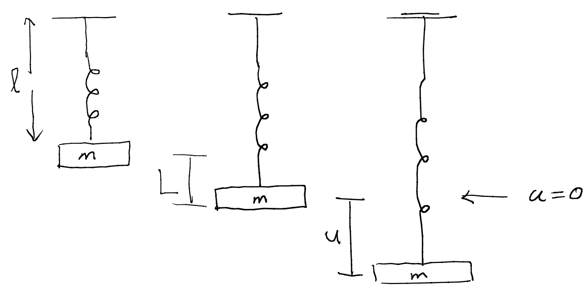
11. Some applications of second order differential equations

The first application we consider is the motion of a mass on a spring.

Consider an object of mass m on a spring suspended vertically as in the next figure.



Assume that the unstretched spring has length ℓ and that gravity pulls the mass down with a force equal to mg in magnitude with $g = 32ft/sec^2 \approx 9.8m/sec^2$. We also assume Hooke's law which says that the force on the object exerted by the spring has magnitude kx where k is a positive constant and x is the displacement of the spring from its unstretched state. If the spring is extended, then the force is exerted toward the spring, while it is exerted away from the spring if the spring is compressed. Let L be the amount the spring is stretched when the mass is in equilibrium. Let u denote the displacement of the mass from equilibrium.

At equilibrium, the force upward due to the spring must equal the force downward due to gravity, so we get

$$kL = mg.$$

By Newton's law of motion, we have that

$$\begin{array}{rcl}\text{mass times acceleration} & = & \text{total force at time } t \\ & & \text{on object at position } u\end{array}$$

If we take the downward direction as positive, and ignore frictional effects, then we get the differential equation of motion

$$\begin{aligned} m\ddot{u} &= mg - k(u+L). \\ &= mg - ku + kL \\ &= -ku \end{aligned}$$

or

$$m\ddot{u} + ku = 0, \text{ or } \ddot{u} + \frac{k}{m}u = 0.$$

This is a second order d.e. with constant coefficients. If we set

$$\omega_0 = \sqrt{\frac{k}{m}},$$

then we have the general solution

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

To graph this function, it is convenient to write it as

$$u(t) = R\cos(\omega_0 t - \delta)$$

Here, we have

$$R\cos(\omega_0 t - \delta) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta).$$

Hence, we have

$$R\cos(\delta) = A,\ \ R\sin(\delta) = B,$$

so,

$$R = \sqrt{A^2 + B^2},\ \ \tan(\delta) = \frac{B}{A}.$$

The quantities involved above have the following names

- ω_0 = fundamental frequency, $T = \frac{2\pi}{\omega_0}$ = period
- R = amplitude
- δ = phase angle or phase shift

The graph of the solution is as in the next figure where we plot $\omega_0 t$ horizontally, and u vertically. This graph is a cosine funtion whose maximum height is R. Its period gets shorter as ω_0 increases. The cosine function with maximum at $t = 0$ is translated to the left by δ units.

Next, we consider the d.e. for a spring with friction. For instance, the mass may have to encounter air resistence in its motion. A typical physical assumtion is that the frictional force points opposite to the direction of motion and has magnitude proportional to the speed.

Without external forces, this leads to the d.e.

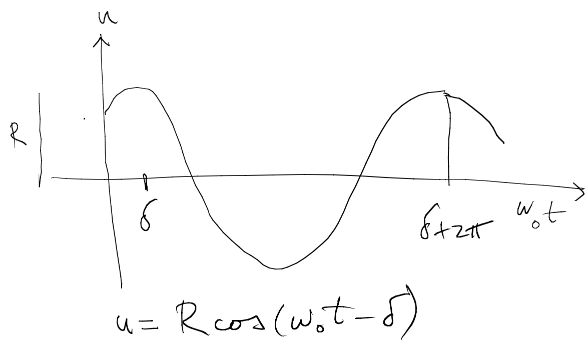
$$m\ddot{u} + \gamma\dot{u} + ku = 0.$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0$$

with roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$



We have three cases for the general solution.

Case 1: $\gamma^2 > 4km$. This is called overdamped. The general solution has the form

$$u(t) = Ae^{r_1 t} + Be^{r_2 t}$$

where

$$r_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m},$$

and

$$r_2 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}.$$

Case 2: $\gamma^2 = 4km$. This is called critically damped. The general solution has the form

$$u(t) = Ae^{r_1 t} + Bte^{r_1 t}$$

where

$$r_1 = -\frac{\gamma}{2m}.$$

Case 3: $\gamma^2 < 4km$. This is under-damped. If $\mu = \sqrt{4km - \gamma^2}$, then the general solution has the form

$$u(t) = e^{-\frac{\gamma}{2m}t}(A\cos(\mu t) + B\sin(\mu t)) = Re^{-\frac{\gamma}{2m}t}(\cos(\mu t - \delta),$$

where $R = \sqrt{A^2 + B^2}$, and $\delta = \tan(\frac{B}{A})$.

Typical Problem: A mass weighing 4 pounds stretches a spring two inches. The mass is in a medium which resists the motion with a force equivalent to 6 pounds when the mass has a speed of 3 feet per second. Write an equation describing the motion of the spring. If the spring is extended a distance of 2 feet and the mass is released from rest, how long does it take the mass to return to its lowest subsequent position? What height is that position?

We have $mg = 4$,and, at equilibrium we have

$$4 = mg = kL$$

We use units of feet and pounds. Thus

$$L = \text{ two inches } = \text{ } 1/6 \text{ feet,}$$

Thus,

$$4 = k/6$$

or $k = 24$.

Also the frictional force has magnitude $\gamma\dot{u} = \gamma 3 = 6$, so $\gamma = 2$.

The equation of motion is

$$m\ddot{u} + 2\dot{u} + 24u = 0.$$

Since g is 32 ft/sec^2, we have

$$m = \frac{4}{32} = \frac{1}{8}.$$

Hence,

$$\frac{1}{8}\ddot{u} + 2\dot{u} + 24u = 0,$$

or

$$\ddot{u} + 16\dot{u} + 192u = 0.$$

The characteristic equation is $r^2 + 16r + 192 = 0$, and its roots are

$$r = \frac{-16 \pm \sqrt{256 - 768}}{2} = -8 \pm 8\sqrt{2}i.$$

The general solution is

$$u = e^{-8t}(A\cos(8\sqrt{2}t) + B\sin(8\sqrt{2}t)).$$

For the initial conditions, we have $u(0) = 2,\ \dot{u}(0) = 0$.
So, we get $A = 2, -8A + 8\sqrt{2}B = 0$, or $B = \sqrt{2}$.
Thus the motion is

$$u(t) = e^{-8t}(2\cos(8\sqrt{2}t) + \sqrt{2}\sin(8\sqrt{2}t)).$$

The lowest position occurs at the the time t_ℓ equal to the period of the oscillation which is

$$t_\ell = \frac{2\pi}{8\sqrt{2}}.$$

The height is $u(t_\ell)$.

Note that, surprisingly, the differential equations for a simple electric circuit turn out to be the same as for the mass-spring system.