

Higher dimensional linear homogeneous systems with constant coefficients

To get the explicit form of the solutions of higher dimensional linear systems with constant coefficients requires methods of linear algebra which are beyond the scope of this course. Basically, one has to compute what is called the Jordan canonical form of the matrix.

We will discuss the methods one can use here, but we will not provide the rigorous justification.

Consider the system

$$\mathbf{x}' = A\mathbf{x}.$$

First, one finds the eigenvalues r_1, r_2, \dots, r_s of the matrix A . We will consider only real eigenvalues. When there are complex eigenvalues one proceeds in a similar way to get complex solutions and then one takes the real and imaginary parts (taking into account multiplicities) as described above in the two dimensional case.

So, assume that all eigenvalues of A are real.

It turns out that every vector \mathbf{v} in \mathbf{R}^n can be uniquely expressed as a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_s \mathbf{v}_s$$

where each \mathbf{v}_i satisfies

$$(A - r_i I)^m \mathbf{v}_i = \mathbf{0}$$

for some positive integer m .

In linear algebra terms, the vectors \mathbf{v}_i are called generalized eigenvectors of A , and the result just mentioned is the statement that \mathbf{R}^n is the direct sum of the generalized eigenspaces of A .

It then turns out that it is enough to find solutions in each generalized eigenspace separately.

So, we assume that r is a single real eigenvalue for A , and that V is the generalized eigenspace for r ; i.e., V is the set of vectors \mathbf{v} such that there is some $m > 0$ such that $(A - rI)^m \mathbf{v} = \mathbf{0}$.

Procedure to find all solutions in V :

First find a maximal set of linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_\ell$ for the eigenvalue r . Once one knows r this simply involves solving systems of linear equations.

One gets solutions of the form

$$\mathbf{x}(t) = e^{rt} \sum_{k=0}^{m_i} \frac{t^k}{k!} \mathbf{v}_{i,k}$$

where

$$\mathbf{v}_{i,k+1} = (A - rI)\mathbf{v}_{i,k}$$

and

$$\mathbf{v}_{i,m_i} = \mathbf{v}_i$$

We find the $\mathbf{v}_{i,k}$ one at a time solving systems of linear equations. Thus, we find \mathbf{v}_{i,m_i-1} as a solution ξ to

$$(A - rI)\xi = \mathbf{v}_i.$$

We repeat to find \mathbf{v}_{i,m_i-2} as a solution ξ to

$$(A - rI)\xi = \mathbf{v}_{i,m_i-1},$$

and we continue.

We do this as long as we can find solutions of the systems of linear equations. The process will stop in q_i steps where q_i is no larger than the multiplicity of the eigenvalue r as a root of the characteristic polynomial of A .

All solutions obtained in this way using all the eigenvectors associated to r will give a set of linearly independent solutions in the generalized eigenspace of r .

As we have said above, this procedure is finding the Jordan canonical form for the matrix A on the generalized eigenspace of r . We refer to linear algebra textbooks (or more advanced differential equations textbooks) for more details on this.