Some sample problems for Exam-4

1. Find the solution of the following heat equations.

   (a) $100\ u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0$
   $u(0,t) = 0 = u(1,t), \quad t > 0$
   $u(x,0) = 2sin(2\pi x) - 3sin(5\pi x), \quad 0 \le x \le 1.$

   (b) $25\ u_{xx} = u_t, \quad 0 < x < 2, \quad t > 0$
   $u(0,t) = 0 = u(2,t), \quad t > 0$
   $u(x,0) = sin(\frac{\pi x}{2}) + 3\ sin(3\pi x), \quad 0 < x < 2$

   (c) $16\ u_{xx} = u_t, \quad 0 < x < 3, \quad t > 0$
   $u(0,t) = 0 = u(3,t), \quad t > 0$
   $u(x,0) = x, \quad 0 < x < 3$

We begin with the general heat equation on $[0, L]$

$$\alpha^2 u_{xx} = u_t, \quad u = XT$$

$$\Rightarrow \alpha^2 X'' T = X T' = -\sigma$$

$$\alpha^2 \frac{X''}{X} = \frac{T'}{T} = -\sigma$$

$$X'' + \frac{\sigma}{\alpha^2} X = 0, \quad T = e^{-\sigma t}$$

$$X = \sum c_n e^{-\sigma_n t} \sin\left(\frac{\pi n x}{L}\right)$$

where $\sigma_n = \left(\frac{\pi n \alpha}{L}\right)$

(a) $L = 1,\ \alpha^2 = 100,\ n = 2,\ \sigma_n = 400\pi^2$
$n = 5,\ \sigma_n = 2500\pi^2$

$$u(x,t) = 2e^{-400\pi^2 t}\sin(2\pi x) - 3e^{-2500\pi^2 t}\sin(5\pi x)$$

(b) $L = 2, \ \alpha^2 = 25, \ \sigma_n = \left(\frac{n\pi\alpha}{L}\right)^2$

$$= \left(\frac{n\pi 5}{2}\right)^2$$

$n = 1, \ n = 6$

$$\sigma_1 = \frac{25}{4}\pi^2, \quad \sigma_6 = \frac{36 \times 25\pi^2}{4} = 225\pi^2$$

$$u = C_1 e^{-\frac{25}{4}\pi^2 t} \sin\left(\frac{\pi x}{2}\right) + C_6 e^{-225\pi^2 t} \sin(3\pi x)$$

$C_1 = 1, \ C_6 = 3$

(c) $L = 3, \ \alpha^2 = 16$

$$u = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi 4}{9}\right)^2 t} \sin\left(\frac{n\pi x}{3}\right)$$

$$C_n = \frac{2}{3}\int_0^3 x \sin\left(\frac{n\pi x}{3}\right) dx$$

[Note: $\int x \sin(ax)\,dx = -\frac{x\cos(ax)}{a} + \frac{\sin(ax)}{a^2}$]

$$C_n = \frac{2}{3}\left[-\frac{3}{n\pi} x \cos\left(\frac{n\pi x}{3}\right)\right]_0^3$$

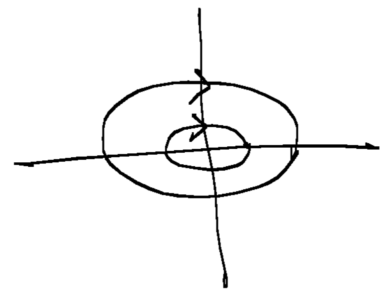
$$= -\frac{6}{n\pi}\cos(n\pi) = (-1)^{n+1}\frac{6}{n\pi}$$

2. Find the general solution to each of the following systems. In each case, sketch the solution curves

(a)

$$\dot{x} = 2y$$
$$\dot{y} = -2x$$

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$



$r^2 + 4 = 0$, $r = \pm 2i$

$$\text{Gen soln} = c_1 \operatorname{Re}\left(e^{2it}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) + c_2 \operatorname{Im}\left(e^{2it}\begin{pmatrix} 1 \\ i \end{pmatrix}\right)$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\left[\cos(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + c_2\left[\cos(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

(b)

$$\dot{x} = x + y$$
$$\dot{y} = x - 2y$$

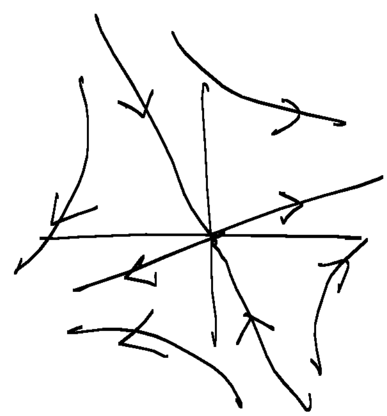
$$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$r^2 + r - 3$

$$r = \frac{-1 \pm \sqrt{1+12}}{2} = -\frac{1}{2} \pm \frac{\sqrt{13}}{2}$$

Gen soln

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\left(-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)t}\begin{pmatrix} 1 \\ -\frac{3}{2}+\frac{\sqrt{13}}{2} \end{pmatrix} + c_2 e^{\left(-\frac{1}{2}-\frac{\sqrt{13}}{2}\right)t}\begin{pmatrix} 1 \\ -\frac{3}{2}-\frac{\sqrt{13}}{2} \end{pmatrix}$$



(c)

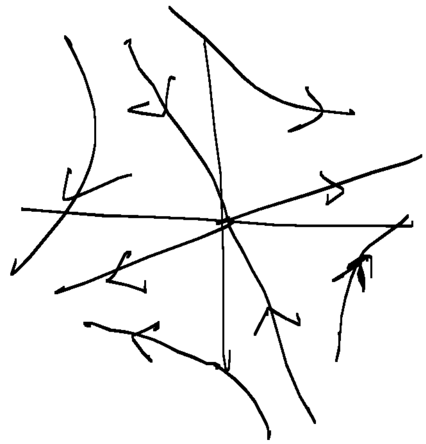
$$A=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$\begin{aligned}\dot{x} &= x+y\\ \dot{y} &= x-y\end{aligned}$$

$$r^2-2=0, \quad r=\pm\sqrt{2}$$

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=c_1e^{\sqrt{2}t}\begin{pmatrix}1\\\sqrt{2}-1\end{pmatrix}$$

$$+c_2e^{-\sqrt{2}t}\begin{pmatrix}1\\-\sqrt{2}-1\end{pmatrix}$$

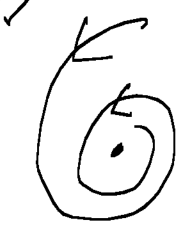


(d)

$$\begin{aligned}\dot{x} &= -2x-y\\ \dot{y} &= x-2y\end{aligned} \qquad A=\begin{pmatrix}-2&-1\\1&-2\end{pmatrix}$$

$$r^2+4r+5, \quad r=\frac{-4\pm\sqrt{16-20}}{2}$$

$$=-2\pm i$$



Gen solution:

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=c_1\,\mathrm{Re}\left(e^{(-2+i)t}\begin{pmatrix}1\\-i\end{pmatrix}\right)$$

$$+c_2\,\mathrm{Im}\left(e^{(-2+i)t}\begin{pmatrix}1\\-i\end{pmatrix}\right)$$

$$=c_1e^{-2t}\left(\cos(t)\begin{pmatrix}1\\0\end{pmatrix}-\sin(t)\begin{pmatrix}0\\-1\end{pmatrix}\right)$$

$$+c_2e^{-2t}\left(\cos(t)\begin{pmatrix}0\\-1\end{pmatrix}+\sin(t)\begin{pmatrix}1\\0\end{pmatrix}\right]$$

3. Find the eigenvalues and associated eigenvectors for each of the following matrices A.

(a)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$r^2 - r - 5$, $r = \dfrac{1 \pm \sqrt{1+20}}{2}$

$= \dfrac{1}{2} \pm \dfrac{\sqrt{21}}{2}$

$\dfrac{1}{2} + \dfrac{\sqrt{21}}{2}$, $\begin{pmatrix} 1 \\ -\dfrac{3}{6} + \dfrac{\sqrt{21}}{6} \end{pmatrix}$

$\dfrac{1}{2} - \dfrac{\sqrt{21}}{2}$, $\begin{pmatrix} 1 \\ -\dfrac{3}{6} - \dfrac{\sqrt{21}}{6} \end{pmatrix}$

(b)

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$r^2 - 6r + 9$, $(r-3)^2$

$r = 3$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(c)

$$A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

$r^2 - 4r + 13$, $r = \dfrac{4 \pm \sqrt{16 - 52}}{2}$

$= \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

$2+3i$, $\begin{pmatrix} 1 \\ i \end{pmatrix}$ $2-3i$, $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

4. Using the method of elimination, find the general solution to the system

$$\begin{aligned} \dot{x} &= 4x + y \\ \dot{y} &= x + 3y \end{aligned}$$

$r^2 - 7r + 11$, $r = \dfrac{7 \pm \sqrt{49 - 44}}{2}$

$= \dfrac{7 \pm \sqrt{5}}{2}$, $r_1 = \frac{7}{2} + \frac{\sqrt{5}}{2}$

$r_2 = \frac{7}{2} - \frac{\sqrt{5}}{2}$

$x = \dot{y} - 3y$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$x = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$
$- 3\left[c_1 e^{r_1 t} + c_2 e^{r_2 t}\right]$

5. Set up the equations for the function $v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$ appearing in the variation of parameters method for finding a particular solution to the system

$$\begin{aligned} \dot{x} &= 4x + y + e^{2t} \\ \dot{y} &= x + 3y - e^{t} \end{aligned}$$

$r^2 - 7r + 11$

$r = \frac{7}{2} \pm \frac{\sqrt{5}}{2}, \quad r_1 = \frac{7}{2} + \frac{\sqrt{5}}{2}, \quad r_2 = \frac{7}{2} - \frac{\sqrt{5}}{2}$

$$e^{r_1 t}\begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{5}}{2} \end{pmatrix} v_1' + e^{r_2 t}\begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} v_2' = \begin{pmatrix} e^{2t} \\ -e^{t} \end{pmatrix}$$

6. For each of the following functions, state whether the function is odd, even, or neither.

(a) $f(x) = x^2 sin(x)$ odd

(b) $f(x) = x^2 - x sin(x)$ even

(c) $f(x) = x^2 + sin(x)$ neither

(d) $f(x) = 2x^3 - x^5 cos(x)$ odd

7. True or False

F (a) The sum of an even function and an odd function is odd

T (b) The product of an even function and an odd function is odd

T (c) The sum of two even functions is even

T (d) The method of separation of variables can be used to reduce the partial differential equation $x^2 u_{xxx} + t u_{tt} = 0$ to two ordinary differential equations.

$u = XT$

$x^2 X''' T + t X T'' = 0$

$$x^2 \frac{X'''}{X} = -\frac{tT''}{T} = \sigma$$

8. Consider the periodic function $f(x)$ of period 4 which satisfies

$$f(x) = \begin{cases} x+2 & \text{for} \quad -2 < x < 0 \\ 2-x & \text{for} \quad 0 \le x \le 2 \end{cases}$$

Find the Fourier series of $f(x)$ on the interval $[-2, 2]$. , period $= 4 = 2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$b_n = 0$ since f is even.

$$a_n = \frac{1}{2}\int_{-2}^{0} (x+2)\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{4}{n^2\pi^2} - \frac{4\cos(n\pi)}{n^2\pi^2}$$

$a_0 = 2$