12a. Review of Sequences

We review some concepts about sequences and series as preparation for the study of power series methods for differential equations.

Let k be an integer. The *terminal set* of integers starting at k is the set $k, k+1, k+2, \ldots$.

Examples of such terminal sets are

$$2, 3, 4, \ldots$$

$$-3, -2, -1, 0, 1, 2, 3, \ldots$$

$$0, 1, 2, 3, 4, \ldots$$

$$1, 2, 3, 4, 5, \ldots$$

A sequence of real numbers is a mapping from some terminal set, A, of integers into the real numbers. The set A is called the *index set* of the sequence.

If $A = \{k, k+1, k+2, \ldots\}$, then we usually write the sequence as

$$\{a_i : i \geq k, \}$$

$$a_k, a_{k+1}, a_{k+2}, \ldots$$

$$\{a_n\}_{n \geq k}$$

$$\{a_i\}_{i \geq k}$$

$$(a_i)_{i \geq k}$$

Logically, the sequence is really the mapping $i \to a_i$ where i varies in $\{k, k+1, k+2, \ldots\}$, but we usually leave this as understood, and don't mention it further.

Similarly, one discusses sequences of complex numbers, vectors in $\mathbf{R}^n$, functions, etc. In all cases these are simply mappings from some terminal set of integers into the corresponding set.

Note that if a_k, a_{k+1}, a_{k+2} is a sequence with terminal set $A = \{k, k+1, k+2, \ldots\}$, and $B = \{m, m+1, m+2, \ldots\}$ is some other terminal set, then we can always relable the indices so that the sequence uses B as its index set. We simply set $b_m = a_k,\ b_{m+1} = a_{k+1}, \ldots$. Thus, the particular choice of terminal set (or index set) is not important.

Definition. Let k be an integer, let $(a_n)_{n \geq k}$ be a sequence of complex numbers, and a be a complex number. We say that the sequence (a_i) *converges* to a if, for every $\epsilon > 0$ there is an index $N(\epsilon)$ such that if $n > N(\epsilon)$, then

$$\mid a_n - a \mid < \epsilon.$$

This concept is logically correct, but there are better words to describe its intuitive meaning.

Let us call ϵ a *prescribed error* and $N(\epsilon)$ its *associated index.*

Then, the definition can be rewritten as

> For every prescribed error ϵ there is an associated index $N(\epsilon)$ such that if n is beyond $N(\epsilon)$ in the index set, then a_n is closer to a than ϵ.

This means that as one goes further and further in the sequence, the terms bunch up around a. How close they bunch up is measured by the error ϵ and its associated index $N(\epsilon)$.

Note that, since real numbers are also complex numbers, the notion of convergence of complex numbers includes that of real numbers.

When (a_n) converges to a, we write

$$a_n \to a \text{ as } n \to \infty,$$

or

$$\lim_{n\to\infty} a_n = a$$

We also call a the *limit of the sequence* (a_n) .

Sometimes we are only interested in determining if a sequence converges and we do not care what the actually limit is.

So, we say that the sequence (a_n) *converges* if it converges to some a (but we don't necessarily specify what a is).

Examples

1. Let $a_n = \frac{1}{n}$ for $n \ge 1$. Then, $a_n \to 0$ as $n \to \infty$.
2. Let $a_n = (-1)^n$ for $n = 0, 1, 2, 3, \ldots$. Then (a_n) does not converge.

The following is a general fact concerning complex numbers.

Suppose that

$$\lim_n a_n = a, \ \lim_n b_n = b$$

and c is any complex number.

Then,

$$\lim_n (a_n + b_n) = a + b,$$

$$\lim_n (a_n b_n) = ab,$$

and

$$\lim_n c\ a_n = c\ a.$$

Also, if $b \neq 0$, then

$$\lim_n (a_n / b_n) = a/b$$

Further, if f is a continuous function from the complex numbers into the reals or complexes, then

$$f(\lim_n a_n) = \lim_n f(a_n)$$

In the case of real numbers, we can use the order property to get some interesting facts about convergence.

Let (a_n) be a sequence of real numbers. We say that (a_n) is *non-decreasing* if $a_n \le a_{n+1}$ for all n (in the index set of (a_n)). Similarly, we say that (a_n) is *non-inceasing* if $a_n \ge a_{n+1}$ for all n.

Definition. A subset B of the real numbers is *bounded above* if there is some real number K such that $x \le K$ for each $x \in B$. We call the number K an *upper bound* for B.

Let $B \subset \mathbf{R}$ be bounded above. The *least upper bound* for B is the unique real number α such that

1. α is an upper bound for B, and
2. if β is any upper bound for B, then $\alpha \le \beta$.

The least upper bound of B is called the *supremum* of B, and is denoted by

$$\sup B \text{ or } lub\ B.$$

It is a basic axiom of the real numbers that every non-empty subset $B \subset \mathbf{R}$ which is bounded above has a least upper bound.

If the set B is not bounded above, we define $\sup B = +\infty$. In this way every non-empty subset of $\mathbf{R}$ has a supremum. The supremum is in $\mathbf{R} \bigcup \{+\infty\}$.

In a similar way, we can define when a non-empty subset B of $\mathbf{R}$ is bounded below and define its *greatest lower bound*. This is usually called the *infimum* of B, and denoted

$$\inf B \text{ or } glb\ B.$$

If B is not bounded below, we define $\inf B = -\infty$.

The set $\mathbf{R} \bigcup \{-\infty, +\infty\}$ is usually called the set of *extended* real numbers.

Thus, any non-empty subset of $\mathbf{R}$ has a supremum or infimum which is an extended real number.

There are some other useful concepts for dealing with bounded sets of real numbers.

If $S = (a_n)_{n \ge 0}$ is a sequence, then a subsequence of S is a sequence $T = (b_i)_{i \ge 1}$ such that for every i there is an n_i such that $b_i = a_{n_i}$. Thus, we move along in the sequence S and choose elements for the sequence T keeping the order.

The following properties of the real numbers are of basic importance.

1. Every non-decreasing sequence of real numbers $a_1 \le a_2 \le \dots$ which is bounded above converges to its supremum.
2. Every non-increasing sequence of real numbers $a_1 \ge a_2 \ge \dots$ which is bounded below converges to its infimum.

Let $S = (a_n)$ be a sequence of real numbers.
For each n let

$$b_n = \sup_{k \geq n} a_k$$

Case 1: The set $\{a_n\}$ not bounded above.
In this case, then for each n, we have

$$b_n = +\infty$$

and we define

$$\limsup_{n \to \infty} a_n = +\infty = \inf_n \sup_{k \geq n} a_k$$

Case 2: The set $\{a_n\}$ is bounded above.
Then the sequence (b_n) is non-increasing. Hence, it converges to its infimum, and we define

$$\limsup_{n \to \infty} a_n = +\infty = \inf_n b_n = \lim_n b_n = \inf_n \sup_{k \geq n} a_k$$

We define

$$\liminf_{n \to \infty} a_k = \sup_n \inf_{k \geq n} a_k$$

It turns out that

$$\liminf_{n \to \infty} a_k = -\infty$$

if and only if a_k is not bounded below, and that

$$\liminf_{n \to \infty} a_k = -\limsup_{n \to \infty}(-a_k)$$

Also, we have

$$\lim_n a_n \text{ exists iff } \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n$$

Review of Series

Let $(a_n)_{n\geq 1}$ be a sequence of real or complex numbers.

The expression

$$a_1 + a_2 + \ldots \tag{1}$$

is called the *infinite series* (or just *series*) associated to the sequence $(a_n)_{n\geq 1}$. We can of course take any terminal set of integers for this concept instead of $\{1, 2, 3, \ldots\}$, so we also get

$$a_1 + a_2 + a_3 + \ldots$$

$$a_{-2} + a_{-1} + a_0 + a_1 + \ldots$$

as infinite series.

Given the series (1), the number

$$S_n = a_1 + a_2 + \ldots + a_n$$

is called the $n-$th partial sum of (1).

We say that the series (1) *converges* if the sequence of partial sums

$$S_1, S_2, \ldots$$

converges.

When that happens, if $S = \lim_n S_n$, then we usually write

$$\sum a_n = \sum_{n\geq 1} a_n = \sum_{n=1}^{\infty} a_n = S$$

If the series of absolute values

$$\mid a_1 \mid + \mid a_2 \mid + \ldots$$

converges, then we say that $a_1 + a_2 + \ldots$ is *absolutely convergent*

Note that absolute convergence implies convergence, but not conversely.

For instance, $\sum_n (-1)^n/n$ converges, but is not absolutely convergent.

The following (called the *ratio test*) is a useful method of establishing when a series converges absolutely.

Consider the infinite series (1) above where we assume that all of the $a_n's$ are not zero.

Form the sequence of ratios

$$\frac{| a_2 |}{| a_1 |}, \frac{| a_3 |}{| a_2 |}, \ldots, \frac{| a_{n+1} |}{| a_n |}, \ldots \tag{2}$$

Ratio Test for convergence and divergence: Given the series (1), assume that the limit

$$\lim_{n \to \infty} | \tfrac{a_{n+1}}{a_n} | = \rho$$

exists.

If $\rho < 1$, then the series (1) converges absolutely.

If $\rho > 1$, then then series (1) diverges.

If $\rho = 1$, then there is no information from the ratio test (i.e., there are examples with $\rho = 1$ which give convergence, and also examples of $\rho = 1$ which give divergence).

The ratio test applies in many cases, but notice that if infinitely many of the terms a_n are zero, then it cannot be applied. This can be remedied by another test called the *root test*.

Root Test for convergence and divergence: Given the series (1), consider the sequence of roots

$$| a_1 |, | a_2 |^{\frac{1}{2}}, \ldots, | a_n |^{\frac{1}{n}}, \ldots$$

and let

$$\rho = \limsup_{n \to \infty} | a_n |^{\frac{1}{n}}.$$

If $\rho < 1$, then the series (1) converges absolutely.

If $\rho > 1$, then then series (1) diverges.

If $\rho = 1$, then there is no information from the ratio test (i.e., there are examples with $\rho = 1$ which give convergence, and also examples of $\rho = 1$ which give divergence).

For some reason, the root test is not mentioned in many elementary calculus texts. The reader can find information about it in the book: *Principles of Mathematical Analysis* by Walter Rudin.

Power Series

An infinite series of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n\geq 0} c_n(x-a)^n \tag{3}$$

is called an *power series* or a *Taylor series* about the base point a. If $a = 0$, it is sometimes called a *Maclaurin series*.

For a series (3), there is an extended real number $0 \leq R \leq \infty$ such that

1. for $\mid x-a \mid < R$, the series converges absolutely,
2. for $\mid x-a \mid > R$, the series diverges.
3. for $\mid x-a \mid = R$, there is no information.

The number R is called the *radius of convergence* of the power series (3). If $R > 0$, then we can consider the function

$$f(x) = \sum_{n\geq 0} c_n(x-a)^n \tag{4}$$

as a well defined real or complex number. If all of the c_n are real, then we call the interval $(a-R, a+R)$ the *interval of convergence* of (3). If the $c_n's$ are complex, then we call the open disk $\{z \in \mathbf{C} : \mid z-a \mid < R\}$ the *circle of convergence* of (3).

We say the function f is *represented by* the series (3), or that (3) is the *Taylor series* of f with base point a.

There are the following useful properties of Taylor series inside their radius of convergence.

Assume that all of the $c_n's$ are real and $f(x)$ is given by (4) and the radius of convergence R of the associated Taylor series is positive.

Then,

1. The function $f(x)$ is infinitely differentiable in the interval $(a-R, a+R)$,
2. the series

   $$\sum_{n\geq 1} nc_n(x-a)^{n-1} \tag{5}$$

   has the same radius of convergence R as (3), and the sum of the series (5) represents $f'(x)$ at each $x \in (a-R, a+R)$.

3. the $n-$th derivative of f at a satisfies

$$c_n = \frac{f^{(n)}(a)}{n!} \tag{6}$$

4. the series

$$\sum_{n\geq 0} c_n \frac{(x-a)^{n+1}}{n+1} \tag{7}$$

has the same radius of convergence R as (3), and the sum of the series (5) represents

$$\int_a^x f(t)dt$$

at each $x \in (a-R, a+R)$.

Statement (2) shows that the derivative of an abolutely convergent power series can be gotten by taking the sum of the derivatives of the terms in the original power series. One refers to this as *term by term* differentialtion.

Analogously, statement (4) shows that integrals can also be taken as the sum of the integrals of the terms of the original series. Again, one refers to this as *term by term* integration.

A real-valued function $f : \mathbf{R} \to \mathbf{R}$ defined on an open interval I is called *real analytic* if for each $a \in I$, there is a taylor series with a as base point,

$$\sum_{n\geq 0} c_n (x-a)^n$$

and a positive number $R = R(a) > 0$ such that

1. the interval $(a-R, a+R)$ is contained in I,
2. the series converges absolutely for $\mid x-a \mid < R$, and
3. $$f(x) = \sum_{n\geq 0} c_n (x-a)^n$$

for each $x \in (a-R, a+R)$.

Thus, near each point $a \in I$, one can write $f(x)$ as an absolutely convergent taylor series with base point a. Note that the radii of convergence of the various taylor series obtained depend on the point a and may actually be larger than $R(a)$.

Real analytic functions are infinitely differentiable. Also, for a given base point a, the Taylor series respresentation of f is unique since the coefficients c_n are given by (6).

Examples of real analytic functions are $sin(x), cos(x), exp(x)$.

It is important to note that real analyticity is stronger than being infinitely differentiable. That is, there are C^∞ functions which are not real analytic.

For instance, let

$$f(x) = \begin{cases} exp(-\frac{1}{x}) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

It can be verified that f is infinitely differentiable on all all of $\mathbf{R}$, $f(0) = 0$ and $f^{(k)}(0) = 0$ for all $k > 0$. Thus, if f were representable as a Taylor series with a positive radius of convergence and base point $a = 0$, then all of the coefficients c_n would have to be 0. But that would imply that $f(x) = 0$ for all x in a small interval about zero. This is obviously not the case. So, indeed f is not real analytic in any open interval which contains 0.