

13a. Laplace Transform

Def Given a function $f(t)$, define

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

This exists (and is finite) if $\exists \sigma > 0, \alpha > 0, C_1 > 0$

such that $|f(t)| \leq C_1 e^{\alpha t}$ for $t \geq C_1$

[$f(t)$ grows at most exponentially
as $t \rightarrow \infty$]

Basic Properties:

1) $\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$
(linearity)

2) $\mathcal{L}(c) = \frac{c}{s}$ if c is a constant

3) $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ for a positive integer n

3) $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ for a real a

$$4) \mathcal{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

$$5) \mathcal{L}(\sin(at)) = \frac{a}{s^2 + a^2}$$

$$6) \mathcal{L}(\sinh(at)) = \frac{a}{s^2 - a^2}$$

$$7) \mathcal{L}(\cosh(at)) = \frac{s}{s^2 - a^2}$$

$$8) \text{ If } \mathcal{L}(f(t)) = F(s), \text{ then}$$

$$\mathcal{L}(t f(t)) = -F'(s) = -\frac{dF}{ds}$$

$$\Rightarrow \mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n F}{ds^n}$$

$$9) \text{ If } \mathcal{L}(f(t)) = F(s), \text{ then}$$

$$\mathcal{L}(e^{at} f(t)) = F(s-a)$$

Examples of computation of \mathcal{L} and \mathcal{L}^{-1}

$$\mathcal{L}: f(t) \rightarrow F(s)$$

inverse Laplace
transform

$$\mathcal{L}^{-1}: F(s) \rightarrow f(t)$$

$$1) \mathcal{L}(2 - 3t^2 + t^4) = \frac{2}{s} - 3 \frac{2}{s^3} + \frac{4!}{s^5}$$

$$2) \mathcal{L}(3 \sin(5t) - 2 \cos(2t))$$

$$= 3 \cdot \frac{5}{s^2+25} - 2 \frac{5}{s^2+4} = \frac{15}{s^2+25} - \frac{25}{s^2+4}$$

$$3) \mathcal{L}(e^{-4t} t^5) = \frac{5!}{(s+4)^6}$$

$$4) \mathcal{L}(t e^{3t}) = -F'(s) \text{ where}$$

$$F(s) = \frac{1}{s-3}$$

$$S_0 = \frac{1}{(s-3)^2}$$

Some inverse transforms

$$1) \quad \mathcal{L}^{-1}\left(\frac{1}{s^2 - 6s + 5}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{(s-5)(s-1)}\right)$$

Use partial fractions $\frac{1}{(s-5)(s-1)} = \frac{1}{s-5} - \frac{1}{s-1}$

$$\mathcal{L}^{-1} = \frac{1}{4}(e^{5t} - e^t)$$

$$\leftarrow \frac{1}{4}\left(\frac{1}{s-5} - \frac{1}{s-1}\right)$$

alternate method - complete the square

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s^2-6s+5}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s^2-6s+9+5-9}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{(s-3)^2-4}\right) \\ &= e^{3t} \mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right) = \frac{1}{2} e^{3t} \mathcal{L}^{-1}\left(\frac{2}{s^2-4}\right) \\ &= \frac{1}{2} e^{3t} \sinh(2t) \end{aligned}$$