $[a,b]$ and assume that $(t, u(t)) \in D$ for all $t \in [a,b]$. If $x(t)$ is a C^1 $n-$vector valued function on $[a,b]$ such that $\mid x(a) \mid \leq u(a)$, and, for $t \in [a,b]$ we have $(t, \mid x(t) \mid) \in D$ and*

$$\mid \dot{x}(t) \mid \leq \omega(t, \mid x(t) \mid),$$

then $\mid x(t) \mid \leq u(t)$ for all $t \in [a,b]$.

Proof.

By a previous Lemma, we have $D_r(\mid x(t) \mid)$ exists and is no larger than $\mid \dot{x}(t) \mid$. Thus, $\mid x(t) \mid$ satisfies the differential inequality $D_r(v(t)) \leq \omega(t, v(t))$ on $[a,b]$. By the previous theorem, we have $\mid x(t) \mid \leq u(t)$ for all t. QED.

Proposition. *Suppose $\omega(t,u)$ is continuous and locally Lipschitz in u in a domain $D \subset \mathbf{R}^2$, and let $u(t)$ be a non-negative solution of $\dot{u} = \omega(t,u)$ on an interval $[a,b)$ such that $\{(t,y) : t \in [a,b), 0 \leq y \leq u(t)\}$ is in D. Let $f(t,x)$ be continuous and locally Lipschitz in x in all of $\mathbf{R}^{n+1}$ and suppose that*

$$\mid f(t,x) \mid \leq \omega(t, \mid x \mid)$$

for $t \in [a,b)$ and $(t, \mid x \mid) \in D$.

Then, the solutions of

$$\dot{x} = f(t,x), \ \mid x(a) \mid \leq u(a)$$

exist on all of $[a,b)$ and $\mid x(t) \mid \leq u(t)$ for $t \in [a,b)$.

Proof.

Since $\mathbf{R}^{n+1}$ has no boundary, the only way a solution to $\dot{x} = f(t,x)$ with $\mid x(a) \mid \leq u(a)$ can fail to exist on $[a,b)$ is that there is some $c \in (a,b)$ such that $x(t)$ is defined for $a \leq t < c$ and

$$\limsup_{t \to c-} |\ f(t, x(t))\ | = \infty. \tag{6}$$

Now, $u(t)$ exists for all $t \in [a, c]$ and

$$E \stackrel{\text{def}}{=} \{(t, y) : t \in [a, c], 0 \leq y \leq u(t)\}$$

is a compact subset of D. So, $\omega(t, y)$ is bounded on E. Thus, there is a constant $K > 0$ so that $\omega(t, y) \leq K$ for all $(t, y) \in E$.

Let $c_1 \in (a, c)$ be such that

$$|\ f(c_1, x(c_1))\ | > K. \tag{7}$$

Now,

$$D_r(|\ x(t)\ |) \leq |\ \dot{x}(t)\ | = |\ f(t, x(t))\ | \leq \omega(t, |\ x(t)\ |), |\ x(a)\ | \leq u(a),$$

and both $u(t)$ and $x(t)$ exist for all $t \in [a, c_1]$.

So, by the Corollary above, we have that $|\ x(t)\ | \leq u(t)$ for all $t \in [a, c_1]$.

Thus,

$$F \stackrel{\text{def}}{=} \{(t, |\ x(t)\ |) : t \in [a, c_1]\} \subseteq E$$

and, in particular, $|\ f(c_1, x(c_1))\ | \leq \omega(c_1, |\ x(c_1)\ |) \leq K$ which contradicts (**??**).

This proves the theorem. QED.

Now we give a result on the existence of long time solutions to ODE's.

Lemma. *Consider the scalar differential equation* $\dot{u} = \phi(t)\psi(u)$ *where* ϕ *is defined and continuous on the half infinite interval* $[\alpha, \infty)$ *and* ψ *is defined, non-negative, and locally Lipschitz on the half infinite interval* $[0, \infty)$*. If*

$$\int_0^\infty \frac{du}{\psi(u)} = \infty,$$

then, for any $t_0 > \alpha, u_0 > 0$*, the IVP* $\dot{u} = \phi(t)\psi(u),\ u(t_0) = u_0$ *has a unique solution* $u(t)$ *defined on the whole interval* $[t_0, \infty)$*.*

Proof. Suppose the (right) maximal solution $u(t)$ of the IVP only exists on the interval $[t_0, c)$. If $u(t)$ remained bounded as $t \to c-$, then, continuity of ψ on closed bounded intervals would imply that $\phi(t)\psi(u(t))$ remains bounded on $[t_0, c)$. Consequently, $u(t)$ would have a limit as $t \to c-$ and then, we could get a non-tivial continuation of $u(t)$ to an interval $[t_0, c+\epsilon)$ for some positive ϵ. This would contradict the maximality, so there must exist a sequence $t_n \in [t_0, c)$ with $t_n \to c-$ and $u(t_n) \to \infty$.

Now, since $\frac{du}{dt}(t) = \phi(t)\psi(u(t))$, we have

$$\int_{u_0}^{u(t)} \frac{dv}{\psi(v)} = \int_{t_0}^{t} \phi(s)ds.$$

Putting t_n into this formula gives

$$\int_{u_0}^{u(t_n)} \frac{dv}{\psi(v)} = \int_{t_0}^{t_n} \phi(s)ds.$$

By assumption on ψ, the left hand side becomes infinite as $n \to \infty$, but the right hand side remains bounded by $\int_{t_0}^{c} \mid \phi(s) \mid ds$. This is a contradiction, so the solution $u(t)$ exists on all of $[t_0, \infty)$. QED

Theorem. *If ϕ, ψ are as in the lemma, and $f(t,x)$ is a continuous, locally Lipschitz $\mathbf{R}^n$-valued function on $\mathbf{R}^{n+1}$ such that $\mid f(t,x) \mid \leq \phi(t)\psi(\mid x \mid)$ for all $t \geq t_0$ and all x, then any solution $x(t)$ to the IVP $\dot{x} = f(t,x)$, $x(t_0) = x_0$ with $\mid x_0 \mid \leq u_0$ can be extended to the whole interval $[t_0, \infty)$.*

Proof. This follows immediately from the preceding results. QED.

Corollary. *Suppose that $A(t)$ is an $n \times n$ matrix-valued function of t and $h(t)$ is an $n-$vector valued function of t and that both $A(t), h(t)$ are defined and continuous on the whole line $\mathbf{R}$. Let $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}$. Then, the linear differential equation*

$$\dot{x} = A(t)x + h(t)$$

has a unique solution $x(t)$ with $x(t_0) = x_0$ which exists for all t.

Proof. It suffices to bound

$\mid A(t)x + h(t) \mid$ by a function of the form $\phi(t)\psi(\mid u \mid)$ with ϕ continuous on $\mathbf{R}$, $\psi(u), C^1$ on $\mathbf{R}$, and $\int^{\infty} \frac{dv}{\psi(v)} = \infty$.

We can use $\phi(t) = \max(\mid A(t) \mid, \mid h(t) \mid)$ and $\psi(u) = u + 1$. QED