Now, consider a matrix A with distinct real eigenvalues r_1, r_2.

Let $\mathbf{v}_1,\ \mathbf{v}_2$ be eigenvectors associated to r_1, r_2, repectively.

We claim that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

If not, then there is a constant α such that $\mathbf{v}_1 = \alpha\mathbf{v}_2$.

So,

$$A\mathbf{v}_1 = r_1\mathbf{v}_1,$$

and

$$\begin{aligned} A\mathbf{v}_1 &= A(\alpha\mathbf{v}_2) \\ &= \alpha A\mathbf{v}_2 \\ &= \alpha r_2\mathbf{v}_2 \\ &= r_2\alpha\mathbf{v}_2 \\ &= r_2\mathbf{v}_1. \end{aligned}$$

This gives

$$r_1\mathbf{v}_1 = r_2\mathbf{v}_1,$$

and, since $\mathbf{v}_1 \neq \mathbf{0}$, we have $r_1 = r_2$ which contradicts the assumption that $r_1 \neq r_2$.

Hence, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Let Q be the linear change of coordinates defined by

$$Q(\mathbf{e}_1) = \mathbf{v}_1, \; Q(\mathbf{e}_2) = \mathbf{v}_2.$$

Letting

$$diag(r_1, r_2) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix},$$

it follows that

$$AQ = Qdiag(r_1, r_2),$$

or

$$Q^{-1}AQ = diag(r_1, r_2).$$

It then follows that the change of coordinates $\mathbf{x} = Q\mathbf{u}$ transforms the system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

into the system

$$\dot{\mathbf{u}} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \mathbf{u}. \tag{2}$$

Since we know how to draw the solutions of (1), we get that the solutions of (2) are then obtained by a linear change of coordinates.

We illustrate this will several examples.

We also mention how to sketch solutions in the cases in which $r_1 = r_2$ or $r_1 = \alpha + i\beta$ with $\beta \neq 0$.

Some Techniques for graphing solution curves.

Summary: $\dot{x} = a_{11}x + a_{12}y$

$\dot{y} = a_{21}x + a_{22}y$

matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

char poly: $r^2 - \mathrm{tr}(A)\,r + \det(A)$

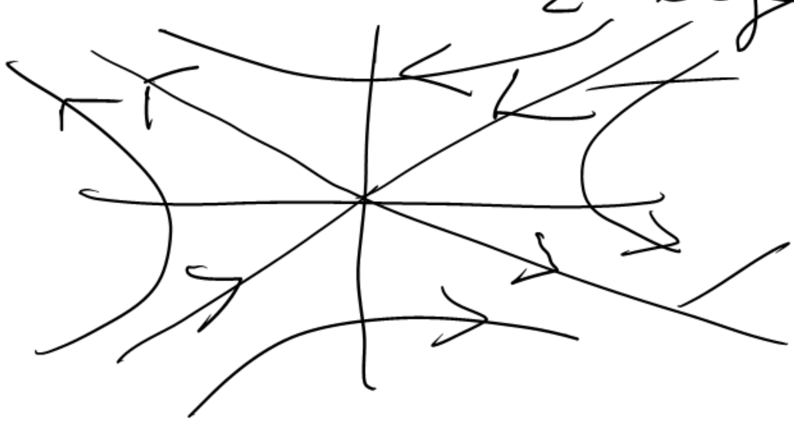
roots r_1, r_2

Case 1: $r_1 \neq r_2$, real.

(a) $r_1 > 0 > r_2$ saddle

Let $v_1 = r_1$-eigenvector

$v_2 = r_2$-eigenvector



Note: directions of v_1, v_2 could be any two different lines through $(0,0)$

- We will see how to make things more accurate below

Case 2: $r_1 < r_2 < 0$ — sink

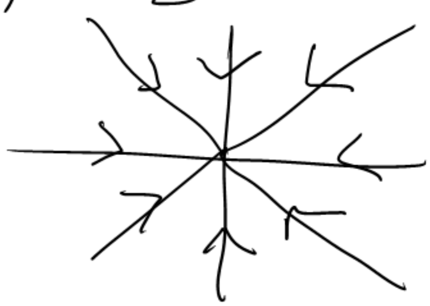
$v_1 \swarrow \quad \searrow v_2$ — stable node

v_1 – strong

solutions tangent to weak

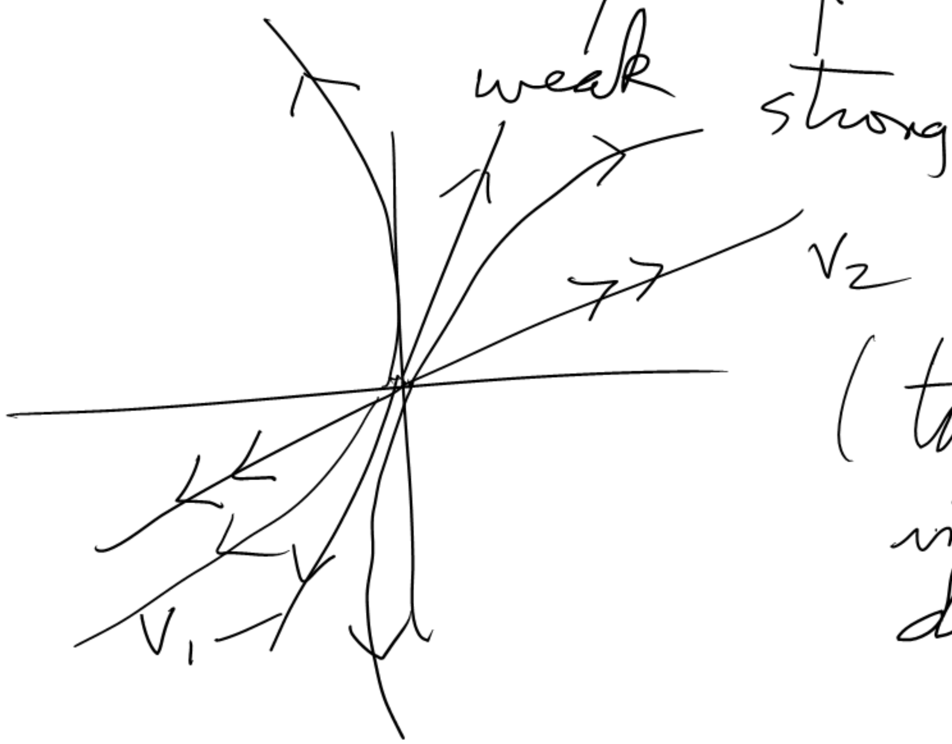
weak

Case 3: $r_1 = r_2 < 0$



— sink

Case 4: $0 < r_1 < r_2$ – source – unstable node

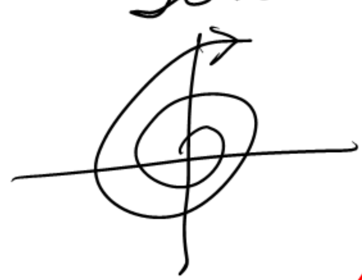


(turn Case 2 in opposite direction)

– Case 5: $0 < r_1 = r_2$ – similar (but opposite direction) to Case 5

– Case 6: $r_1 = a + bi$, $b \neq 0$

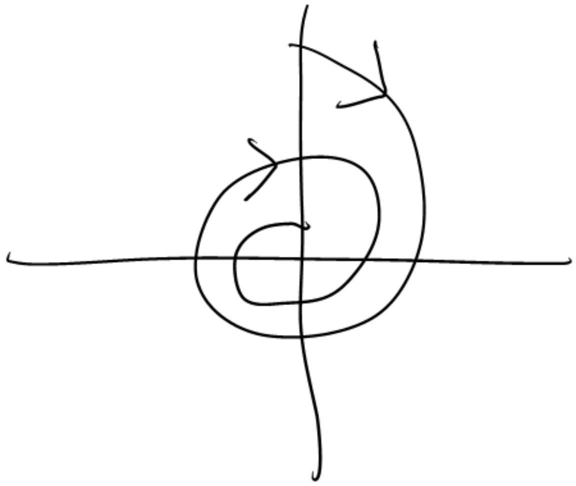
$a > 0$ – spiral source

either  or 

(describe difference below in examples)

Case 7: $r_1 = a + bi$, $b \neq 0$

$a < 0$ — spiral sink



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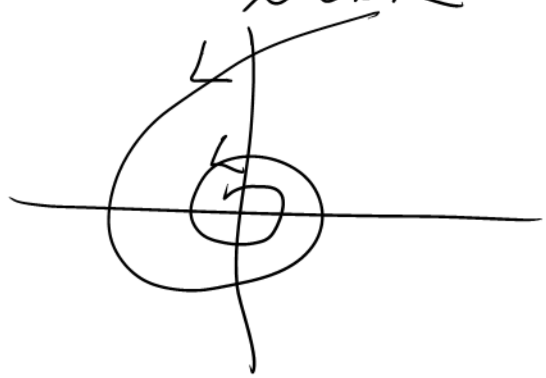


(describe difference below in examples)

Consider the system

$$\dot{x} = 2x - y$$
$$\dot{y} = -x + y$$

The solutions are curves in the plane $(x(t), y(t))$ and $(\dot{x}(t), \dot{y}(t))$ is the tangent to the curve at the point $(x(t), y(t))$

We know how to do the analytic solution

---

1) Find characteristic polynomial

matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ $\quad r^2 - 3r + 1$

roots, $r_1 = \frac{3+\sqrt{5}}{2}$, $r_2 = \frac{3-\sqrt{5}}{2}$

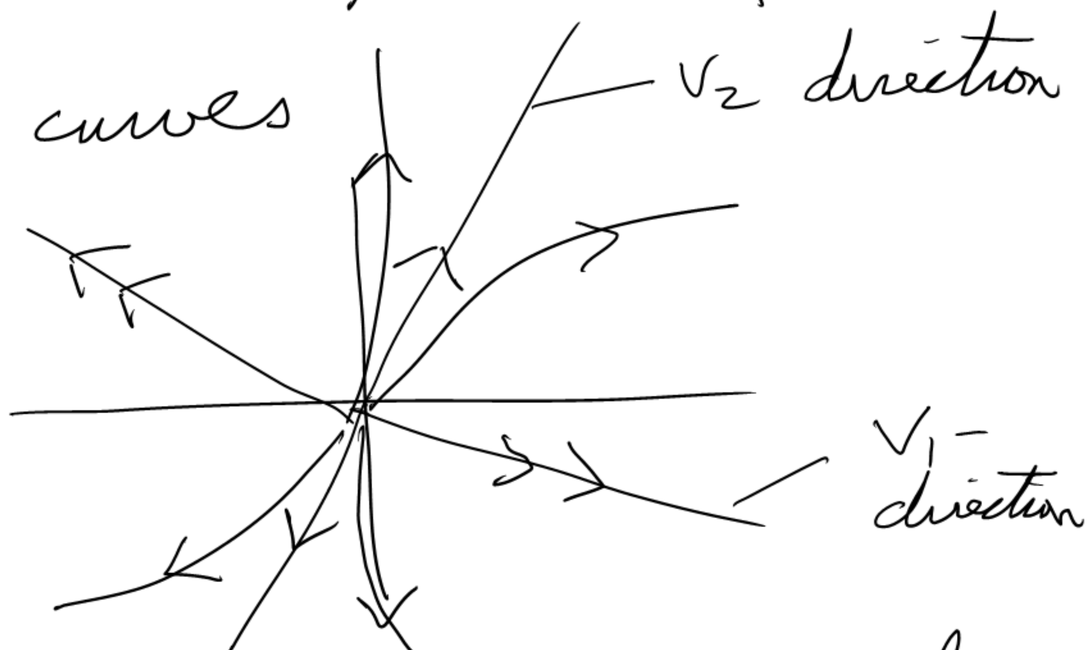
eigenvectors $v_1 = \begin{pmatrix} 1 \\ \frac{r_1 - 2}{-1} \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ \frac{r_2-2}{-1} \end{pmatrix}$

Using $\sqrt{5} \simeq 2.2$, we get

$r_1 \simeq 2.6$, $r_2 \simeq 0.25$

$v_1 \simeq \begin{pmatrix} 1 \\ -0.6 \end{pmatrix}$, $v_2 \simeq \begin{pmatrix} 1 \\ 1.9 \end{pmatrix}$

Solution curves



- When two positive eigenvalues, solutions go out and are tangent to the weakest expansion.

2) $\dot{x} = 2x - y$ Matrix $A = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$

$\dot{y} = -3x + y$

- charpoly: $r^2 - 3r - 1$

$r_1 = \frac{3+\sqrt{13}}{2}$, $r_2 = \frac{3-\sqrt{13}}{2}$

$\approx \frac{3+3.5}{2}$ $\approx \frac{3-3.5}{2}$

≈ 3.2 ≈ -0.25

eigenvectors: $v_1 = \begin{pmatrix} 1 \\ \frac{r_1 - 2}{-1} \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ \frac{r_2 - 2}{-1} \end{pmatrix}$

$\approx \begin{pmatrix} 1 \\ -1.2 \end{pmatrix}$ $\approx \begin{pmatrix} 1 \\ 2.25 \end{pmatrix}$

direction of v_2

direction of v_1

- saddle

3) $\dot{x} = x - y$ $\quad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\dot{y} = x + y$,

char poly: $r^2 - 2r + 2$

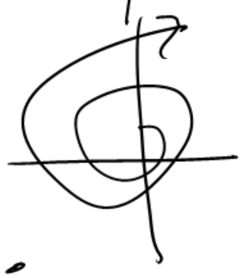
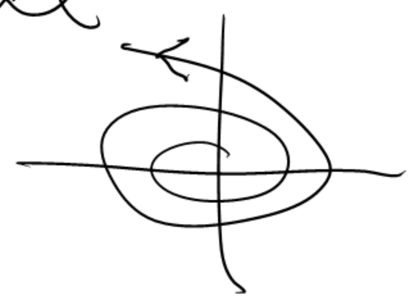
$r_1 = \frac{2 + \sqrt{-4}}{2}$, $r_2 = \frac{2 - \sqrt{-4}}{2}$

$= 1 + i$, $1 - i$

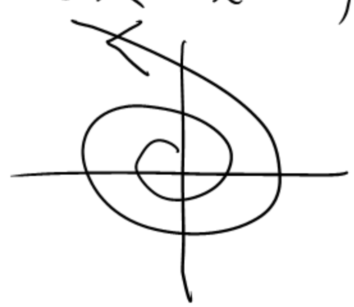
real part > 0

complex eigenvalue

$\Longrightarrow$ outward spiral

– either  or 

Look at $\dot{x}$ on $x = 0$, $y > 0$, $\dot{x} < 0$

$\Longrightarrow$  – actual solutions

4) $\dot{x} = -x - y$ | $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

$\dot{y} = x - y$

charpoly: $r^2 + 2r + 2$

$r_1 = \dfrac{-2 + \sqrt{-4}}{2} = -1 + i$

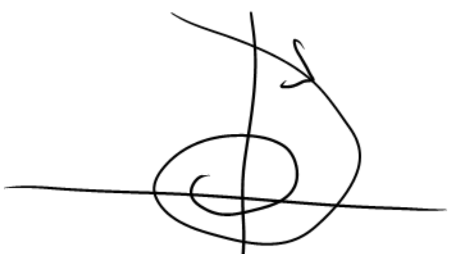
$r_2 = -1 - i$

not needed for picture

complex eigenvalue

negative real part

$\Longrightarrow$ inward spiral



I

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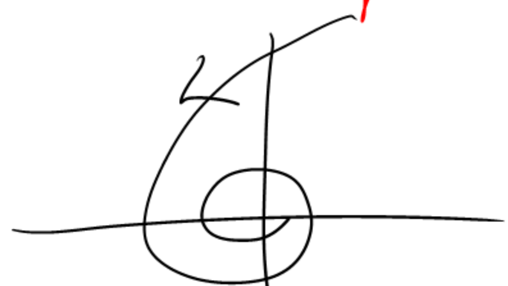


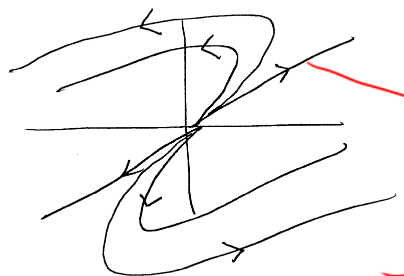
II

which? look at $\dot{x}$ on $\{x = 0, y > 0\}$

$\dot{x} < 0 \Longrightarrow \underline{\overline{II}}$

Typical picture of solutions with real eigenvalue r_1 of multiplicity two and one-dimensional eigenspace.

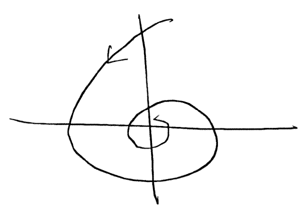
$r_1 > 0$



eigen-direction

$r_1 > 0 \Rightarrow$ outward

For $A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\alpha < 0$



or