## 17. Linear Homogeneous Systems with Constant Coefficients

Consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

where $A$ is a constant $n \times n$ matrix and $\mathbf{x}$ is an $n-$vector in $\mathbf{R}^n$.

In the one-dimensional (scalar case), we found solutions using exponential functions, so it seems reasonable to try to find a solution of the form

$$\mathbf{x}(t) = e^{rt}\xi$$

where $r$ is a real constant and $\xi$ is a non-zero constant vector.

Plugging in, we get

$$\dot{\mathbf{x}}(t) = re^{rt}\xi = Ae^{rt}\xi$$

for all $t$. Since $e^{rt}$ is never zero, we can cancel it and get

$$r\xi = A\xi$$

or

$$(rI - A)\xi = \mathbf{0}. \tag{2}$$

Thus, $r$ is a scalar such that there is a non-zero vector $\xi$ such that $\xi$ is a solution of the system of linear equations (2).

This only holds for special $r's$ and special $\xi's$.

**Definition.** Given an $n \times n$ matrix $A$, we call a number $r$ and *eigenvalue* of $A$ is there is a non-zero vector $\xi$ such that

$$A\xi = r\xi.$$

The vector $\xi$ is called an *eigenvector* for the eigenvalue $r$.

Note that $r$ is an eigenvalue of $A$ if and only if $det(rI - A) = 0$. The function $z(r) = det(rI - A)$ is a polynomial of degree $n$ in $r$ and is called the **characteristic polynomial** of $A$. Thus, eigenvalues of $A$ are the roots of the characteristic polynomial of $A$.

Remark.

1. Some general facts about eigenvalues and eigenvectors.

   (a) Let $A$ be an $n \times n$ matrix and let $r_1$ be an eigenvalue of $A$. Let $\xi$ and $\eta$ be eigenvectors associated to $r_1$. Then, for arbitrary scalars $\alpha, \beta$, we have that $\alpha\xi + \beta\eta$ is also an eigenvector associated to $r_1$ provided that it is not the zero vector.

Proof.

Let $\mathbf{v} = \alpha\xi + \beta\eta$ and assume this is not $\mathbf{0}$.

We have

$$\begin{aligned}
A(\mathbf{v}) &= A(\alpha\xi + \beta\eta) \\
&= \alpha A\xi + \beta A\eta \\
&= \alpha r_1\xi + \beta r_1\eta \\
&= r_1(\alpha\xi + \beta\eta) \\
&= r_1\mathbf{v}
\end{aligned}$$

Therefore $\mathbf{v}$ is also an eigenvector as required.

(b) Let $r_1 \neq r_2$ be distinct eigenvalues of $A$ with associated eigenvectors $\xi, \eta$, respectively. Then, $\xi$ is not a multiple of $\eta$.

Proof.

Assume that $\xi = \alpha\eta$ for some $\alpha$. Since both vectors are not $\mathbf{0}$, we must have $\alpha \neq 0$.

Now,

$$\begin{aligned}
A\xi &= r_1\xi \\
&= r_1\alpha\eta,
\end{aligned}$$

$$A\xi = A\alpha\eta = \alpha A\eta = \alpha r_2\eta,$$

So,

$$r_1 \alpha \eta = r_2 \alpha \eta.$$

Since $\alpha \neq 0$, and $\eta \neq \mathbf{0}$, we get $r_1 = r_2$ which is a contradiction.

2. A real matrix may not have any real eigenvalues, but always has complex eigenvalues.

There is a simple formula for the characteristic polynomial of a $2 \times 2$ matrix.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then,

$$rI - A = \begin{pmatrix} r - a_{11} & -a_{12} \\ -a_{21} & r - a_{22} \end{pmatrix}$$

So,

$$
\begin{aligned}
det(rI - A) &= (r - a_{11})(r - a_{22}) - a_{12}a_{22} \\
&= r^2 - a_{11}r - a_{22}r + a_{11}a_{22} - a_{12}a_{21} \\
&= r^2 - tr(A)r + det(A)
\end{aligned}
$$

where we define $tr(A) = a_{11} + a_{22}$ and $det(A) = a_{11}a_{22} - a_{12}a_{21}$.

Let us find some characteristic polynomials and eigen-values for the following matrices.

**Example 1**

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

We have $z(r) = r^2 - 3r + 1$. $r = \frac{3 \pm \sqrt{5}}{2}$.

**Example 2**

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

We have $z(r) = r^2 - 5r + 7$. $r = \frac{3 \pm \sqrt{25-28}}{2} = \frac{3 \pm \sqrt{-3}}{2} = \frac{3 \pm 3i}{2}$.

Next, we compute the eigenvectors associated to the eigenvalues.

**Example 1a.**

Return to

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Let $r_1 = \frac{3 + \sqrt{5}}{2}$.

Let

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

be the associated eigenvector.

Then, we have

$$(r_1 - a_{11})\xi_1 - a_{12}\xi_2 = 0.$$

The other equation is not necessary since the matrix $(r_1 I - A)$ is singular.

Thus, we have the condition

$$\xi_2 = \frac{(r_1 - a_{11})\xi_1}{a_{12}}$$

for $\xi$ to be an eigenvector for $r_1$. We can take $\xi_1 = 1$, and get

$$\xi = \begin{pmatrix} 1 \\ \frac{r_1 - a_{11}}{a_{12}} \end{pmatrix}.$$

Similarly, for $r_2$, we get

$$\xi = \begin{pmatrix} 1 \\ \frac{r_2 - a_{11}}{a_{12}} \end{pmatrix}.$$

**Example 2a.**

In this case, we have complex eigenvalues, so there is no real eigenvector. Let $r_1 = \alpha + i\beta$ and $r_2 = \alpha - \beta$ be the two roots. We can get complex eigenvectors for $r_1$ and $r_2$ in a manner analgous to that of Example 1a. For $r_1$ or $r_2$, we simply look for a complex vector

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

with complex numbers $\xi_1$ and $\xi_2$ such that

$$A\xi = r_1\xi,$$

and

$$A\xi = r_2\xi.$$

This gives the complex vector

$$\xi = \begin{pmatrix} 1 \\ \frac{r_1 - a_{11}}{a_{12}} \end{pmatrix}$$

for $r_1$, and the complex vector

$$\xi = \begin{pmatrix} 1 \\ \frac{r_2 - a_{11}}{a_{12}} \end{pmatrix}$$

for $r_2$.

We will see next how to use this for solving systems of two linear differential equations.

# 1 Two dimensional homogeneous systems of linear differential equations with constant coefficients

Consider the system

$$\begin{aligned} \dot{x} &= a_{11}\, x + a_{12}\, y \\ \dot{y} &= a_{21}\, x + a_{22}\, y \end{aligned}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a constant $2 \times 2$ real matrix.

We compute the eigenvalues $r_1, r_2$. These are the roots of the characteristic polynomial

$$r^2 - tr(A)r + det(A).$$

**Case 1:** Both roots are real and distinct. Say these are $r_1 \neq r_2$.

Step 1. Compute the eigenvectors $\mathbf{v}_1$ for $r_1$ and $\mathbf{v}_2$ for $r_2$, repectively.

Then, we get solutions of the form

$$\mathbf{x}_1 = e^{r_1 t}\mathbf{v}_1, \ \mathbf{x}_2 = e^{r_2 t}\mathbf{v}_2.$$

These turn out to be linearly independent, so the general solution is

$$\mathbf{x}(t) = \alpha_1 e^{r_1 t}\mathbf{v}_1 + \alpha_2 e^{r_2 t}\mathbf{v}_2$$

where $\alpha_1$ and $\alpha_2$ are constants.

**Case 2:** Both roots are real and equal. Say the common root is $r_1$.

We get one solution $\mathbf{x}_1(t)$ of the form

$$\mathbf{x}_1(t) = e^{r_1 t}\mathbf{v}_1$$

where $\mathbf{v}_1$ is an eigenvector for $r_1$.

Next, we have two subcases

**Subcase 2a:** There are two linearly independent eigenvectors, say $\xi, \eta$ for the eigenvalue $r_1$.

In this case the general solution is

$$\mathbf{x}(t) = e^{r_1 t}(\alpha_1 \xi + \alpha_2 \eta).$$

An example of this is the system

$$\dot{x} = 2x$$
$$\dot{y} = 2y$$

with general solution

$$\mathbf{x}(t) = e^{2t}\left(\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

**Subcase 2b:** All eigenvectors for $r_1$ are multiples of $\mathbf{v}_1$.

In this case we proceed as follows.

Let us try to find another linearly independent solution of the form

$$\mathbf{x}_2(t) = e^{r_1 t}\mathbf{v}_0 + te^{r_1 t}\mathbf{v}_1.$$

We get

$$\begin{aligned}
\dot{\mathbf{x}}_2(t) &= r_1 e^{r_1 t} \mathbf{v}_0 + t r_1 e^{r_1 t} \mathbf{v}_1 + e^{r_1 t} \mathbf{v}_1 \\
&= A(e^{r_1 t} \mathbf{v}_0 + t e^{r_1 t} \mathbf{v}_1),
\end{aligned}$$

or

$$r_1 \mathbf{v}_0 + t r_1 \mathbf{v}_1 + \mathbf{v}_1 = A(\mathbf{v}_0 + t \mathbf{v}_1).$$

Setting the constant and terms with $t$ equal we get that $\mathbf{v}_1$ is an eigenvector, and $\mathbf{v}_0$ satisfies the linear system

$$(A - r_1 I)\mathbf{v}_0 = \mathbf{v}_1. \tag{3}$$

Finding the solution $v_0$, we can in fact obtain a second linearly independent solution of the above form.

Thus, the general solution has the form

$$\mathbf{x}(t) = \alpha_1 e^{r_1 t} \mathbf{v}_1 + \alpha_2 e^{r_1 t}(\mathbf{v}_0 + t \mathbf{v}_1).$$

where $\mathbf{v}_1$ is an eigenvector associated to $r_1$ and $\mathbf{v}_0$ satisfies (3).

Note that this involves solving the two systems of equations

$$(A - r_1 I)\mathbf{v}_1 = \mathbf{0}, \ (\mathbf{A} - \mathbf{r_1 I})\mathbf{v_0} = \mathbf{v_1}.$$

**Example.**
Consider the system

$$\dot{x} = x$$
$$\dot{y} = x + y$$

The matrix is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

with characteristic equation

$$r^2 - 2r + 1 = (r-1)^2.$$

Hence, $r = 1$ is a root of multiplicity two.

We see that $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector so we get one non-zero solution as

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For the second independent solution we have

$$\mathbf{x}_1(t) = e^t \mathbf{v}_0 + t e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where

$$(A - I)\mathbf{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4}$$

The solutions to this last equation are all vectors of the form $\begin{pmatrix} 1 \\ \xi_2 \end{pmatrix}$ with $\xi_2$ arbitrary, so we can pick $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and get a second linearly independent solution as

$$\mathbf{x}_2(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t).$$

**Remark.** Note that we used the method above when there are not two linearly independent eigenvectors for the eigenvalue 1. We did not check whether this is the case, so why does this work? The answer is that if there were indeed two linearly independent eigenvectors for the eigenvalue 1, then the system (4) would not have had any solutions, so the fact that we could solve the system justifies the approach. (The proof of this requires more linear algebra and will have to be deferred to a more advanced course.)

This method generalizes to $n$ dimensional systems with eigenvalues of multiplicity greater than one although the linear algebra required is more complicated.

We will see that the method of elimination described below is more efficient for two dimensional systems with a multiple root.

**Case 3:** The roots are $\alpha \pm i\beta$ where $\beta \neq 0$.

Here we use complex variables. We have that $\mathbf{x}_c(t) = e^{(\alpha+i\beta)t}\xi$ is a complex solution where $\xi$ is a complex eigenvector associated to the eigenvalue $\alpha + i\beta$. The real and imaginary parts give linearly independent solutions. Then the general real solution is a linear combination of these independent solutions.

Let us do an example.

Consider the system

$$\begin{aligned} \dot{x} &= 2\,x - 3\,y \\ \dot{y} &= 2\,x + 4\,y \end{aligned}$$

The matrix $A$ is given by

$$\begin{pmatrix} 2 & -3 \\ 2 & 4 \end{pmatrix}.$$

The characterstic polynomial is

$$r^2 - 6r + 14$$

with roots

$$r = \frac{6 \pm \sqrt{36 - 56}}{2} = 3 \pm i\sqrt{5}.$$

We seek a complex eigenvector $\xi = (\xi_1, \xi_2)$ for the eigenvalue $r = 3 + i\sqrt{5}$.

We get the equation

$$(rI - A)\xi = \mathbf{0}.$$

or

$$\begin{pmatrix} r - 2 & 3 \\ -2 & r - 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because the matrix is singular, we need only consider the first row equation

$$(r - 2)\xi_1 + 3\xi_2 = 0$$

Setting $\xi_1 = 1$, we get

$$\begin{aligned} \xi_2 &= \frac{(2 - r)}{3} \\ &= \frac{2 - (3 + i\sqrt{5})}{3} \\ &= \frac{-1 - i\sqrt{5}}{3} \end{aligned}$$

Thus, we get a complex solution of the form

$$\begin{aligned} \mathbf{x}_c &= e^{(3+i\sqrt{5})t} \begin{pmatrix} 1 \\ \frac{-1-i\sqrt{5}}{3} \end{pmatrix} \\ &= e^{(3+i\sqrt{5})t} \left( \begin{pmatrix} 1 \\ \frac{-1}{3} \end{pmatrix} + i \begin{pmatrix} 0 \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \right) \end{aligned}$$

The real and imaginary parts are

$$R_1 \overset{\text{def}}{=} e^{3t}\left(\cos(\sqrt{5}t)\begin{pmatrix} 1 \\ \frac{-1}{3} \end{pmatrix} - sin(\sqrt{5}t)\begin{pmatrix} 0 \\ -\frac{\sqrt{5}}{3} \end{pmatrix}\right)$$

$$I_1 \overset{\text{def}}{=} e^{3t}\left(\sin(\sqrt{5}t)\begin{pmatrix} 1 \\ \frac{-1}{3} \end{pmatrix} + \cos(\sqrt{5}t)\begin{pmatrix} 0 \\ -\frac{\sqrt{5}}{3} \end{pmatrix}\right)$$

The general real solution is

$$\mathbf{x}(t) = \alpha_1 R_1 + \alpha_2 I_1.$$

## 2    An alternate method for 2 dimensional systems: Elimination and reduction to scalar equations

First, we give some examples to describe the elimination method to solve two dimensional systems.

**Example 1:**

Consider the system

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= x - y \end{aligned}$$

We can write

$$y = \dot{x} - x \tag{5}$$

from the first equation and substitute into the second equation getting

$$\dot{y} = \ddot{x} - \dot{x}$$
$$= x - (\dot{x} - x)$$

This gives the second order scalar equation for $x$

$$\ddot{x} - 2x = 0.$$

We know how to solve this. The characteristic equation is $r^2 - 2$ with roots $r = \pm\sqrt{2}$ and general solution

$$x(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}.$$

Then, we get $y$ from (5) as

$$y(t) = \dot{x} - x$$
$$= c_1\sqrt{2}e^{\sqrt{2}t} - c_2\sqrt{2}e^{-\sqrt{2}t} - c_1 e^{\sqrt{2}t} - c_2 e^{-\sqrt{2}t},$$

so, the general solution to the system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} \\ c_1\sqrt{2}e^{\sqrt{2}t} - c_2\sqrt{2}e^{-\sqrt{2}t} - c_1 e^{\sqrt{2}t} - c_2 e^{-\sqrt{2}t} \end{pmatrix}$$
$$= c_1 e^{\sqrt{2}t} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} + c_2 e^{-\sqrt{2}t} \begin{pmatrix} 1 \\ -\sqrt{2} - 1 \end{pmatrix}.$$

This method can most often be used for two dimensional homogeneous systems.

Which method is best?

In my opinion, this method is best when the eigenvalue is real of multiplicity two and the matrix method is best in the other cases.

**Example.**

Let us apply the method of elimination to the system

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= x + y \end{aligned}$$

we considered above.

We get

$$x = \dot{y} - y$$

$$\dot{x} = \ddot{y} - \dot{y} = x = \dot{y} - y.$$

or

$$\ddot{y} - 2\dot{y} + y = 0.$$

The general solution is

$$y(t) = c_1 e^t + c_2 t e^t.$$

This gives

$$
\begin{aligned}
x(t) &= \dot{y} - y \\
&= c_1 e^t + c_2 e^t + c_2 t e^t - (c_1 e^t + c_2 t e^t) \\
&= c_2 e^t
\end{aligned}
$$

and the general solution

$$
\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} c_2 e^t \\ c_1 e^t + c_2 t e^t \end{pmatrix} \\
&= c_1 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ t \end{pmatrix}.
\end{aligned}
$$

Finally, we describe some general aspects of the method of elimination.

We consider the system

$$
\begin{aligned}
x' &= ax + by \\
y' &= cx + dy
\end{aligned}
$$

We assume, that either $b$ or $c$ is not $0$. Otherwise, the system is diagonal and easily solvable.

Assuming $b \neq 0$, we use the elimination method to find a second order equation for $x(t)$. We will see that the characteristic polynomial for this second order equation is the same as the characteristic polynomial of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The latter characteristic polynomial is

$$r^2 - (a+d)r + ad - bc.$$

Now,

$$
\begin{aligned}
x'' &= ax' + by' \\
&= ax' + bcx + bdy \\
&= ax' + dx'((bc - ad)x
\end{aligned}
$$

or

$$x'' - (a+d)x' + ad - bc = 0.$$

Now, we can find the general solution $x(t)$ of this second order equation, and then get $y(t)$ in the original system from

$$y = \frac{x' - ax}{b}$$

If, $b = 0$ but $c \neq 0$, we use the elimination method to find a second order equation for $y(t)$. We get the general solution for $y(t)$, and then get $x(t)$ from

$$x = \frac{y' - dy}{c}.$$