18. Geometry of two dimensional homogeneous systems

The solutions of two dimensional homogeneous systems are curves in the plane. In this section we wish to describe the geometric structure of these curves.

First, it will be convenient to obtain formulas for linear changes of coordinates.

Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis vectors in $\mathbf{R}^2$, and let $\mathbf{v}_1, \mathbf{v}_2$ be two linearly independent vectors.

We know that every vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbf{R}^2$ can be written uniquely as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2.$$

The numbers $x_1,\ x_2$ are simply the usual coordinates of $\mathbf{x}$. How do we get $\alpha_1,\ \alpha_2$? Using 2×2 matrices, it is easy to get the answer.

Let

$$\mathbf{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} c \\ d \end{pmatrix},$$

and consider the matrix

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose first column is $\mathbf{v}_1$ and whose second column is $\mathbf{v}_2$.

Because the vectors $\mathbf{v}_1, \mathbf{v}_2$ were linearly independent, the matrix Q is invertible.

Note that $Q(\mathbf{e}_1) = \mathbf{v}_1$ and $Q(\mathbf{e}_2) = \mathbf{v}_2$.

So,

$$\begin{aligned}\mathbf{x} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \\ &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2\end{aligned}$$

which is the same as the matrix equation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = Q^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and we can find the constants α_1 and α_2 in terms of x_1 and x_2 by multiplying $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ by the inverse of the matrix Q.

There is a simple formula for the inverse of a 2×2 matrix

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which we now give.
Write $det(Q) = ad - bc$ as usual.
Then,

$$Q^{-1} = \frac{1}{det(Q)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \qquad (1)$$

Def. A linear change of coordinates in the plane $\mathbf{R}^2$ is a change of coordinates of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where Q is an invertible 2×2 matrix.
Written out in coordinates, if $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get

$$\begin{aligned} x_1 &= a\, u_1 + b\, u_2 \\ x_2 &= c\, u_1 + d\, u_2 \end{aligned}$$

It is easy to see how this transforms a homogeneous linear system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where A is a 2×2 matrix.
Letting $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we have

$$\begin{aligned}\dot{\mathbf{x}} &= Q\dot{\mathbf{u}}\\ &= A\mathbf{x}\\ &= AQ\mathbf{u},\end{aligned}$$

so we get

$$Q\dot{\mathbf{u}} = AQ\mathbf{u}$$

or

$$\dot{\mathbf{u}} = Q^{-1}AQ\mathbf{u}.$$

Hence, the differential equation gives rise to a new linear homogeneous differential equation in the $\mathbf{u}-$coordinates whose matrix is

$$B = Q^{-1}AQ.$$

A similar formula holds for n dimensional matrices.

In general two matrices A and B which have the property that there is an invertible matrix Q such that $Q^{-1}AQ = B$ are called *similar*. This is an important concept and is studied more deeply in courses in linear algebra.

We want to use the concept to help in describing the geometric structure of the solutions of linear homogenous 2×2 systems with constant coefficients.

Let us begin by considering the special system

$$\begin{aligned}\dot{x} &= r_1 x\\ \dot{y} &= r_2 y.\end{aligned}$$

We have the general solution

$$\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix} c_1 e^{r_1 t}\\ c_2 e^{r_2 t}\end{pmatrix}.$$

If either c_1 or c_2 are 0, then the solution is on a vertical or horizontal line, so we assume that both c_1 and c_2 are not 0.

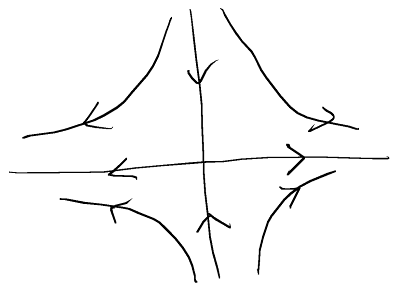
We have

$$\begin{aligned} y &= c_2 e^{r_2 t}\\ &= c_2 \frac{c_1^{\frac{r_2}{r_1}}}{c_1^{\frac{r_2}{r_1}}} e^{r_2 t}\\ &= c_2 \frac{c_1^{\frac{r_2}{r_1}}}{c_1^{\frac{r_2}{r_1}}} (e^{r_1 t})^{\frac{r_2}{r_1}}\\ &= \frac{c_2}{c_1^{\frac{r_2}{r_1}}} (x^{\frac{r_2}{r_1}})\\ &= A x^{\alpha}\end{aligned}$$

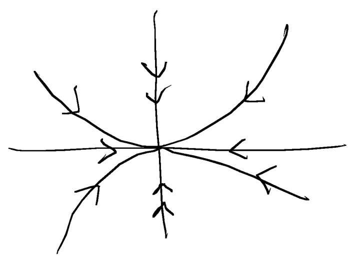
where A is a constant depending of c_1, c_2, and $\alpha = \frac{r_2}{r_1}$. We draw some of these curves for various r_1, r_2.

Examples

1) $\dot{x} = 2x$
$\dot{y} = -3y$



2) $\dot{x} = -x$
$\dot{y} = -2y$



3) $\dot{x} = 2x$
$\dot{y} = 2y$