

# A) Matrix Methods for Systems of ODE's.

Consider the ODE in  $\mathbb{R}^n$  given by

$$\underline{\dot{x}} = A\underline{x} \quad (1)$$

where  $A$  is an  $n \times n$  real or complex matrix and

$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  is an  $n \times 1$  matrix (or vector) in  $\mathbb{R}^n$  or  $\mathbb{C}^n$

We also consider the associated IVP

given by  $\underline{\dot{x}} = A\underline{x}$ ,  $\underline{x}(0) = \underline{a}$ ,  $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

The general solution to (1)

is an expression

$$\underline{x}(t) = c_1 \underline{x}_1(t) + \dots + c_n \underline{x}_n(t) \quad (2)$$

where  $\{\underline{x}_1(t), \dots, \underline{x}_n(t)\}$  is a set of  $n$  linearly independent solutions and  $c_1, \dots, c_n$  are constants

The general solution can be written in matrix form as

$$\underline{x}(t) = \underline{\Phi}(t) \underline{c}$$

with  $\underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ ,  $\underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ ,  $\underline{\Phi}(t) = \begin{bmatrix} \underline{x}_1(t) & & \\ & \ddots & \\ & & \underline{x}_n(t) \end{bmatrix}$

with  $\underline{x}_i(t)$  equal to the  $i$ -th column of  $\underline{\Phi}(t)$

Here  $\underline{\Phi}(t)$  is called a fundamental matrix for (1)

B) The 2-dim case - Methods of finding independent solutions  
- Solving IVP's.

Consider  $\underline{\dot{x}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underline{x}$ ,  $\underline{x}(0) = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}$

Step 1. Find eigenvalues and eigenvectors of  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

characteristic polynomial of  $A$

$$z(r) = r^2 - (a+d)r + ad - bc$$

let  $r_1, r_2$  be the roots (eigenvalues of  $A$ )

Case 1:  $r_1 > r_2$ , both real,  $b \neq 0$

We call  $r_1$  the first eigenvalue

$r_2$  the 2<sup>nd</sup> eigenvalue

Associated eigenvectors,  $r_1 \leftrightarrow \begin{pmatrix} 1 \\ r_1 - a \\ b \end{pmatrix} = \underline{v}_1$

$r_2 \leftrightarrow \begin{pmatrix} 1 \\ r_2 - a \\ b \end{pmatrix} = \underline{v}_2$

Get two linearly independent solutions

$$\underline{x}_1(t) = e^{r_1 t} \underline{v}_1, \quad \underline{x}_2(t) = e^{r_2 t} \underline{v}_2$$

first fundamental solution

2<sup>nd</sup> fundamental solution

General solution  $\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t)$

$$= c_1 e^{r_1 t} \underline{v}_1 + c_2 e^{r_2 t} \underline{v}_2$$

Writing  $\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ ,  $\underline{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ ,  $\underline{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$

$x_1(t) = x_1 = c_1 e^{r_1 t} v_{11} + c_2 e^{r_2 t} v_{21}$  (real form)  
 $x_2(t) = x_2 = c_1 e^{r_1 t} v_{12} + c_2 e^{r_2 t} v_{22}$  of solution)

Example 1 Solve the IVP

$$\underline{x}' = \begin{pmatrix} 3 & 8 \\ -1 & -6 \end{pmatrix} \underline{x}, \quad \underline{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

char poly.  $r^2 + 3r - 10 = (r+5)(r-2)$   
 $r = -5, 2$ ,  $r_1 = 2$ ,  $r_2 = -5$

$$\underline{x}_1(t) = e^{2t} \underline{v}_1, \quad \underline{x}_2(t) = e^{-5t} \underline{v}_2$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2-3}{8} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{8} \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ \frac{-8}{8} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

First fund soln:  $\underline{x}_1(t) = e^{2t} \begin{pmatrix} 1 \\ -\frac{1}{8} \end{pmatrix}$ , 2<sup>nd</sup> fund soln  
 $\underline{x}_2(t) = e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Gen soln:  $\underline{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -\frac{1}{8} \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

IVP conditions:  $\underline{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

plug  $t=0$  into  $\underline{x}(0)$  to get

$$\begin{aligned} c_1 + c_2 &= 3 \\ -\frac{c_1}{8} - c_2 &= 4 \end{aligned}$$

$$W = \begin{vmatrix} 1 & 1 \\ -\frac{1}{8} & -1 \end{vmatrix} = -1 + \frac{1}{8} = -\frac{7}{8}, \quad c_1 = \frac{\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}}{W} = 8, \quad c_2 = \frac{\begin{vmatrix} 1 & 3 \\ -\frac{1}{8} & 4 \end{vmatrix}}{W} = -5$$

Solution  $\underline{x}(t) = 8e^{2t} \begin{pmatrix} 1 \\ -\frac{1}{8} \end{pmatrix} - 5e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

real form:  $x_1 = 8e^{2t} - 5e^{-5t}$

$x_2 = -e^{2t} + 5e^{-5t}$

Example 2  $\underline{x}' = \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix} \underline{x}$ ,  $\underline{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

char poly  $r^2 - 8r + 17$ ,  $r = \frac{8 \pm \sqrt{64 - 68}}{2} = 4 \pm i$

$r_1 = 4 + i$ ,  $r_2 = 4 - i$

independent complex solutions:

$$\begin{aligned} x_{c1}(t) &= e^{(4+i)t} \begin{pmatrix} 1 \\ \frac{4+i-4}{-1} \end{pmatrix} & x_{c2}(t) &= e^{(4-i)t} \begin{pmatrix} 1 \\ \frac{4-i-4}{-1} \end{pmatrix} \\ &= e^{(4+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} & &= e^{(4-i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} \end{aligned}$$

First fundamental soln =  $\text{Re} \left( e^{(4+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)$

$x_1(t) = \text{Re} \left( e^{4t} e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)$

$= e^{4t} (\cos(t) + i \sin(t)) \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]$

$= e^{4t} \left[ \cos(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]$

2<sup>nd</sup> fund soln:  $x_2(t) = \text{Im} \left( e^{4t} e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)$

$= e^{4t} \left[ \sin(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos(t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]$

Gen soln:  $\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t)$

$$\underline{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow c_1 = 3, c_2 = -4$$

Ans:  $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{4t} (3 \cos(t) - 4 \sin(t)) \\ e^{4t} (3 \sin(t) + 4 \cos(t)) \end{pmatrix}$

real form  $\begin{cases} x_1 = e^{4t} (3 \cos(t) - 4 \sin(t)) \\ x_2 = e^{4t} (3 \sin(t) + 4 \cos(t)) \end{cases}$

Case 2:  $r_1 > r_2, b = 0, c \neq 0$

$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ , eigenvector for  $r_1$ ,

$$(A - r_1 I) \underline{v} = \underline{0}, \underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a - r_1 & 0 \\ c & d - r_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Use 2<sup>nd</sup> row of matrix equation

$$c v_1 + (d - r_1) v_2 = 0$$

Set  $v_2 = 1, c v_1 = r_1 - d, v_1 = \frac{r_1 - d}{c}$

So, can use  $\underline{v} = \begin{pmatrix} \frac{r_1 - d}{c} \\ 1 \end{pmatrix}$

First fund soln:  $x_1(t) = e^{r_1 t} \begin{pmatrix} \frac{r_1 - d}{c} \\ 1 \end{pmatrix}$

2<sup>nd</sup> fund soln:  $x_2(t) = e^{r_2 t} \begin{pmatrix} \frac{r_2 - d}{c} \\ 1 \end{pmatrix}$

Case 3:  $r_1 = r_2$  (both real)

Case 3a:  $b = c = 0$ ,  $\dot{x}_1 = r_1 x_1$   
 $\dot{x}_2 = r_1 x_2$

Gen soln:  $x_1(t) = e^{r_1 t} c_1$   
 $x_2(t) = e^{r_2 t} c_2$

as vector eqn:

$$\underline{x}(t) = c_1 e^{r_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{r_1 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Case 3b:  $b \neq 0$

Try to find soln of form

$$\underline{x}(t) = e^{rt} \underline{w} + t e^{rt} \underline{v}, \quad \underline{v} \neq \underline{0}, \underline{w} \neq \underline{0}$$

Get  $\dot{\underline{x}} = r e^{rt} \underline{w} + e^{rt} \underline{v} + r t e^{rt} \underline{v}$   
 $= A e^{rt} \underline{w} + t e^{rt} A \underline{v}$

$$\Rightarrow r \underline{w} + \underline{v} = A \underline{w}, \quad A \underline{v} = r \underline{v},$$
$$(A - rI) \underline{w} = \underline{v}, \quad (A - rI) \underline{v} = \underline{0}$$

So,  $r = \text{eigenval}$  and  $\underline{v} = \text{eigvec assoc with } r_1$   
 $= r_1 \text{ here}$

Take  $\underline{v} = \begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix}$

let  $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{pmatrix} (a - r_1)w_1 + b w_2 \\ c w_1 + (d - r_1)w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix}$$

or  $(a - r_1)w_1 + b w_2 = 1$

$$c w_1 + (d - r_1)w_2 = \frac{r_1 - a}{b}$$

Set  $w_1 = 1$ ,  $(a-r_1) + bw_2 = 1$ ,  $w_2 = \frac{r_1 - a + 1}{b}$   
- solves first equation

Need to show  $c + (d-r_1)\left(\frac{r_1 - a + 1}{b}\right) = \frac{r_1 - a}{b}$

Mult by  $b$ :  $cb + (d-r_1)(r_1 - a + 1) \stackrel{?}{=} r_1 - a$

$$cb + dr_1 - ad + d - r_1^2 + ar_1 - r_1 = r_1 - a$$
$$= \underbrace{-(r_1^2 - (a+d)r_1 + ad - bc)}_0 - r_1 + d = r_1 - a$$

$$\Leftrightarrow 2r_1 - (a+d) = 0$$

But  $r^2 - (a+d)r + ad - bc = (r - r_1)^2$   
 $\Rightarrow r_1$  is a root of  $2r - (a+d) = 0$

So OK.

---

If  $c \neq 0$ ,  $b = 0$ , get similar  
 $\underline{w}, \underline{v}$  by  $\underline{v} = \begin{pmatrix} \frac{r_1 - d}{c} \\ 1 \end{pmatrix}$

$$\underline{w} = \begin{pmatrix} \frac{r_1 - d + 1}{c} \\ 1 \end{pmatrix}$$

Example 3:

Solve the IVP

$$\underline{\dot{x}} = \begin{pmatrix} 8 & 4 \\ -9 & -4 \end{pmatrix} \underline{x}, \quad \underline{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

char poly  $r^2 - 4r + 4 = (r-2)^2$

$r_1 = 2$ , multiplicity 2

First fund soln:

$$\underline{x}_1(t) = e^{2t} \begin{pmatrix} 1 \\ \frac{r_1 - 8}{4} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$$

2nd fund soln:  $\underline{x}_2(t) = te^{2t} \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ \frac{r_1 - 8 + 1}{4} \end{pmatrix}$

$$= te^{2t} \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ -\frac{5}{4} \end{pmatrix}$$

Gen soln:  $\underline{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$

$$+ c_2 \left[ e^{2t} \begin{pmatrix} 1 \\ -\frac{5}{4} \end{pmatrix} + te^{2t} \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} \right]$$

$$\underline{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -\frac{5}{4} \end{pmatrix}$$

$$W = \begin{vmatrix} 1 & 1 \\ -\frac{3}{2} & -\frac{5}{4} \end{vmatrix} = -\frac{5}{4} + \frac{3}{2} = \frac{1}{4}$$

$$c_1 = \frac{\begin{vmatrix} 2 & 1 \\ 3 & -\frac{5}{4} \end{vmatrix}}{\frac{1}{4}} = \frac{-\frac{5}{2} - 3}{\frac{1}{4}} = -22,$$

$$c_2 = \frac{\begin{vmatrix} 1 & 2 \\ -\frac{3}{2} & 3 \end{vmatrix}}{\frac{1}{4}} = \frac{3+3}{\frac{1}{4}} = 24$$

## Second order equations as systems

Consider the second order IVP

$$\left. \begin{aligned} u'' + p(t)u' + q(t)u &= g(t) \\ u(0) &= u_0, u'(0) = u_0' \end{aligned} \right\} (1)$$

Write (1) as an equivalent system.

Set  $u' = v$

Get

$$u' = v$$

$$v' = -q(t)u - p(t)v + g(t)$$

$$u(0) = u_0$$

$$v(0) = u_0'$$

So

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

$$\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ u_0' \end{bmatrix}$$