Hamiltonian Systems in $\mathbf{R}^{2n}$

Let $H : \mathbf{R}^{2n} \to \mathbf{R}$ be a C^k function, $k \geq 1$. Write coordinates $(q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $\mathbf{R}^{2n}$.

A system of differential equations of the form

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \quad i = 1, \ldots, n \tag{1}$$

is called a *Hamiltonian system* with n degrees of freedom and Hamiltonian function H. We also write X_H for the vector field defined by (1).

Sometimes we write the shortened form of (1) as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}$$

where $q = (q_1, \ldots, q_n), p = (p_1, \ldots, p_n)$.

If we define $\nabla H = (\frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n})$ and

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the $n \times n$ identity matrix,

then (1) has the form

$$\dot{z} = J \nabla H(z). \tag{2}$$

The matrix J above is called the standard symplectic matrix. It is one of the normal forms of a non-degenerate alternating bilinear form on $\mathbf{R}^{2n}$. Because of equation (2), one sometimes refers to a Hamiltonian system as a symplectic gradient. However, the orbit structure of a hamiltonian system is vastly different from that of a gradient system.

Proposition. *If X_H is a Hamiltonian system with Hamiltonian H, then H is constant on orbits.*

Proof.

For any solution curve $\gamma(t) = (q(t), p(t))$ we have

$$\frac{dH(q(t), p(t))}{dt} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i$$

$$\begin{aligned} &= \sum_{i=1}^{n} H_{q_i} H_{p_i} + H_{p_i}(-H_{q_i}) \\ &= 0 \end{aligned}$$

QED.

Classical Mechanical Systems in $\mathbf{R}^n$

Let $x = (x_1, \ldots, x_n)$ denote points in $\mathbf{R}^n$, and let $U : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function. Let $m_i > 0, i = 1, \ldots, n$ be n positive real constants.

The system

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}, \ i = 1, \ldots, n \tag{3}$$

is called a *conservative mechanical system* with potential function U in $\mathbf{R}^n$. The constants represent the masses of the system, and the function U plays the role of potential energy. The system (3) is a formulation of Newton's law of motion which, in words, says that *mass times acceleration* equals *force* and the force is the negative gradient of the potential energy function. Note that the potential function is a function of position alone (not velocity) and can be an arbitrary C^1 function.

Let $c_i > 0, i = 1, \ldots, n$ denote some other constants.

The system

$$m_i \ddot{x}_i + c_i \dot{x}_i = -\frac{\partial U}{\partial x_i}, \ i = 1, \ldots, n \tag{4}$$

is called a *dissipative mechanical system* with potential function U and frictional constants c_i.

Given (3) or (4), we set $v = (v_1, \ldots, v_n) = (\dot{x}_1, \ldots, \dot{x}_n)$ and form the function

$$T(x, v) = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 + U(x)$$

This is called the total energy function (or simply the energy function) of the system. The function $K(v) = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2$ is called the *Kinetic Energy* of the system. It is a function of velocity alone.

The equations (3), (4) are second order systems.

We can form the associated first order systems

$$\begin{array}{rcl} \dot{x}_i & = & v_i \\ m_i \dot{v}_i & = & -\frac{\partial U}{\partial x_i} \end{array} \tag{5}$$

and

$$\begin{array}{rcl} \dot{x}_i & = & v_i \\ m_i \dot{v}_i & = & -c_i v_i - \frac{\partial U}{\partial x_i} \end{array} \tag{6}$$

Proposition. *There is a coordinate system on* $\mathbf{R}^{2n}$ *in which the conservative system (3) becomes a Hamiltonian system.*

Proof.

Let $q_i = x_i, p_i = m_i v_i$.

Then,

$$H(q,p) = T(x,v) = \frac{1}{2}\sum_{i=1}^{n} \frac{p_i^2}{m_i} + U(q_1, \ldots, q_n)$$

and (3) becomes

$$\begin{array}{rcccl} \dot{q}_i & = & \frac{p_i}{m_i} & = & \frac{\partial H}{\partial p_i} \\ \dot{p}_i & = & m_i \dot{v}_i & = & -\frac{\partial H}{\partial q_i} \end{array}$$

QED.

Facts.

1. The critical points of a classical mechanical system are the points (x, v) with x a critical point of U and $v = 0$.

2. The total energy function $T(x, v)$ is a Lyapunov function for a conservative mechanical system and a strict Lyapunov function for a dissipative mechanical system.

3. If x_0 is a strict relative minimum of the potential function U, then $(x_0, 0)$ is a stable equilibrium of the system (5) and an asymptotically stable equilibrium of the system (6).

The fact that the energy function $T(x, v)$ is a Lyapunov function for a mechanical system frequently helps us to get a picture of the the solutions without solving the equation.

To illustrate this phenomenon, let us consider systems with one degree of freedom.

These have the form

$$\ddot{x} + f(x) = 0 \tag{7}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a real-valued function of one real variable.

Writing $U(x) = \int_0^x f(s)ds$, we get a total energy function of the form

$$T(x, v) = \frac{1}{2}v^2 + U(x)$$

Let us consider some examples.

1. Harmonic oscillator
$$T(x, v) = \frac{v^2}{2} + \frac{x^2}{2}$$
The orbits are circles around the origin $(0, 0)$ which is a single stable equilibrium.

2. Pendulum
$$T(x, v) = \frac{v^2}{2} + k(1 - cos(x))$$
for some constant $k > 0$.
The critical points are $(\pm n\pi, 0)$. The stable ones are $(2\pi n, 0)$ and the saddles are $(\pi(2n + 1), 0)$.

3. Duffing equation
$$T(x, v) = \frac{v^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}$$
There are three critical points at $(0, 0), (-1, 0), (1, 0)$. The origin is a saddle and the others are centers (a center is a critical point surrounded by periodic orbits).

When one adds friction to each of the above equations, the orbits cross the level sets of T instead of lying in them.