1. Find the general real solution to the differential equation $\dot{x} = Ax$ for each of the following matrices A. Also, draw a rough sketch of the orbits in each case.

   (a)
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   (b)
$$A = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$$

   (c)
$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (d)
$$A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Let
$$\eta(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

   Prove that $\eta(x)$ is a C^∞ function from $\mathbf{R}$ to $\mathbf{R}$.

3. Let $\eta_1(x) = \eta(x)\eta(1-x)$, where η is as in the previous exercise, and let
$$\xi(x) = \frac{\int_0^x \eta_1(t)dt}{\int_0^1 \eta_1(t)dt}$$

   The function $\xi(x)$ is a C^∞ function from $\mathbf{R}$ to $\mathbf{R}$ such that $\xi(x) \geq 0\ \forall x, \xi(x) = 0$ for $x \leq 0$, and $\xi(x) = 1$ for $x \geq 1$. Using a modification of ξ prove that, for any $a < b$, there is a C^∞ function $\rho : \mathbf{R} \to \mathbf{R}$ such that $\rho(x) \geq 0\ \forall x, \rho(x) = 0$ for $x \leq a$, and $\rho(x) = 1$ for $x \geq b$.

4. (a) Let $x \neq y$ be distinct points in $\mathbf{R}^n$. Prove that there is a C^∞ function ρ from $\mathbf{R}^n$ to $\mathbf{R}$ such that $\rho(u) \in [0, 1]$ $\forall u$, $\rho(x) = 0$ and $\rho(y) = 1$.

   (b) Let γ_1 and γ_2 be distinct circles in the plane $\mathbf{R}^2$, both of which centered at the origin. Prove that there is a C^∞ function ρ from $\mathbf{R}^2$ to $\mathbf{R}$ such that $\rho(x) \geq 0$ for all x, $\rho(x) = 0$ for $x \in \gamma_1$, and $\rho(x) = 1$ for $x \in \gamma_2$.

5. Show that according as $ad - bc > 0$ or $ad - bc < 0$, the index of the origin with respect to the linear vector field $f_0(x, y) = (ax+by, cx+dy)$ is ± 1.

6. Suppose that $f(x, y) = (f_1(x, y), f_2(x, y))$ is a C^1 vector field with an isolated critical point at $0 \in \mathbf{R}^2$ and the derivative of f at 0 is the linear map f_0 in exercise 5. Show that if $ad - bc > 0$, then the index of f at 0 is +1 while if $ad - bc < 0$, then the index at 0 of f is -1.

7. Let $f(z) = z^k$ where $z = x + iy$ and z^k means the complex number z is multiplied by itself k-times. Consider f as a vector field in $\mathbf{R}^2$. Show that the index of f at 0 is k.

8. Let $f(z) = \bar{z}^k$ where $z = x + iy$ and $\bar{z}^k$ means the complex conjugate of z multiplied by itself k times. Consider f as a vector field in $\mathbf{R}^2$. Show that the index of f at 0 is $-k$. Recall that if $z = x + iy$, then $\bar{z} = x - iy$ where $i = \sqrt{-1}$.

9. Give an example of a C^∞ vector field f in the plane which has the unit circle as the only non-trivial closed orbit.

   Hint: Consider the use of polar coordinates.