1 Introduction

This course will cover basic material about ordinary differential equations.

Before we go into the course material itself, it is worth noting that there are many useful resources available on the internet for the study of differential equations. In particular, there are sample problem sets, sample tests, and useful software available. Among the most useful software are programs like Maxima, Mathematica, Maple, and Matlab. The first three of these programs enable one to solve almost all of the problems in the present course, so they can be very useful for checking answers.

Now we begin with the subject which will concern us in this course.

Differential Equations: equations involving functions and their derivatives.

The *order* of a differential equation is the highest order of derivative that occurs in the equation.

Examples:

1. $$f(t, y, y', \ldots, y^{(n)}) = 0.$$

   where $f : \mathbf{R}^{n+2} \to \mathbf{R}$ is a real-valued function of $n+2$ variables. Order $= n$

2. $$y'' + ty' + y^2 = 0$$

   order $= 2$

3. $$y' = (t^2 + 1)y$$

   order $= 1$

The above differential equations are called *scalar* differential equations because they involve a single equation involving a real-valued function and its derivatives.

We will later consider *systems* of differential equations. These involve more than one differential equation. For instance, an example of a two-dimensional system is the following.

$$\begin{aligned} x' &= x^2 + 2y + t^2 \\ y' &= 2x - y - sin(t) \end{aligned}$$

We begin our study with first order scalar differential equations

Consider

$$y' = f(t, y) \tag{1}$$

A solution to (1) is a differentiable function $t \to y(t)$ defined on a real interval $I = \{t : \alpha < t < \beta\}$ such that, for all $t \in I$, we have

$$y'(t) = f(t, y(t)).$$

Examples:

1. $y' = 2y$ solution $y(t) = ae^{2t}$ where a is an arbitrary constant

   Note that $y(0) = a$. Thus, we have a one-parameter family of solutions, and we get a particular solution by specifying the *value* at a single real number.

   If we know y at any fixed value, we can get the unique value of a by solving an algebraic equation.

   If $y(1) = 2$, then we get

$$\begin{aligned} y(t) &= ae^{2t} \\ y(1) &= 2 \\ ae^2 &= 2 \\ a &= \frac{2}{e^2} \end{aligned}$$

2. $y = t^2 sin(t)$
   is a solution of $y' = \frac{2y}{t} + y\ cot(t)$

3. $y = a\ cos(t) + b\ sin(t)$ is solution to
   $y'' = -y$.

This is a second order equation, and we usually need two conditions to obtain a unique solution.

These can be expressed as

- initial conditions: $y(0) = y_0, y'(0) = y_1$, or as
- boundary value conditions: $y(0) = 1, y(2\pi) = 2$.
- *any* two initial values for $y(0), y'(0)$ or boundary conditions for $y(\alpha), y(\beta), \alpha \neq \beta$ will suffice to uniquely determine solutions

Fact: For an $n-th$ order scalar differential equation, one typically needs n conditions to uniquely express the solution.

Given a differential equation (1) as above, the *general solution* to (1) is an expression $y(t, c)$ involving an arbitrary constant c such that each function

$y_c(t) = y(t, c)$ with c given a specific value is a solution of (1), and any solution has this form for *some* c.

The constant c is determined by initial conditions. Many examples will follow.

We frequently write the pair of equations

$$y' = f(t, y), \; y(t_0) = y_0$$

which we call an *inital value problem*.

The following important theorem states that, under mild conditions, an initial value problem has a unique solution.

Theorem (Existence-Uniqueness Theorem) *Let $f(t, y)$ be a C^1 function of the variables (t, y) defined in an open set D in the plane $\mathbf{R}^2$. Then, for each $(t_0, y_0) \in D$ there is a unique solution to the initial value problem*

$$y' = f(t, y), y(t_0) = y_0$$

Most of this course consists of

1. learning how to find solutions to various differential equations, and

2. studying applications of certain differential equations

Direction Fields

It will turn out that it is difficult to solve many differential equations.

Some insight can be gained by the method of *direction fields*

Consider the equation $y' = f(t, y)$.

and the curve $(t, y(t))$ where $y(\cdot)$ is a solution

At the point $(t, y(t)$ in the plane, the number $f(t, y(t))$ is the slope of the tangent line.

Thus, if we draw a small line segment at in the direction of $f(t, y)$ at (t, y) and *fit* these together, we get approximations to the solution to (1).

Frequently, we can obtain information about limiting behavior of solutions as $t \to \infty$ is this way.

Examples.

We use the program Maxima to generate direction fields for various d.e.'s

1. $y' = y$

   Note that we can determine that solutions $y(t)$ with $y(0) > 0$ approach ∞ as $t \to \infty$ while those with $y(0) < 0$ approach $-\infty$ as $t \to \infty$.

2. The other d.e.'s are $y' = \frac{3-y}{2}$, $y' = 1 - 2ty$, $y' = \frac{2e^{-t}+y}{2}$. Again various behaviors as $t \to \infty$ can be determined.

These are shown in the next figures

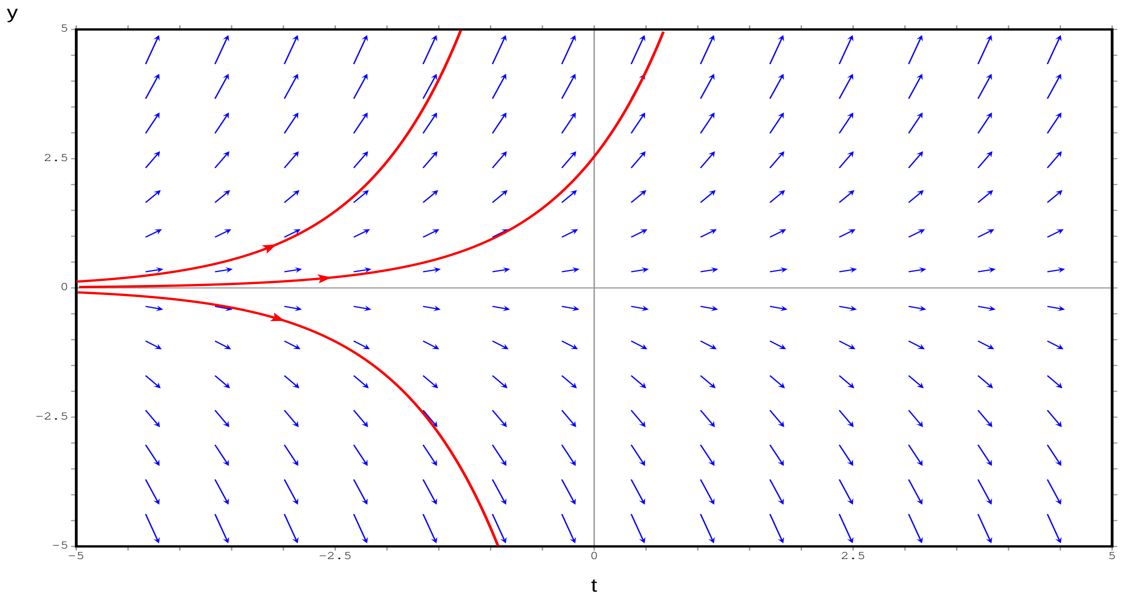

Figure 1: $y' = y$, $t \in [-1, 1]$, $y \in [-1, 1]$

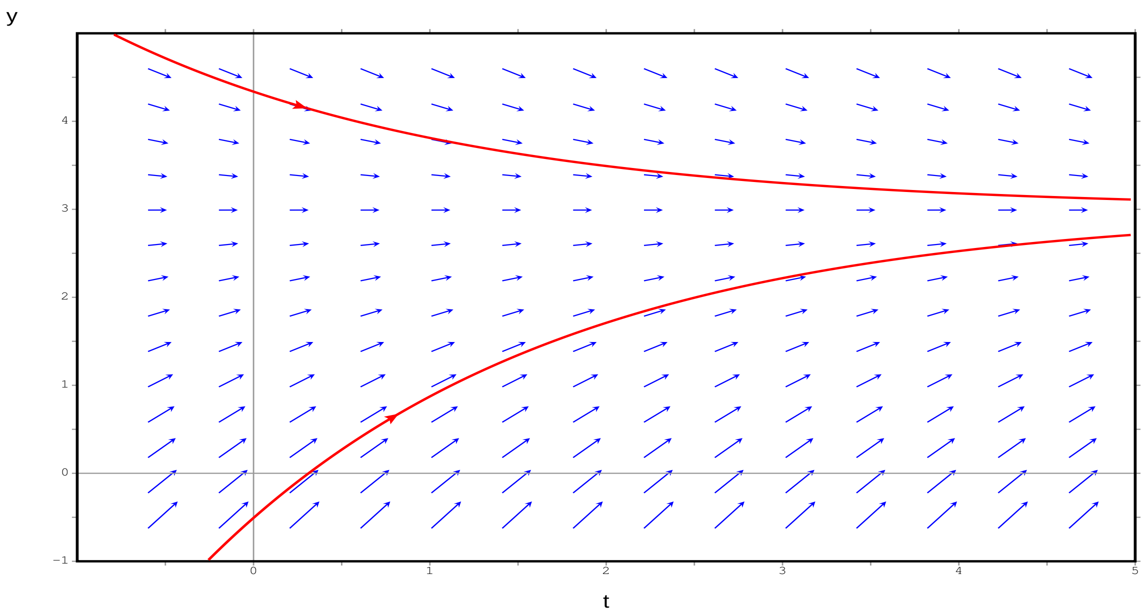

Figure 2: $y' = \frac{3-y}{2}$, $t \in [-1, 5]$, $y \in [-1, 5]$

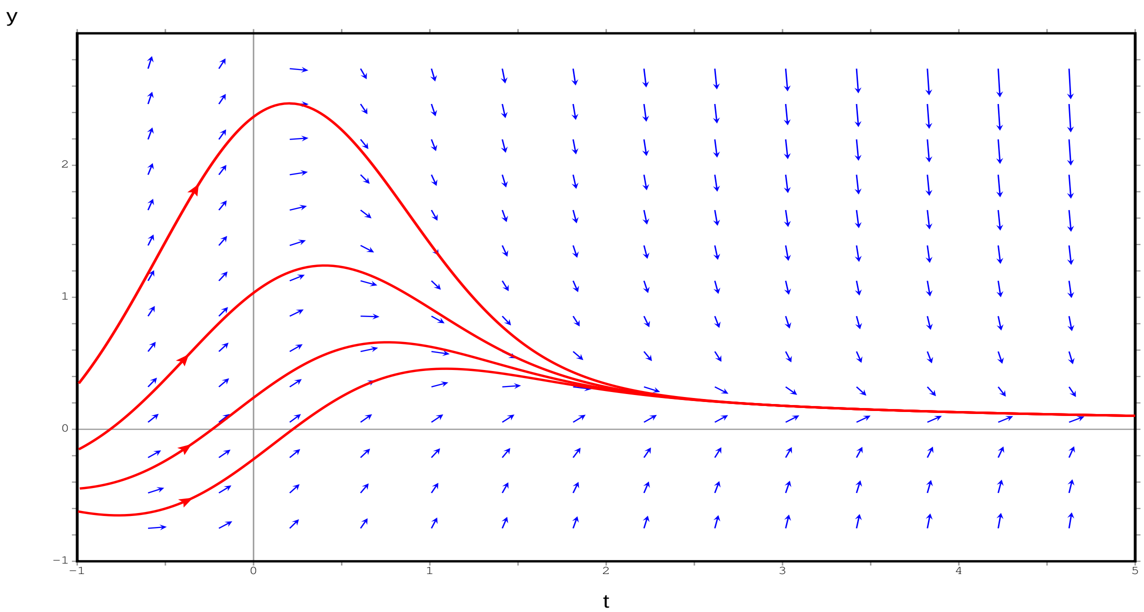

Figure 3: $y' = 1 - 2ty$, $t \in [-1, 5]$, $y \in [-1, 3]$

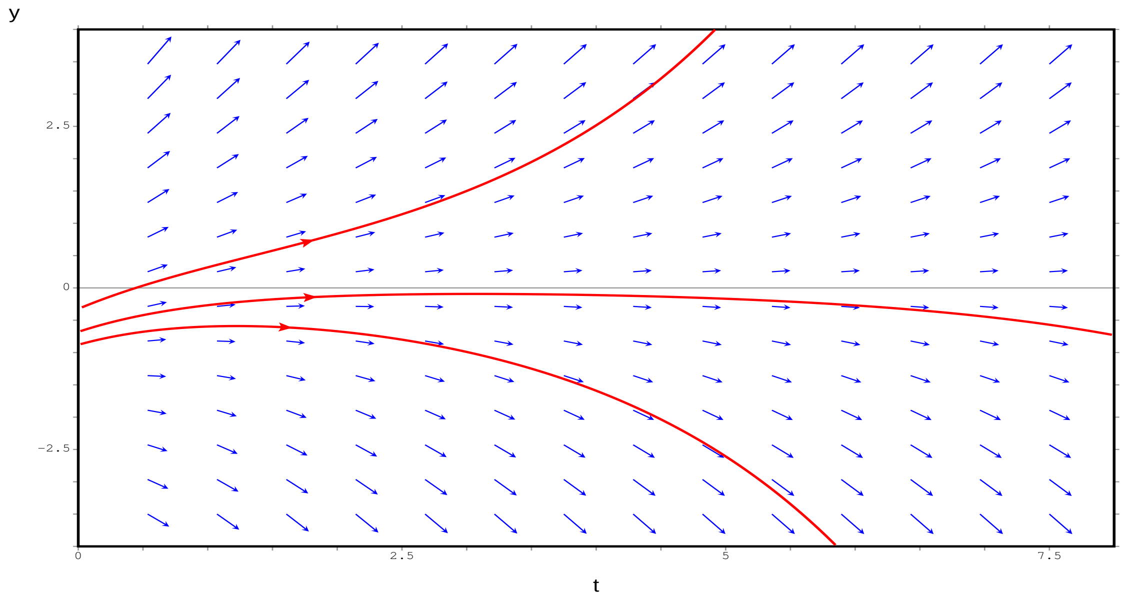

Figure 4: $y' = \frac{2e^{-t}+y}{2}$, $t \in [0,8]$, $y \in [-4,4]$