GROUP REPRESENTATIONS AND SYMMETRIC FUNCTIONS

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I. Group representations: A. Modules

All groups G will be finite and all vector spaces V will be over $\mathbb{C}$.

A *matrix representation (rep) of a group* G is a group homomorphism

$$X : G \to GL_d(\mathbb{C}).$$

A G*-module* is a vector space V, $\dim V = d$, with a group homomorphism

$$\rho : G \to GL(V).$$

This gives a linear action of G on V: $g\mathbf{v} = \rho(g)\mathbf{v}$.

The parameter d is called the *degree* or *dimension* of the rep. We will freely go between matrix rep's and G-modules.

The *group algebra* is the G-module

$$\mathbb{C}[\mathbf{G}] = \{\sum_{g \in G} c_g \mathbf{g} \mid c_g \in \mathbb{C}\}$$

with action $g\mathbf{h} = \mathbf{k}$ if $gh = k$ in G. The corresponding matrix rep in the basis $\mathcal{B} = \{\mathbf{g} \mid g \in G\}$ is called the *(left) regular rep.* The corresponding matrices $X(g)$ are permutation matrices (cf. Cayley's Theorem).

Ex. Every group G has the *trivial rep* X^{tri}

$$X^{\text{tri}}(g) = (1) \quad \text{for all } g \in G.$$

A module for this rep is V with $\dim V = 1$ and

$$g\mathbf{v} = \mathbf{v} \quad \text{for all } g \in G, \mathbf{v} \in V.$$

Ex. For a cyclic group $G = \{g, g^2, \ldots, g^n = \epsilon\}$ any 1-dim rep would have $X(g) = (c)$ where

$$(c^n) = X(g^n) = X(\epsilon) = (1).$$

So c is an nth root of 1 and all such nth roots give 1-dim rep's.

If $n = 2$ then the group algebra is $\mathbb{C}[\mathbf{G}] = \{c_1\epsilon + c_2\mathbf{g}\}$ with action $g\epsilon = \mathbf{g}, g\mathbf{g} = \epsilon$. So the left regular rep is

$$X(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Changing basis to $\{\epsilon + \mathbf{g}, \epsilon - \mathbf{g}\}$ gives an equivalent rep

$$Y(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is a direct sum of the rep's for $\sqrt{1} = \pm 1$.

If G acts on a set S then one obtains a representation by linearly extending to the vector space

$$\mathbb{C}[\mathbf{S}] = \{\sum_{s \in S} c_s \mathbf{s} \mid c_s \in \mathbb{C}\}.$$

The basis $\mathbf{S}$ gives a rep by permutation matrices.

<u>Ex.</u> Given any group G, a subgroup $H \leq G$, and a set of all distinct left cosets

$$S = \{t_1 H, \ldots, t_l H\}$$

there is an action $g\mathbf{t}_j\mathbf{H} = \mathbf{t}_i\mathbf{H}$ if $gt_jH = t_iH$. The module $\mathbb{C}[\mathbf{S}]$ is called a *coset rep.* If $H = G$ (resp. $H = \{\epsilon\}$) then it's the trivial (resp. regular) rep.

<u>Ex.</u> The symmetric group $\mathfrak{S}_n$ acts by definition on

$$S = \{1, 2, \ldots, n\}.$$

The corresponding module $\mathbb{C}[\mathbf{1}, \ldots, \mathbf{n}]$ is the *defining rep.* If $n = 2$ then $(1,2)\mathbf{1} = \mathbf{2}, (1,2)\mathbf{2} = \mathbf{1}$ so

$$X(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X((1,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Also $\mathfrak{S}_n$ has the 1-dim *sign rep*

$$X(\pi) = (\operatorname{sgn} \pi) \quad \operatorname{sgn} \pi = \begin{cases} +1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd.} \end{cases}$$

B. Reducibility and Maschke's Theorem

A *submodule* W of G-module V, $W \leq V$, is a subspace closed under G's action. Every G-module has *trivial submodules* $W = \{\mathbf{0}\}, V$. Module V is *irreducible (irr)* or an *irrep* if it has no non-trivial submodules. Every 1-dim module is an irrep.

<u>Ex.</u> The group algebra $\mathbb{C}[\mathbf{G}]$ has submodule

$$W = \mathbb{C}[\sum_{g \in G} \mathbf{g}]$$

since $h \sum_g \mathbf{g} = \sum_g \mathbf{g}$. This W gives the trivial rep.

If $G = \mathfrak{S}_n$ then we can get the sign rep with

$$U = \mathbb{C}[\sum_{\pi \in \mathfrak{S}_n} (\operatorname{sgn} \pi)\pi].$$

<u>Ex.</u> If $G = \mathfrak{S}_n$ and $V = \mathbb{C}[\mathbf{1}, \ldots, \mathbf{n}]$ then

$$W = \mathbb{C}[\mathbf{1} + \mathbf{2} + \cdots + \mathbf{n}]$$

is a submodule for the trivial rep. Consider the inner product on V: $\langle \mathbf{i}, \mathbf{j} \rangle = \delta_{i,j}$ (Kronekcer δ). Then

$$W^{\perp} = \{\sum_i c_i \mathbf{i} \mid \sum_i c_i = 0\}$$

is also a submodule and $V = W \oplus W^{\perp}$ with $W, W^{\perp}$ irr. (Clear for W, not for $W^{\perp}$.)

A G-module V is *completely reducible* if

$$V = W^{(1)} \oplus \cdots \oplus W^{(k)}$$

where each $W^{(i)}$ is irr.

Theorem 1 (Maschke) *If G is finite then every complex G-module V is completely reducible.*

Proof. If V is irr, we are done. If not, let W be a non-trivial submodule. Pick a basis for V $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ with corresponding inner product $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{i,j}$. Now define another inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle' = \sum_{g \in G} \langle g\mathbf{v}, g\mathbf{w} \rangle$$

which is G-invariant:

$$\langle h\mathbf{v}, h\mathbf{w} \rangle' = \sum_{g \in G} \langle gh\mathbf{v}, gh\mathbf{w} \rangle = \sum_{g \in G} \langle g\mathbf{v}, g\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle'.$$

Now $W^{\perp}$ (with respect to $\langle \cdot, \cdot \rangle'$) is a submodule since if $\mathbf{v} \in W^{\perp}, \mathbf{w} \in W$, and $g \in G$ then

$$\langle g\mathbf{v}, \mathbf{w} \rangle' = \langle \mathbf{v}, g^{-1}\mathbf{w} \rangle' = 0.$$

So $V = W \oplus W^{\perp}$ and done by induction on $\dim V$. ∎

Note: 1. Maschke may not be true if $|G| = \infty$ or the field is different from $\mathbb{C}$.

2. Henceforth we can just concentrate on irreps.

C. G-homomorphisms and Schur's Lemma

A *G-homomorphism (hom)* of G-modules V, W is a linear map $\theta : V \to W$ such that for all $g \in G, \mathbf{v} \in V$

$$\theta(g\mathbf{v}) = g\theta(\mathbf{v}).$$

A bijective θ is called a *G-isomorphism (iso)* and then V, W are *G-equivalent (equiv)*, $V \cong W$. Turning everything into matrices

$$\begin{aligned}
&TX(g)\mathbf{v} = Y(g)T\mathbf{v} \quad \text{for all } g \in G, \mathbf{v} \in \mathbb{C}^d \\
&\quad \Rightarrow TX(g) = Y(g)T \quad \text{for all } g \in G \\
&\quad \stackrel{\text{def}}{\Rightarrow} TX = YT.
\end{aligned}$$

<u>Ex.</u> Let $V = \mathbb{C}[\mathbf{v}]$ be the trivial rep and $W = \mathbb{C}[\mathbf{G}]$ be the group algebra. Then a G-hom is $\theta : V \to W$ defined by

$$\theta(\mathbf{v}) = \sum_{g \in G} \mathbf{g}.$$

<u>Ex.</u> Let $G = \mathfrak{S}_2$, let $V = \mathbb{C}[\mathbf{1}, \mathbf{2}]$ be the defining rep and $W = \mathbb{C}[\epsilon, (\mathbf{1}, \mathbf{2})]$ be the group algebra. Then $\theta : V \to W$ by $\mathbf{1} \mapsto \epsilon,\ \mathbf{2} \mapsto (\mathbf{1}, \mathbf{2})$ is an $\mathfrak{S}_2$-iso, e.g.,

$$\theta((1,2)\mathbf{2}) = \theta(\mathbf{1}) = \epsilon = (1,2)(\mathbf{1},\mathbf{2}) = (1,2)\theta(\mathbf{2}).$$

Lemma 2 (Schur) *If V, W are irreducible modules and $\theta : V \to W$ is a G-homomorphism then either*

1. θ is a G-isomorphism or

2. θ is the zero map.

Proof. Since θ is a G-hom, $\ker\theta$ and $\operatorname{im}\theta$ are G-submodules of V and W, respectively. Since V, W are irr, $\ker\theta = \{\mathbf{0}\}$ or V and $\operatorname{im}\theta = \{\mathbf{0}\}$ or W. If $\ker\theta = \{\mathbf{0}\}$ and $\operatorname{im}\theta = W$ then θ is a G-iso. All other cases lead to the zero map. ∎

Schur's Lemma is valid for infinite groups and arbitrary fields. For $\mathbb{C}$ more is true.

Corollary 3 *If X is an irreducible matrix representation (irrep) of G over $\mathbb{C}$ and T commutes with X then $T = cI$, $c \in \mathbb{C}$.*

Proof. Let c be an eigenvalue of T. Then

$$TX = XT \quad \Rightarrow \quad (T - cI)X = X(T - cI).$$

By Schur, $T - cI$ is invertible or zero and the former can't happen by the choice of c. ∎

D. The endomorphism algebra

A G-module V has *endomorphism algebra*

$$\text{End}\, V = \{\theta : V \to V \mid \theta \text{ is a } G\text{-homomorphism}\}.$$

For a d-dim matrix representation X this becomes

$$\text{End}\, X = \{T \in \text{Mat}_d \,|\, TX = XT\}.$$

To describe $\text{End}\, X$ we use block matrix operations

$$S \oplus T = \left(\begin{array}{c|c} S & 0 \\ \hline 0 & T \end{array}\right), \quad S \otimes T = (S_{i,j}T) \text{ where } S = (S_{i,j}).$$

Suppose that X decomposes as

$$X = X^{(1)} \oplus X^{(2)} \oplus \cdots \oplus X^{(l)}$$

where the $X^{(i)}$ are irr. Let $T = (T_{i,j}) \in \text{End}\, X$ have the same block form. Then $XT = XT$ implies $T_{i,j}X^{(j)} = X^{(i)}T_{i,j}$ so

$$T_{i,j} = \begin{cases} 0 & \text{if } X^{(i)} \not\cong X^{(j)} \text{ (Schur)} \\ c_{i,j}I & \text{if } X^{(i)} \cong X^{(j)} \text{ (Cor).} \end{cases}$$

Renaming the irreps to collect equiv ones and letting $d_i = \dim X^{(i)}$

$$X = \oplus_{i=1}^{k} m_i X^{(i)} \Rightarrow$$
$$\text{End}\, X = \{\oplus_{i=1}^{k}(M_{m_i} \otimes I_{d_i}) \mid M_{m_i} \in \text{Mat}_{m_i}\ \forall i\}.$$

Otherwise put

$$\operatorname{End} X \cong \oplus_{i=1}^{k} \operatorname{Mat}_{m_i}.$$

The center $Z_{\operatorname{Mat}_m} = \{cI \mid c \in \mathbb{C}\}$ and so

$$Z_{\operatorname{End} X} = \{\oplus_{i=1}^{k} c_i I_{m_i d_i} \mid c_i \in \mathbb{C} \text{ for all } i\} \cong \operatorname{Diag}_k,$$

where Diag_k are the diagonal matrices in Mat_k.

Summarizing and taking dimensions:

Theorem 4 *Let X be a matrix rep of G with*

$$X = m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)}$$

where the $X^{(i)}$ are inequiv, irr and with dimensions $\dim X^{(i)} = d_i$. *Then*

1. $\operatorname{End} X \cong \oplus_{i=1}^{k} \operatorname{Mat}_{m_i}$,

2. $Z_{\operatorname{End} X} \cong \operatorname{Diag}_k$,

3. $\dim X = m_1 d_1 + m_2 d_2 + \cdots + m_k d_k$,

4. $\dim(\operatorname{End} X) = m_1^2 + m_2^2 + \cdots + m_k^2$,

5. $\dim Z_{\operatorname{End} X} = k$.

E. Group characters and inner products

Matrix rep X has *character (char)* $\chi : G \to \mathbb{C}$ where

$$\chi(g) = \operatorname{tr} X(g).$$

A G-module also has a unique character since any two bases give conjugate matrix reps.

<u>Ex.</u> If $\dim X = 1$ then χ is called a *linear char* and

$$\chi(gh) = \operatorname{tr} X(gh) = \operatorname{tr} X(g) \operatorname{tr} X(h) = \chi(g)\chi(h).$$

<u>Ex.</u> If $V = \mathbb{C}[\mathbf{G}]$ (regular rep) then the char is

$$\chi^{\text{reg}}(g) = |\{\mathbf{h} \ : \ g\mathbf{h} = \mathbf{h}\}| = \begin{cases} |G| & \text{if } g = \epsilon \\ 0 & \text{else.} \end{cases}$$

<u>Ex.</u> If $V = \mathbb{C}[\mathbf{1}, \ldots, \mathbf{n}]$ (defining rep of $\mathfrak{S}_n$), then

$$\chi^{\text{def}}(\pi) = \text{number of fixed points of } \pi.$$

Proposition 5 *Let group G have matrix representation X with $\dim X = d$ and character χ.*

1. $\chi(\epsilon) = d$,
2. *If K is a conjugacy class: $g, h \in K \Rightarrow \chi(g) = \chi(h)$,*
3. *If rep Y has char ψ: $X \cong Y \Rightarrow \chi = \psi$.* $\blacksquare$

Character χ is a *class function* since it is constant on conjugacy classes K. Let $\chi_K = \chi(g), g \in K$. The *character table* of G has rows indexed by the irreps (χ^{tri} first) columns indexed by the conjugacy classes ($\{\epsilon\}$ first) and entries χ_K. It is square.

<u>Ex.</u> If $G = \mathfrak{S}_3$ then we have

	$\{\epsilon\}$	$\{(1,2);(1,3);(2,3)\}$	$\{(1,2,3);(1,3,2)\}$
χ^{tri}	1	1	1
χ^{sgn}	1	-1	1
χ^{mys}	?	?	?

The *inner product* of $\chi, \psi : G \to \mathbb{C}$ is

$$\langle \chi, \psi \rangle \;:=\; \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} \;=\; \frac{1}{|G|} \sum_K |K| \chi_K \overline{\psi_K}.$$

If G-module V has char ψ then an orthonormal basis for V with respect to a G-invariant inner product on V gives matrices for ψ which are unitary and

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

If $G = \mathfrak{S}_n$ then g and g^{-1} are conjugate and so

$$\langle \chi, \psi \rangle = \frac{1}{n!} \sum_K |K| \chi_K \psi_K.$$

Theorem 6 (Character relations, the 1st kind) *If χ, ψ are irreducible characters of a group G then*

$$\langle \chi, \psi \rangle = \delta_{\chi,\psi}. \quad (*)$$

Proof sketch. Let χ, ψ come from reps X, Y. Let $Z = (z_{i,j})$ and $W = |G|^{-1} \sum_{g \in G} X(g) Z Y(g^{-1})$. Then $XW = WY$ and by Schur's Lemma

$$W = \begin{cases} 0 & \text{if } X \not\cong Y, \\ cI & \text{if } X \cong Y. \end{cases}$$

Since this is true for all Z, one can get equations relating the entries of X and Y giving (*). ■

Corollary 7 *Let $X \cong \oplus_{i=1}^{k} m_i X^{(i)}$ where the $X^{(i)}$ are pairwise inequiv with char's $\chi^{(i)}$.*

1. $\chi = m_1 \chi^{(1)} + m_2 \chi^{(2)} + \cdots + m_k \chi^{(k)}$,

2. $\langle \chi, \chi^{(i)} \rangle = m_i$,

3. $\langle \chi, \chi \rangle = m_1^2 + m_2^2 + \cdots + m_k^2$,

4. X *is irr* $\Leftrightarrow \langle \chi, \chi \rangle = 1$ *(use 3)*,

5. *If Y has char ψ then $X \cong Y \Leftrightarrow \chi = \psi$ (use 2).*■

Ex. Let $G = \mathfrak{S}_3$ and $V = \mathbb{C}[\mathbf{1}, \mathbf{2}, \mathbf{3}]$ (defining rep) with char $\chi = \chi^{\text{def}}$. Then

$$\begin{aligned} &\chi(\pi) = \text{number of fixed points of } \pi \\ &\chi(\epsilon) = 3,\ \chi((1,2)) = 1,\ \chi((1,2,3)) = 0. \end{aligned}$$

Also

$$\begin{aligned} &\chi = m_1\chi^{\text{tri}} + m_2\chi^{\text{sgn}} + m_3\chi^{\text{mys}} \quad \text{where} \\ &m_1 = (1\cdot 3\cdot 1 + 3\cdot 1\cdot 1 + 2\cdot 0\cdot 1)/3! = 1 \\ &m_2 = (1\cdot 3\cdot 1 + 3\cdot 1(-1) + 2\cdot 0\cdot 1)/3! = 0. \end{aligned}$$

So

$$\chi = \chi^{\text{tri}} + m_3\chi^{\text{mys}}.$$

Consider the character

$$\begin{aligned} &\psi = \chi - \chi^{\text{tri}}, \\ &\psi(\epsilon) = 2,\ \psi((1,2)) = 0,\ \psi((1,2,3)) = -1. \end{aligned}$$

Then ψ is irreducible since

$$\langle \psi, \psi \rangle = (1\cdot 2^2 + 3\cdot 0^2 + 2(-1)^2)/3! = 1.$$

So $m_3 = 1$ and $\chi^{\text{mys}} = \psi$ giving the complete table

	$\{\epsilon\}$	$\{(1,2);(1,3);(2,3)\}$	$\{(1,2,3);(1,3,2)\}$
χ^{tri}	1	1	1
χ^{sgn}	1	-1	1
χ^{mys}	2	0	-1

Note: For general n, $\chi^{\text{def}} - \chi^{\text{tri}}$ is irreducible.

F. Decomposing the group algebra

Proposition 8 *Let* $\mathbb{C}[\mathbf{G}] = \oplus_i m_i V^{(i)}$ *where the* $V^{(i)}$ *are a complete list of all inequiv irreps*

1. $m_i = \dim V^{(i)}$ *(so all irreps occur),*

2. $\sum_i (\dim V^{(i)})^2 = |G|$*,*

3. # of irreps = # of conjugacy classes K *of* G*.*

Proof sketch. 1. Let $\chi = \chi^{\text{reg}}$. Then

$$m_i = \frac{\sum_{g \in G} \chi(g)\chi^{(i)}(g^{-1})}{|G|} = \frac{\chi(\epsilon)\chi^{(i)}(\epsilon)}{|G|} = \dim V^{(i)}.$$

2. Follows from 1. 3.# of irreps $= \dim Z_{\text{End}\,\mathbb{C}[\mathbf{G}]}$.

$$\text{End}\,\mathbb{C}[\mathbf{G}] = \{\phi_{\mathbf{v}} \ : \ \phi_{\mathbf{v}}(\mathbf{w}) = \mathbf{wv}\} \cong \mathbb{C}[\mathbf{G}].$$

Now $\mathbf{z} \in Z_{\mathbb{C}[\mathbf{G}]}$ iff $\mathbf{z} = \mathbf{hzh}^{-1}$ for all $\mathbf{h} \in \mathbf{G}$. So for each conjugacy class K of G, $Z_{\mathbb{C}[\mathbf{G}]}$ has a basis element

$$\mathbf{z}_K = \sum_{k \in K} \mathbf{k}. \quad \blacksquare$$

Corollary 9 *1. The character table of G is square.*

2. The irr characters χ of G form an orthonormal basis for the space $R(G)$ of class functions on G.

3. (Character relations of the second kind) If K, L are conjugacy classes of G and χ is irreducible

$$\sum_{\chi} \chi_K \overline{\chi_L} = \frac{|G|}{|K|} \delta_{K,L}.$$

Proof. 1 and 2 follow from part 3 of the Proposition and the character relations of the first kind.

3. The relations of the first kind also give that the modified character table $U = \left(\sqrt{|K|/|G|}\chi_K\right)$ has orthonormal rows, thus orthonormal columns. ■

<u>Ex.</u> We can find χ^{mys} for $\mathfrak{S}_3$ another way. By the Proposition, part 2,

$$1^2 + 1^2 + \chi^{\text{mys}}(\epsilon)^2 = 3! \Rightarrow \chi^{\text{mys}}(\epsilon) = 2.$$

The other two entries are found using the relations of the second kind. For example, taking $K = \{\epsilon\}$ and $L = \{(1,2), \ldots\}$

$$0 = 1 \cdot 1 + 1(-1) + 2\chi_L^{\text{mys}} \Rightarrow \chi_L^{\text{mys}} = 0.$$

G. Representations of products and subgroups

If X, Y are matrix reps of G, H respectively then the *tensor product rep* of $G \times H$ is

$$(X \otimes Y)(g, h) = X(g) \otimes Y(h).$$

Proposition 10 *1. $X \otimes Y$ is a rep of $G \times H$. If X, Y are irreps then so is $X \otimes Y$.*

2. As $X^{(i)}, Y^{(j)}$ run over complete lists of inequiv irreps for G, H resp, $X^{(i)} \otimes Y^{(j)}$ runs over a complete list of inequiv irreps for $X \otimes Y$.

3. If $X, Y, X \otimes Y$ have characters $\chi, \psi, \chi \otimes \psi$ resp then

$$(\chi \otimes \psi)(g, h) = \chi(g)\psi(h).$$

Proof of 2. Suppose that $X^{(i)}, Y^{(j)}$ have chars $\chi^{(i)}, \psi^{(j)}$ resp. Then inequivalence follows from

$$\begin{aligned}\langle \chi^{(i)} \otimes \psi^{(j)}, \chi^{(k)} \otimes \psi^{(l)} \rangle &= \langle \chi^{(i)}, \chi^{(k)} \rangle \langle \psi^{(j)}, \psi^{(l)} \rangle \\ &= \delta_{i,k}\delta_{j,l} = \delta_{(i,j),(k,l)}.\end{aligned}$$

For list completeness, just check we have the right number of irreps. Let $k(\cdot) =$# of conjugacy classes.

$$\begin{aligned}\#\text{ of irreps of } G \times H &= k(G \times H) = k(G)k(H) \\ &= (\#\text{ of irreps of } G)(\#\text{ of irreps of } H).\end{aligned}$$

If $H \leq G$ and X is a rep of G then the *restriction* of X to H, $X \downarrow_H = X \downarrow_H^G$, is

$$X \downarrow_H (h) = X(h).$$

It is clear the $X \downarrow_H$ is a rep, but if X is irr then $X \downarrow_H$ need not be. For example, if X is the 2-dim irrep of $\mathfrak{S}_3$ and $H = \{\epsilon, (1,2)\} := \mathfrak{S}_{\{1,2\}} \cong \mathfrak{S}_2$ then

$$X \downarrow_H \cong X^{\text{tri}} \oplus X^{\text{sgn}}.$$

If Y is a rep of H then $Y(g) := 0$ for $g \notin H$ doesn't give a rep. But if $G = \uplus_i t_i H$ then the *induction* of Y to G, $Y \uparrow^G = Y \uparrow_H^G$, has block matrices

$$Y \uparrow^G (g) = (Y(t_i^{-1} g t_j)).$$

<u>Ex.</u> Consider $1 \uparrow^G$ for the trivial char 1 of H. Then

$$1(g) = \begin{cases} 1 & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

So

$$1(t_i^{-1} g t_j) = 1 \iff t_i^{-1} g t_j \in H \iff g t_j H = t_i H.$$

So $1 \uparrow^G$ equals the coset rep $\mathbb{C}[\mathcal{H}]$ in the standard basis $\mathcal{H} = \{\mathbf{t_1 H}, \ldots, \mathbf{t_l H}\}$ and so consists of permutation matrices. In general, $Y \uparrow^G$ consists of block permutation matrices.

Proposition 11 *1.* $Y\uparrow^G$ *is a representation of* G *which may be reducible even if* Y *is an irrep of* H*.*

2. Two transversals of H *give equiv induced reps.*

3. If $Y, Y\uparrow^G$ *have chars* $\psi, \psi\uparrow^G$ *resp then*

$$\psi\uparrow^G(g) = \frac{1}{|H|}\sum_{x\in G}\psi(x^{-1}gx).$$

4. (Frobenius Reciprocity) If χ *is a char of* G *then*

$$\langle \psi\uparrow^G, \chi\rangle = \langle \psi, \chi\downarrow_H\rangle.$$

Proof of 4. We have

$$\begin{aligned}
\langle \psi\uparrow^G, \chi\rangle &= \frac{1}{|G|}\sum_{g\in G}\psi\uparrow^G(g)\chi(g^{-1}) \\
&= \frac{1}{|G||H|}\sum_{x,g\in G}\psi(x^{-1}gx)\chi(g^{-1}) \\
&= \frac{1}{|G||H|}\sum_{x,y\in G}\psi(y)\chi(xy^{-1}x^{-1}) \\
&= \frac{1}{|G||H|}\sum_{x,y\in G}\psi(y)\chi(y^{-1}) \\
&= \frac{1}{|H|}\sum_{y\in G}\psi(y)\chi(y^{-1}) \\
&= \frac{1}{|H|}\sum_{y\in H}\psi(y)\chi(y^{-1}) = \langle \psi, \chi\downarrow_H\rangle.
\end{aligned}$$

H. The group determinant

Indeterminates $\{c_g | g \in G\}$ give the *group matrix*

$$\Gamma = (c_{g^{-1}h})_{g,h\in G}.$$

In the case $G = \{g, g^2, \ldots, g^n = \epsilon\}$, Γ is a *circulant*.

<u>Ex.</u> If $n = 3$ with rows and cols indexed ϵ, g, g^2

$$\Gamma = \begin{pmatrix} c_\epsilon & c_g & c_{g^2} \\ c_{g^2} & c_\epsilon & c_g \\ c_g & c_{g^2} & c_\epsilon \end{pmatrix} := \begin{pmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{pmatrix}.$$

Theorem 12 (Frobenius) *If the irreps G are $X^{(i)}$, $\dim X^{(i)} = d_i$, $1 \le i \le k$, then*

$$\det \Gamma = \prod_{i=1}^{k} \Delta_i^{d_i} \text{ with } \Delta_i := \left| \sum_{g\in G} X^{(i)}(g) c_g \right| \text{ irr.} \blacksquare$$

Corollary 13 $|c_{j-i}| = \prod_{\zeta^n=1} (c_0 + c_1\zeta + \cdots + c_{n-1}\zeta^{n-1}).\ \blacksquare$

<u>Ex.</u> $\begin{vmatrix} c_0 & c_1 \\ c_1 & c_0 \end{vmatrix} = c_0^2 - c_1^2 = (c_0 + c_1)(c_0 - c_1).$

Open Problem. Find a combinatorial proof of the corollary: The det counts $\mathfrak{S}_n$ with weight wt_1. The product counts $\mathcal{F} = \{f : \{1, \ldots, n\} \to \{1, \ldots, n\}\}$ with weight wt_2. Partition $\mathfrak{S}_n = \uplus_i S_i$, $\mathcal{F} = \uplus_j F_j$ s.t. $\sum_{f\in F_j} \mathrm{wt}_2 f = 0$ for certain F_j and for the rest there's a weight preserving bijection with the S_i.

II. Reps of $\mathfrak{S}_n$: A. Permutation modules

The number of irreps of $\mathfrak{S}_n$ is the number of conjugacy classes which is the same as the number of *partitions* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *of* n, $\lambda \vdash n$, i.e.,

$$\lambda \in (\mathbb{Z}^+)^l \text{ is weakly decreasing and } \textstyle\sum_i \lambda_i = n.$$

To every λ is associated a *Young subgroup*

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots$$

The corresponding coset rep M^λ (for $1 \uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}$) is called a *permutation module*. These are not irreducible, but we will find an ordering $>$ of partitions such that

$$M^\lambda = S^\lambda \oplus \bigoplus_{\mu > \lambda} K_{\mu\lambda} S^\mu$$

where the S^μ are irreps and the $K_{\mu\lambda}$ multiplicities.

To conveniently describe M^λ: The *Ferrers diagram* of λ is the set of dots or cells

$$\lambda = \{(i,j) \in (\mathbb{Z}^+)^2 \mid 1 \le j \le \lambda_i\}.$$

<u>Ex.</u>

$$(4,4,2) = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \end{array} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & (2,3) & \\ \hline & & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}.$$

A *Young tableau of shape* λ or λ-*tableau*, written $t = t^\lambda$ or $\operatorname{sh} t = \lambda$, is a bijection

$$t : \lambda \to \{1, 2, \ldots, n\}, \quad t_{i,j} := t(i, j).$$

A *tabloid*, $\{t\}$, is an equivalence class of tableaux with the same corresponding rows.

<u>Ex.</u> All tableaux of shape $(2, 1)$ are

$$\begin{array}{l}1 \;\; 2,\\ 3\end{array} \quad \begin{array}{l}2 \;\; 1,\\ 3\end{array} \quad \begin{array}{l}1 \;\; 3,\\ 2\end{array} \quad \begin{array}{l}3 \;\; 1,\\ 2\end{array} \quad \begin{array}{l}2 \;\; 3,\\ 1\end{array} \quad \begin{array}{l}3 \;\; 2.\\ 1\end{array}$$

If t is the first tableau in the list

$$\{t\} = \left\{ \begin{array}{l}1 \;\; 2,\\ 3\end{array} \quad \begin{array}{l}2 \;\; 1\\ 3\end{array} \right\} := \begin{array}{l}\overline{\;1 \;\; 2\;}\\ \overline{\;3\;}\\ \hline\end{array}.$$

A $\pi \in \mathfrak{S}_n$ acts on tableau $t = (t_{i,j})$ by $\pi t = (\pi t_{i,j})$ and thus acts on tabloids. With this action

$$M^\lambda = \mathbb{C}[\{\mathbf{t}\} \mid \text{all } \lambda\text{-tabloids } \{t\}].$$

<u>Ex.</u> $\lambda = (n)$ gives the trivial rep

$$M^{(n)} = \mathbb{C}[\,\overline{\underline{\;\mathbf{1} \;\; \mathbf{2} \;\; \cdots \;\; \mathbf{n}\;}}\,].$$

$\lambda = (1, 1, \ldots, 1) := (1^n)$ gives the regular rep

$$M^{(1^n)} \cong \mathbb{C}[\mathfrak{S}_n].$$

$\lambda = (n-1, 1)$ gives the defining rep (ignore 1st row)

$$M^{(n-1,1)} \cong \mathbb{C}[\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}].$$

B. Orderings on partitions

For partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ of n, the *dominance partial order*, $\lambda \trianglerighteq \mu$, is

$$\text{for all } i \geq 1\text{:} \quad \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

and the *lexicographic (lex) total order*, $\lambda > \mu$, is

$$\text{for some } i \geq 1\text{:} \quad \lambda_i > \mu_i \text{ and } \lambda_j = \mu_j \text{ for } j < i.$$

<u>Ex.</u>

$(3,3) \triangleright (3,2,1): \; 3 \geq 3, 3+3 > 3+2, \ldots$
$(3,3), (4,1,1)$ incomp in $\trianglerighteq: \; 3 < 4, 3+3 > 4+1.$
$(4,1,1) > (3,3): \; 4 > 3.$
$(3,3) > (3,2,1): \; 3 = 3, 3 > 2.$

Proposition 14 *1. $\lambda \trianglerighteq \mu$ implies $\lambda \geq \mu$.*

2. (Dominance Lemma, DL) If $\forall i$ entries of row i of tableau s^μ are in different col's of t^λ then $\lambda \trianglerighteq \mu$.

Proof. 2. Sort each column of t^λ so the entries in the first i rows of s^μ lie in the first i rows of t^λ.

$$\begin{aligned} \textstyle\sum_{j \leq i} \lambda_j &= \text{ \# elements in first } i \text{ rows of } t^\lambda \\ &\geq \text{ \# elements in first } i \text{ rows of } s^\mu = \textstyle\sum_{j \leq i} \mu_j. \end{aligned}$$

C. The irreducible Specht modules

If $H \subseteq \mathfrak{S}_n$ then let

$$H^- = \sum_{\pi \in H} (\mathrm{sgn}\, \pi)\pi \in \mathbb{C}[\mathfrak{S}_\mathbf{n}].$$

If tableau t has columns $C_1, \ldots, C_m$ then let

$$\begin{aligned}
&C_t := \mathfrak{S}_{C_1} \times \cdots \times \mathfrak{S}_{C_m} \text{ (the \textit{column group})},\\
&\kappa_t := C_t^- = \kappa_{C_1}\kappa_{C_2}\cdots\kappa_{C_m},\\
&\mathbf{e}_t := \kappa_t\{\mathbf{t}\} \text{ (the \textit{polytabloid}).}
\end{aligned}$$

<u>Ex.</u> If $t = \begin{array}{ccc} 4 & 1 & 2 \\ 3 & 5 & \end{array}$ then

$$\begin{aligned}
C_t &= \mathfrak{S}_{\{3,4\}} \times \mathfrak{S}_{\{1,5\}} \times \mathfrak{S}_{\{2\}},\\
\kappa_t &= \epsilon - (3,4) - (1,5) + (3,4)(1,5)\\
&= (\epsilon - (3,4))(\epsilon - (1,5)),\\
\mathbf{e}_t &= \begin{array}{ccc}\hline \mathbf{4} & \mathbf{1} & \mathbf{2} \\ \hline \mathbf{3} & \mathbf{5} & \\ \hline \end{array} - \begin{array}{ccc}\hline \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \hline \mathbf{4} & \mathbf{5} & \\ \hline \end{array} - \begin{array}{ccc}\hline \mathbf{4} & \mathbf{5} & \mathbf{2} \\ \hline \mathbf{3} & \mathbf{1} & \\ \hline \end{array} + \begin{array}{ccc}\hline \mathbf{3} & \mathbf{5} & \mathbf{2} \\ \hline \mathbf{4} & \mathbf{1} & \\ \hline \end{array}
\end{aligned}$$

Lemma 15 *If $\pi \in \mathfrak{S}_n$ and t is a tableau then*

$$\kappa_{\pi t} = \pi \kappa_t \pi^{-1} \quad \textit{and} \quad \mathbf{e}_{\pi t} = \pi \mathbf{e}_t.$$

Partition λ has *Specht module*

$$S^\lambda = \mathbb{C}[\mathbf{e}_t | \text{ all } \lambda\text{-tableaux } t].$$

Ex. 1. $\lambda = (n)$ gives the trivial rep: Any (n)-tableau t has $\mathbf{e}_t = \begin{array}{c}\hline \mathbf{1} \;\; \mathbf{2} \;\; \cdots \;\; \mathbf{n} \\ \hline\end{array}$ so

$$\pi \mathbf{e}_t = \mathbf{e}_{\pi t} = \mathbf{e}_t.$$

2. $\lambda = (1^n)$ gives the sign rep: For any $t = t^{(1^n)}$

$$\pi \mathbf{e}_t = \pi \mathfrak{S}_n^- \{\mathbf{t}\} = (\operatorname{sgn} \pi)\mathbf{e}_t.$$

3. $\lambda = (n-1, 1)$: Abbreviate $t = t^\lambda$ to the 2nd row

$$\begin{aligned} \mathbf{e}_t &= \begin{array}{l}\hline \mathbf{i} \;\; \cdots \;\; \mathbf{k} \\ \hline \mathbf{j} \\ \hline\end{array} - \begin{array}{l}\hline \mathbf{j} \;\; \cdots \;\; \mathbf{k} \\ \hline \mathbf{i} \\ \hline\end{array} = \mathbf{j} - \mathbf{i}, \\ S^{(n-1,1)} &= \mathbb{C}[\mathbf{j} - \mathbf{i} | 1 \le i < j \le n], \\ &= \{\textstyle\sum_{i=1}^n c_i \mathbf{i} | \sum_{i=1}^n c_i = 0\}. \end{aligned}$$

A G-module U is *cyclic, generated by* $\mathbf{u} \in U$ if

$$U = \mathbb{C}[g\mathbf{u} \mid g \in G].$$

Corollary 16 *S^λ is cyclic generated by any $\mathbf{e}_t \in S^\lambda$.*

Define an $\mathfrak{S}_n$-invariant inner product on M^λ by

$$\langle \{\mathbf{t}\}, \{\mathbf{s}\} \rangle = \delta_{\{t\},\{s\}}.$$

Lemma 17 (Sign Lemma, SL) *Let* $H \le \mathfrak{S}_n$.

1. $\pi \in H \Rightarrow \pi H^- = H^- \pi = (\operatorname{sgn} \pi) H^-$.

2. $\mathbf{u}, \mathbf{v} \in M^\lambda \Rightarrow \langle H^- \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, H^- \mathbf{v} \rangle$.

3. $(b, c) \in H \Rightarrow H^- = k(\epsilon - (b, c))$ *for some* $k \in \mathbb{C}[\mathfrak{S}_\mathbf{n}]$.

4. b, c *in the same row of tableau* s *and* $(b, c) \in H$ $\Rightarrow H^- \{\mathbf{s}\} = \mathbf{0}$.

Corollary 18 *I. If* $\operatorname{sh} t = \lambda, \operatorname{sh} s = \mu$ *with* $\kappa_t \{\mathbf{s}\} \ne 0$ *then* $\lambda \trianglerighteq \mu$. *If* $\lambda = \mu$ *then* $\kappa_t \{\mathbf{s}\} = \pm \mathbf{e}_t$.

II. (James' Submodule Theorem) If U *is a submodule of* M^μ *then* $U \supseteq S^\mu$ *or* $U \subseteq S^{\mu \perp}$.

III. The $S^\mu, \mu \vdash n$, *are all inequiv* $\mathfrak{S}_n$*-irreps over* $\mathbb{C}$.

Proof. I. b, c in the same row of $s \Rightarrow b, c$ not in the same col of t (else $\kappa_t \{\mathbf{s}\} = \mathbf{0}$ by SL4) $\Rightarrow \lambda \trianglerighteq \mu$ (DL). If $\lambda = \mu \Rightarrow \{s\} = \pi \{t\}$ for some $\pi \in \mathfrak{S}_n$ and by SL1

$$\kappa_t \{\mathbf{s}\} = \kappa_t \pi \{\mathbf{t}\} = (\operatorname{sgn} \pi) \kappa_t \{\mathbf{t}\} = \pm \mathbf{e}_t.$$

II. If $\mathbf{u} \in U$ and $t = t^\mu \Rightarrow \kappa_t \mathbf{u} = c \mathbf{e}_t$ for $c \in \mathbb{C}$ by I. If some $c \ne 0 \Rightarrow \mathbf{e}_t \in U$ and $S^\mu \subseteq U$. Else use SL2 to show $U \subseteq S^{\mu \perp}$.

D. The standard tableaux basis for S^λ

Tableau t is *standard* if its rows and col's increase.

Ex. $\begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix}$ is standard; $\begin{matrix} 2 & 3 & 4 \\ 1 & 5 & \end{matrix}$ is not.

Theorem 19 *A basis for S^λ is*

$$\{\mathbf{e}_t \mid t \text{ a standard } \lambda\text{-tableau}\}.$$

Independence. A *composition* is a permutation of a partition. If $\{t\}$ is a tabloid, for $i \geq 1$ let

$$\begin{aligned} \{t\}^i &= \text{tabloid of all entries} \leq i \text{ in } \{t\}, \\ \lambda^i &= \text{the shape of } \{t\}^i, \text{ a composition.} \end{aligned}$$

Ex. If $\{t\} = \begin{array}{ll} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}$ then

$$\begin{array}{lll} \{t\}^1 = \begin{array}{l} \hline \emptyset \\ \hline 1 \\ \hline \end{array} & \{t\}^2 = \begin{array}{l} \hline 2 \\ \hline 1 \\ \hline \end{array} & \{t\}^3 = \begin{array}{ll} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array} \\ \lambda^1 = (0,1) & \lambda^2 = (1,1) & \lambda^3 = (2,1). \end{array}$$

Dominance order on tabloids is

$$\{t\} \trianglerighteq \{s\} \iff \lambda^i \trianglerighteq \mu^i \quad \forall i.$$

Proposition 20 *1. (Tabloid Dominance Lemma) If $k < l$ and k is lower than l in $\{t\}$ then $(k,l)\{t\} \triangleright \{t\}$.*

2. t standard and $\{\mathrm{s}\}$ appears in $\mathbf{e}_t \Rightarrow \{t\} \trianglerighteq \{s\}$.

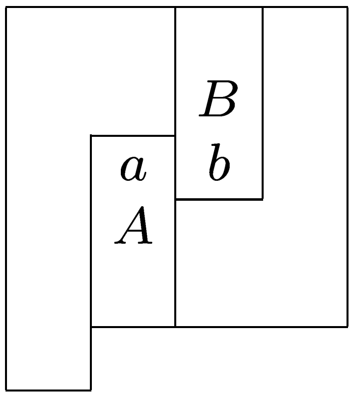
3. The standard $\mathbf{e}_t$ are independent.

Span. To show $\mathbf{e}_t$ a lin comb of standard $\mathbf{e}_s$ one can assume the col's of t increase. (Else $\exists \pi \in C_t$ with col's of πt increasing and $\mathbf{e}_{\pi t} = (\operatorname{sgn} \pi)\mathbf{e}_t$.) If t has row descent $a > b$, it suffices to find tableaux s s.t.

1. $\mathbf{e}_t = -\sum_s (\operatorname{sgn} \pi_s)\mathbf{e}_s$ where $\pi_s t = s$,

2. $[s] \rhd [t]$ for all s, $[s]$ = col tabloid.

A (resp B) := entries of t below a (resp above b).

$t =$ 

.

The s are all tableaux gotten by permuting $A \cup B$ s.t. the elements of $A \cup B$ still increase in their col's.

Ex. If $t = \begin{matrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{matrix}$ with $2 > 1 \Rightarrow A = \{2, 3\}, B = \{1\}$,

$$s_1 = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix} \qquad s_2 = \begin{matrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{matrix}$$

$$\pi_1 = (1, 3, 2) \qquad \pi_2 = (1, 2)$$

$$\mathbf{e}_t = -(\mathbf{e}_{s_1} - \mathbf{e}_{s_2}).$$

E. Young's natural representation

The matrix rep X^λ for S^λ in the standard basis is *Young's natural rep.* Since $(k, k+1), 1 \leq k < n$, generate $\mathfrak{S}_n$ it suffices to compute $M = X^\lambda((k, k+1))$. If t is standard than to find $M_{t,t}$ we must express $(k, k+1)\mathbf{e}_t$ in the standard basis.

1. If $k, k+1$ in the same col of $t \Rightarrow (k, k+1) \in C_t$

$$\therefore (k, k+1)\mathbf{e}_t = -\mathbf{e}_t.$$

2. If $k, k+1$ in the same row of $t \Rightarrow (k, k+1)t$ has row descent $k+1 > k$

$$\therefore (k, k+1)\mathbf{e}_t = \mathbf{e}_t \pm \text{ other } \mathbf{e}_s \text{ with } [s] \rhd [t].$$

3. Else $(k, k+1)t = t'$ where t' is standard

$$\therefore (k, k+1)\mathbf{e}_t = \mathbf{e}_{t'}.$$

<u>Ex.</u> If $\lambda = (2, 1)$ then the standard tableaux are

$$t_1 = \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \quad \text{and} \quad t_2 = \begin{array}{ll} 1 & 2 \\ 3 & \end{array}.$$

If $(k, k+1) = (1, 2)$ then

$$(1,2)\mathbf{e}_{t_1} = \begin{array}{cc} \hline \mathbf{2} & \mathbf{3} \\ \hline \mathbf{1} & \\ \hline \end{array} - \begin{array}{cc} \hline \mathbf{1} & \mathbf{3} \\ \hline \mathbf{2} & \\ \hline \end{array} = -\mathbf{e}_{t_1}.$$

$(1,2)\mathbf{e}_{t_2}$ was essentially computed last slide.

$$\therefore X^{(2,1)}((1,2)) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

F. The Branching and Young Rules

Partition λ has *inner corner* $(i,j) \in \lambda$ if

$$\lambda^- = \lambda \setminus (i,j) \text{ is a partition,}$$

and *outer corner* $(i,j) \notin \lambda$ if

$$\lambda^+ = \lambda \cup (i,j) \text{ is a partition.}$$

<u>Ex.</u> If $\lambda = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \end{array}$ then

$$\lambda^- : \quad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array} \qquad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

$$\lambda^+ : \quad \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & & \\ \bullet & \bullet & & & \end{array} \qquad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \end{array} \qquad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

Theorem 21 (Branching Rule) *If $\lambda \vdash n$ then*

1. $S^\lambda \downarrow_{\mathfrak{S}_{n-1}} \cong \bigoplus_{\lambda^-} S^{\lambda^-}$,
2. $S^\lambda \uparrow^{\mathfrak{S}_{n+1}} \cong \bigoplus_{\lambda^+} S^{\lambda^+}$.

<u>Ex.</u> From the example above

$$\begin{aligned} S^{(4,2,2)} \downarrow_{\mathfrak{S}_7} &\cong S^{(3,2,2)} \oplus S^{(4,2,1)}, \\ S^{(4,2,2)} \uparrow^{\mathfrak{S}_9} &\cong S^{(5,2,2)} \oplus S^{(4,3,2)} \oplus S^{(4,2,2,1)}. \end{aligned}$$

Tableau T is called *semistandard* if it has strictly increasing columns while its rows weakly increase. The *content* of T, $\operatorname{ct} T$, is the composition μ s.t.

$$\mu_i = \text{ \# of } i\text{'s in } T.$$

<u>Ex.</u> $T = \begin{array}{ccc} 1 & 1 & 4 \\ 2 & 4 & \end{array}$ has content $\mu = (2,1,0,2)$.

The *Kostka numbers* are

$K_{\lambda\mu} =$ # of semistandard T, shape λ, content μ.

Theorem 22 (Young's Rule)

$$M^{\mu} \cong \bigoplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} S^{\lambda}.$$

<u>Ex.</u> If $\mu = (2,1,1)$ then the possible $\lambda \trianglerighteq \mu$ are

$$\begin{array}{ccccc}
\lambda: & (2,1,1) & (2,2) & (3,1) & (4) \\
T: & \begin{array}{cc} 1 & 1 \\ 2 & \\ 3 & \end{array} & \begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array} & \begin{array}{ccc} 1 & 1 & 2 \\ 3 & & \end{array} & \begin{array}{cccc} 1 & 1 & 2 & 3 \end{array} \\
 & & & \begin{array}{ccc} 1 & 1 & 3 \\ 2 & & \end{array} &
\end{array}$$

So $M^{(2,1,1)} \cong S^{(2,1,1)} \oplus S^{(2,2)} \oplus 2S^{(3,1)} \oplus S^{(4)}$.

Note: For any μ, $K_{\mu\mu} = 1 = K_{(n)\mu}$.

III. Combinatorics: A. Schensted's algorithm

Let $\mathsf{SYT}(\lambda)$ be the set of standard λ-tableaux and

$$f^\lambda := |\,\mathsf{SYT}(\lambda)| = \dim S^\lambda.$$

For any group with irreps $V^{(i)}$: $\sum_i (\dim V^{(i)})^2 = |G|$. If $G = \mathfrak{S}_n$ the formula can be proved combinatorially

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Proof. Construct the Robinson-Schensted bijection

$$\pi \overset{\mathsf{R-S}}{\longleftrightarrow} (P, Q)$$

where $\pi \in \mathfrak{S}_n$ and $P, Q \in \mathsf{SYT}(\lambda)$ for some λ.

$\pi \overset{\mathsf{R-S}}{\to} (P, Q)$: *Insert* $x \in \mathbb{Z}^+$ into increasing tableau P to get increasing tableau P', $r_x(P) = P'$, by

1. Let $i := 1$

2. If $x >$ every element of row i of P, put it at the end of the row and stop.

3. Else exchange x and the smallest $P_{i,j} > x$. (We say x *bumps* $P_{i,j}$.) Set $i := i + 1$ and go to 2.

<u>Ex.</u> Suppose $x = 2$

$$P = \begin{array}{ll} 1 & 3 \\ 4 & 5 \end{array} \leftarrow \mathbf{2} \quad \begin{array}{lll} 1 & \mathbf{2} & \\ 4 & 5 & \leftarrow \mathbf{3} \end{array} \quad \begin{array}{ll} 1 & 2 \\ \mathbf{3} & 5 \\ \leftarrow \mathbf{4} & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 3 & 5 \\ \mathbf{4} & \end{array} = r_2(P).$$

Now if $\pi = x_1 \ldots x_n$ then construct a sequence of pairs $(\emptyset, \emptyset) = (P_0, Q_0), \ldots, (P_n, Q_n) = (P, Q)$ by

$$\begin{aligned} P_k &= r_{x_k}(P_{k-1}), \\ Q_k &= Q_{k-1} \uplus k \text{ with } k \text{ in } \mathrm{sh}\, P_k \setminus \mathrm{sh}\, P_{k-1}. \end{aligned}$$

<u>Ex.</u> $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \mathbf{3} & \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{5} \end{matrix} \overset{\text{R–S}}{\to} \left(\begin{matrix} \mathbf{1} & \mathbf{4} & \mathbf{5} \\ \mathbf{2} & & \\ \mathbf{3} & & \end{matrix}, \begin{matrix} 1 & 3 & 5 \\ 2 & & \\ 4 & & \end{matrix} \right)$ by

$$\mathbf{P_k}: \ \emptyset, \quad \mathbf{3}, \quad \begin{matrix} \mathbf{2}, \\ \mathbf{3} \end{matrix} \quad \begin{matrix} \mathbf{2} & \mathbf{4}, \\ \mathbf{3} & \end{matrix} \quad \begin{matrix} \mathbf{1} & \mathbf{4}, \\ \mathbf{2} & \\ \mathbf{3} & \end{matrix} \quad \begin{matrix} \mathbf{1} & \mathbf{4} & \mathbf{5} \\ \mathbf{2} & & \\ \mathbf{3} & & \end{matrix} = \mathbf{P}$$

$$Q_k: \ \emptyset, \quad 1, \quad \begin{matrix} 1, \\ 2 \end{matrix} \quad \begin{matrix} 1 & 3, \\ 2 & \end{matrix} \quad \begin{matrix} 1 & 3, \\ 2 & \\ 4 & \end{matrix} \quad \begin{matrix} 1 & 3 & 5 \\ 2 & & \\ 4 & & \end{matrix} = Q$$

$(P, Q) \overset{\text{R–S}}{\to} \pi$: *Delete* $P_{i,j}$, $r^{-1}_{(i,j)} P = (P', x)$, by

1. Remove $x := P_{i,j}$ from its row and set $i := i - 1$.
2. While $i \geq 1$ exchange x and the greatest $R_{i,j} < x$ and set $i := i - 1$.

<u>Ex.</u> Do the Ex on the previous page backwards.

Starting with (P, Q) we obtain the reverse sequence $(P_n, Q_n), \ldots, (P_0, Q_0)$ and $\pi = x_1 \ldots x_n$ by

$$\begin{aligned} Q_{k-1} &= Q_k \setminus k \\ (P_{k-1}, x_k) &= r^{-1}_{(i,j)} P_k \quad \text{where} \quad Q_{i,j} = k. \end{aligned}$$

B. Properties of Robinson-Schensted

If $\pi \overset{\text{R–S}}{\to} (P, Q)$ then the P*-tableau of* π is $P(\pi) = P$ and the Q*-tableau of* π is $Q(\pi) = Q$.

If $\pi = x_1 \ldots x_n$ then $\pi^r = x_n \ldots x_1$.

A *subsequence* of $\pi = x_1 \ldots x_n$, $\sigma \subseteq \pi$, is

$$\sigma = x_{k_1}, x_{k_2}, \ldots, x_{k_m} \quad \text{with} \quad k_1 < k_2 < \ldots < k_m.$$

Proposition 23 *1.* $P(\pi^r) = P(\pi)^t$ *(the transpose)*

2. If $\operatorname{sh} P(\pi) = (\lambda_1, \ldots, \lambda_l)$ *then*

$$\begin{aligned} \lambda_1 &= \textit{length of a longest increasing } \sigma \subseteq \pi, \\ l &= \textit{length of a longest decreasing } \sigma \subseteq \pi. \end{aligned}$$

3. If $\pi \overset{\text{R–S}}{\to} (P, Q)$ *then* $\pi^{-1} \overset{\text{R–S}}{\to} (Q, P)$.

4. $\sum_{\lambda \vdash n} f^\lambda = \#$ *of involutions in* $\mathfrak{S}_n$.

Proof. 1. One can define column insertion $c_y(P)$ and prove $r_x c_y(P) = c_y r_x(P)$. Then

$$\begin{aligned} P(\pi^r) &= r_{x_1} \cdots r_{x_n}(\emptyset) = r_{x_1} \cdots r_{x_{n-1}} c_{x_n}(\emptyset) \\ &= c_{x_n} r_{x_1} \cdots r_{x_{n-1}}(\emptyset) = \ldots = c_{x_n} \cdots c_{x_1}(\emptyset) = P(\pi)^t. \end{aligned}$$

4. By 3: $\pi \overset{\text{R–S}}{\to} (P, P)$ iff $\pi = \pi^{-1}$. So

$$\sum_{\lambda \vdash n} f^\lambda = \# \text{ of P} = \# \text{ of involutions } \pi.$$

When does $P(\pi) = P(\sigma)$?

Ex. For $\mathfrak{S}_3$: $P(123) = 1\ 2\ 3$, $P(321) = (1\ 2\ 3)^t$,

$$P(213) = P(231) = \begin{array}{ll} 1 & 3 \\ 2 & \end{array}, P(132) = P(312) = \begin{array}{ll} 1 & 2 \\ 3 & \end{array}.$$

π, σ differ by a *Knuth transposition* if for $x < y < z$:

1. $\{\pi, \sigma\} = \{x_1 \ldots yxz \ldots x_n,\ x_1 \ldots yzx \ldots x_n\}$, or
2. $\{\pi, \sigma\} = \{x_1 \ldots zxy \ldots x_n,\ x_1 \ldots xzy \ldots x_n\}$.

Also π, σ are *Knuth equivalent*, $\pi \overset{K}{\cong} \sigma$, if

$$\pi = \pi_1, \pi_2, \ldots, \pi_k = \sigma$$

with π_i, π_{i+1} differing by a Knuth transposition $\forall i$.

Ex. $2\ 1\ 3 \overset{K}{\cong} 2\ 3\ 1$ and $1\ 3\ 2 \overset{K}{\cong} 3\ 1\ 2$.

Theorem 24 (Knuth) $P(\pi) = P(\sigma) \iff \pi \overset{K}{\cong} \sigma$.

Proof sketch. "$\Leftarrow$" Type 1 transposition: x's (resp z's) insertion path is weakly left (resp strictly right) of y's so $P(\pi) = P(\sigma)$. Type 2: then π^r, σ^r differ by type 1 and

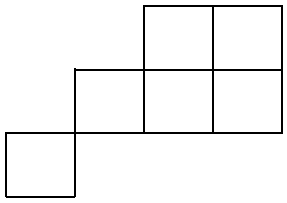
$$P(\pi^r) = P(\sigma^r) \Rightarrow P(\pi)^t = P(\sigma)^t \Rightarrow P(\pi) = P(\sigma).$$

C. Schützenberger's jeu de taquin

If $\mu \subseteq \lambda$ then one has the *skew diagram*

$$\lambda/\mu = \{(i,j) \mid (i,j) \in \lambda,\ (i,j) \notin \mu\}.$$

<u>Ex.</u> If $\mu = (2,1)$ and $\lambda = (4,4,1)$

$\lambda/\mu =$  .

If P is an increasing tableau, sh $P = \lambda/\mu$, a *backward slide* into an inner corner c of μ, $j^c(P) = P'$, is

While $c = (i,j)$ is not an inner corner of λ, exchange c and the smaller of $P_{i+1,j}, P_{i,j+1}$.

<u>Ex.</u> If $c = (1,2)$ then

$$P = \begin{array}{ccc} & \bullet & 1 & 5 \\ & 2 & 3 & 7 \\ 4 & & & \end{array}, \quad \begin{array}{ccc} & 1 & \bullet & 5 \\ & 2 & 3 & 7 \\ 4 & & & \end{array}, \quad \begin{array}{ccc} & 1 & 3 & 5 \\ & 2 & \bullet & 7 \\ 4 & & & \end{array}, \quad \begin{array}{ccc} & 1 & 3 & 5 \\ & 2 & 7 & \bullet \\ 4 & & & \end{array} = j^c(P).$$

A *forward slide* into outer corner $d = (i,j)$ of λ, $j_d(P) = P'$, exchanges d with the larger of the numbers $P_{i-1,j}, P_{i,j-1}$, etc. until an outer corner of μ is reached. Clearly if $j^c(P) = P'$ vacating d then

$$j_d j^c(P) = P \quad \text{and} \quad j^c j_d(P') = P'.$$

Let $\delta_n = (n-1, n-2, \ldots, 1)$. Any $\pi = x_1 \ldots x_n$ has a δ_{n+1}/δ_n-tableau with x_j in $(n-j+1, j)$.

Ex. $\pi = 132$ has tableau $\pi = \begin{array}{|c|c|c|} \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \end{array}$.

A *backward slide sequence* for $P = P_1$ is

$$(c_1, \ldots, c_l) \text{ with } P_{i+1} = j^{c_i}(P_i) \text{ defined } \forall i.$$

If $l = |\mu|$ where $\operatorname{sh} P = \lambda/\mu$ let $j(P) := j^{c_l} \cdots j^{c_1}(P)$.

Ex. (cont) If $c_1 = (2,1), c_2 = (1,2), c_3 = (1,1)$

$$\pi = \begin{matrix} & & 2 \\ & 3 & \\ 1 & & \end{matrix}, \quad \begin{matrix} & & 2 \\ 1 & 3 & \end{matrix}, \quad \begin{matrix} & 2 \\ 1 & 3 \end{matrix}, \quad \begin{matrix} 1 & 2 \\ 3 & \end{matrix} = j(\pi).$$

Theorem 25 (Schützenberger) $j(\pi) = P(\pi)$.

Proof sketch. If P has rows $R_1, \ldots, R_l$ then its *row word* is $\rho(P) = R_l R_{l-1} \ldots R_1$.

Ex. $P = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & \end{matrix}$ has $\rho(P) = 2\ 4\ 6\ 1\ 3\ 5\ 7$.

It is easy to prove $P(\rho(P)) = P$. Furthermore if P is skew and $P' = j^c(P)$ then $\rho(P') \overset{\mathrm{K}}{\cong} \rho(P)$. So

$$\rho(j(\pi)) \overset{\mathrm{K}}{\cong} \rho(\pi) = \pi \overset{\text{apply}P}{\Rightarrow} j(\pi) = P(\pi). \blacksquare$$

D. The hook formula

The *hook* and *hooklength* of $(i,j) \in \lambda$ are

$$H_{i,j} = \{(i',j),(i,j') \in \lambda \mid i' \geq i, j' \geq j\},\ h_{i,j} = |H_{i,j}|.$$

The *arm length* and *leg length* of the hook are

$$a_{i,j} = |\{(i,j') \in \lambda \mid j' > j\}|,\ l_{i,j} = |\{(i',j) \in \lambda \mid i' > i\}|.$$

<u>Ex.</u> In $\lambda = (4^2,3,1)$

$$H_{2,2} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \bullet & a & a \\ \hline & l & & \\ \cline{1-3} & \\ \cline{1-1} \end{array} \text{ and } h_{2,2} = 4,\ a_{2,2} = 2,\ l_{2,2} = 1.$$

Theorem 26 (Frame-Robinson-Thrall) *If we have $\lambda \vdash n$, then*

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

<u>Ex.</u> $(3,2) \vdash 5$ has hooklengths

$$\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2} \end{array}.$$

So $f^{(3,2)} = \frac{5!}{4\cdot3\cdot2\cdot1^2} = 5$ which agrees with

$$\begin{matrix}1\ 2\ 3, \\ 4\ 5\end{matrix} \quad \begin{matrix}1\ 2\ 4, \\ 3\ 5\end{matrix} \quad \begin{matrix}1\ 2\ 5, \\ 3\ 4\end{matrix} \quad \begin{matrix}1\ 3\ 4, \\ 2\ 5\end{matrix} \quad \begin{matrix}1\ 3\ 5. \\ 2\ 4\end{matrix}$$

The Novelli-Pak-Stoyanovskii Proof. Preprint: http://www.math.harvard.edu/~pak/papers

Show $n! = f^\lambda \prod_{(i,j)} h_{i,j}$ with a bijection

$$T \longleftrightarrow (P, J)$$

where $\operatorname{sh} T = \operatorname{sh} P = \operatorname{sh} J = \lambda$, T is any Young tableau, P is standard, and

$$-l_{i,j} \leq J_{i,j} \leq a_{i,j} \qquad \forall (i,j) \in \lambda.$$

$T \to (P, J)$: If T is standard of shape λ/μ and entry $x \in \mathbb{Z}^+$ is in c then $j^c(T)$ has x moving in place of $\bullet$ and terminating when it becomes standard.

<u>Ex.</u> If $c = (1,2)$ contains 6

$$T = \begin{array}{ccc} & \mathbf{6}\ 1\ 5, \\ & 2\ 3\ 7 \\ 4 & \end{array} \quad \begin{array}{ccc} & 1\ \mathbf{6}\ 5, \\ & 2\ 3\ 7 \\ 4 & \end{array} \quad \begin{array}{ccc} & 1\ 3\ 5 \\ & 2\ \mathbf{6}\ 7 \\ 4 & \end{array} = j^c(T).$$

Lex order λ's cells $c_1 > c_2 > \ldots > c_n$. Define

$$T = T_1, \ldots, T_n = P \quad \text{where} \quad T_k = j^{c_k}(T_{k-1}).$$

Define $J_1, \ldots, J_n = J$ by $J_1 = 0$ and if j^{c_k} starts in $c_k = (i,j)$ and ends in (i', j') then $J_k = J_{k-1}$ except

$$(J_k)_{i,l} = \begin{cases} (J_{k-1})_{i,l+1} + 1 & \text{for } j \leq l < j', \\ i - i' & \text{for } l = j'. \end{cases}$$

Ex. For spacing purposes we use $\bar{1}$ for -1.

$$\begin{array}{rcccccccl} T_1 = & 645, & 645, & 645, & 643, & 623, & 123 & = P. \\ & 231 & 213 & 123 & 125 & 145 & 456 & \\ J_1 = & 000, & 000, & 000, & 00\bar{1}, & 0\bar{1}\bar{1}, & 00\bar{1} & = J. \\ & 000 & 010 & 200 & 200 & 200 & 200 & \end{array}$$

$(P, J) \to T$: To reconstruct $(P, J) = (T_n, J_n), \ldots,$ $(T_1, J_1) = (T, 0)$, assume (T_k, J_k) has been constructed. The *possible cells* for $c_k = (i, j)$ in T_k are

$$\mathcal{P} = \{(i', j') | i' \geq i, j' \geq j, (J_k)_{i,j'} \leq 0, i' = i - (J_k)_{i,j'}\}.$$

Define j_d for $d \in \mathcal{P}$ by having the slide stop at c_k. (must prove well-defined) The *code* of j_d replaces each move north (resp west) with N (resp W) written in reverse order.

Ex. For $c_6 = (1, 1)$: $\mathcal{P} = \{(1, 1), (1, 2), (2, 3)\}$ and

$$j_{1,1} : \emptyset, \quad j_{1,2} : W, \quad j_{2,3} = NWW.$$

Lex order the codes using $W < \emptyset < N$. Then

$$T_{k-1} = j_d(T_k) \text{ where } d \in \mathcal{P} \text{ has maximum code.}$$

Also if $c_k = (i, j)$, $d = (i', j')$ then $J_{k-1} = J_k$ except

$$(J_{k-1})_{i,l} = \begin{cases} (J_k)_{i,l-1} - 1 & \text{for } j < l \leq j' \\ 0 & \text{for } l = j. \end{cases}$$

E. The determinantal formula

Theorem 27 (Frobenius) *If* $(\lambda_1, \ldots, \lambda_l) \vdash n$ *then*

$$f^\lambda = n! \det(1/(\lambda_i - i + j)!)$$

where the determinant is $l \times l$ *and* $1/r! = 0$ *if* $r < 0$.

<u>Ex.</u> $f^{(3,2)} = 5! \begin{vmatrix} 1/3! & 1/4! \\ 1/1! & 1/2! \end{vmatrix} = 5.$

Proof. It suffices to show the determinant equals the hook formula. We have

$$\lambda_i + l = h_{i,1} + i \quad \Rightarrow \quad \lambda_i - i + j = h_{i,1} - l + j.$$

So every row of the determinant is of the form

$$[\cdots \quad 1/(h-2)! \quad 1/(h-1)! \quad 1/h!] .$$

After factoring out $\prod_i 1/h_{i,1}!$ we get rows

$$[\cdots \quad h(h-1) \quad h \quad 1]$$

which by column operations can be turned into

$$[\cdots \quad (h-1)(h-2) \quad h-1 \quad 1] .$$

Putting $\prod_i 1/(h_{i,1}-1)!$ back in we get $\prod_i 1/h_{i,1}$ times the det for λ with its first column removed, so we're done by induction. ■

IV. Symmetric functions: A. Bases

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ and also consider $\mathbb{C}[[\mathbf{x}]]$, the corresponding formal power series algebra. Then $\pi \in \mathfrak{S}_n$ acts on $f \in \mathbb{C}[[\mathbf{x}]]$ by

$$\pi f(x_1, x_2, \ldots) = f(x_{\pi 1}, x_{\pi 2}, \ldots),\ \pi(m) := m, m > n.$$

We say f is *symmetric* if

$$\pi f = f, \quad \forall \pi \in \mathfrak{S}_n, \forall n.$$

Each partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ has an associated *monomial symmetric function*

$$m_\lambda = m_\lambda(\mathbf{x}) = \sum x_{i_1}^{\lambda_1} \cdots x_{i_l}^{\lambda_l}$$

where the sum is over all distinct monomials that have exponents $\lambda_1, \ldots, \lambda_l$.

<u>Ex.</u>

$$m_{(2,2,1)} = x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2 + x_1^2 x_2^2 x_4 + \cdots$$

The *algebra of symmetric functions* is

$$\Lambda = \Lambda(\mathbf{x}) = \mathbb{C}[m_\lambda].$$

Note: $f = \prod_{i \ge 1}(1 + x_i)$ is symmetric but isn't in Λ. We have a grading by degree

$$\Lambda = \oplus_{n \ge 0} \Lambda^n, \quad \dim \Lambda^n = p(n),\ \text{the \# of } \lambda \vdash n.$$

$p_n := m_{(n)} = \sum_{i\ge 1} x_i^n$ (power sum).

$e_n := m_{(1^n)} = \sum_{i_1<\ldots<i_n} x_{i_1}\cdots x_{i_n}$ (elementary).

$h_n := \sum_{\lambda\vdash n} m_\lambda = \sum_{i_1\le\ldots\le i_n} x_{i_1}\cdots x_{i_n}$ (complete homo).

Ex.
$$\begin{array}{rcl} p_3 &=& x_1^3+x_2^3+x_3^3+\cdots \\ e_3 &=& x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+\cdots \\ h_3 &=& x_1^3+\cdots+x_1^2x_2+\cdots+x_1x_2x_3+\cdots \end{array}$$

Proposition 28 *We have the generating functions*

1. $E(t) := \sum_{n\ge 0} e_n(\mathbf{x})t^n = \prod_{i\ge 1}(1+x_it).$
2. $H(t) := \sum_{n\ge 0} h_n(\mathbf{x})t^n = \prod_{i\ge 1}\frac{1}{1-x_it}.$
3. $P(t) := \sum_{n\ge 1} p_n(\mathbf{x})t^n = \ln\prod_{i\ge 1}\frac{1}{1-x_it}.$

If $f=p,e,$ or h and $\lambda=(\lambda_1,\ldots,\lambda_l)$ let $f_\lambda=\prod_i f_{\lambda_i}$.

Theorem 29 *Three bases for Λ^n are*

1. $\{e_\lambda \mid \lambda\vdash n\}$, 2. $\{h_\lambda \mid \lambda\vdash n\}$, 3. $\{p_\lambda \mid \lambda\vdash n\}$.

Proof. $1\Rightarrow$ XS2. $|\{h_\lambda\}|=p(n)$ so it suffices to show every e_n is a polynomial in h_k. But $H(t)E(-t)=1$ and taking the coefficient of t^n, $n\ge 1$,

$$\sum_{k=0}^{n}(-1)^k h_{n-k}e_k = 0 \Rightarrow e_n = h_1e_{n-1}-h_2e_{n-2}+\cdots$$

B. Schur functions

For tableau T let $\mathbf{x}^T = \mathbf{x}^\mu = x^{\mu_1} \cdots x^{\mu_m}$ where T's content is $\mu = (\mu_1, \ldots, \mu_m)$. A *Schur function* is

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T$$

summed over all semistandard T of shape λ. Note $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$.

$$\begin{array}{lllllll}
\underline{\text{Ex.}} \quad T: & \begin{smallmatrix}1 & 1\\ 2 & \end{smallmatrix} & \begin{smallmatrix}1 & 2\\ 2 & \end{smallmatrix} & \ldots & \begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix} & \begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix} & \ldots \\
s_{(2,1)} = & x_1^2 x_2 + & x_1 x_2^2 + & \cdots + & 2x_1x_2x_3 + & & \cdots
\end{array}$$

The *alternant* for $\lambda = (\lambda_1, \ldots, \lambda_l)$ is

$$a_\lambda = |x_i^{\lambda_j}|_{1 \le i,j \le l}.$$

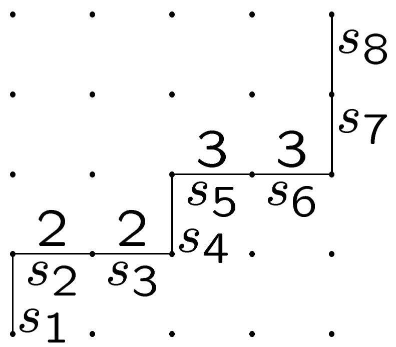
If $\delta = (l-1, l-2, \ldots, 0)$ then $a_\delta =$ Vandermonde. Let χ^λ be an irr character and k_μ be the size of a conjugacy class in $\mathfrak{S}_n$. Let $K_{\lambda\mu}$ be a Kostka number.

Theorem 30 *If $\lambda = (\lambda_1, \ldots, \lambda_l)$ then*

1. $\{s_\lambda \mid \lambda \vdash n\}$ *is a basis of* Λ^n.
2. $s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu}\, m_\mu$.
3. $s_\lambda = \frac{1}{n!} \sum_{\mu \vdash n} k_\mu\, \chi^\lambda_\mu\, p_\mu$.
4. $s_\lambda(x_1, \ldots, x_l) = \frac{a_{\lambda+\delta}}{a_\delta}$.
5. *(Jacobi-Trudi)* $s_\lambda = |h_{\lambda_i - i + j}|_{1 \le i,j \le l}$.

Proof of 5. (Gessel-Viennot-Lindström) A *lattice path* in $\mathbb{Z}^2$ is $p = s_1, s_2, \ldots$ where each s_i is a unit step N or E. Label the E steps by

$$N(s_i) = (\text{number of } N \text{ steps preceding } s_i) + 1.$$

$p =$ 

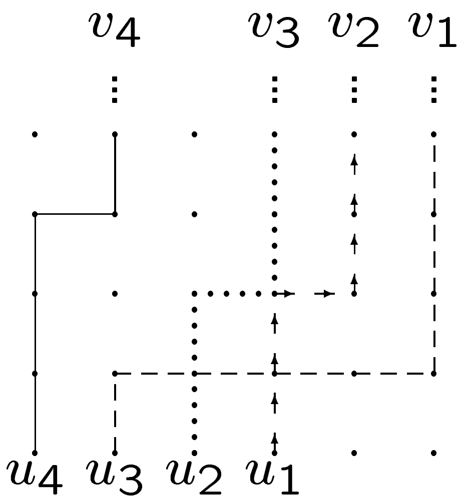
$\mathbf{x}^P = x_2^2 x_3^2.$

If p is from (a,b) to (c,d) write $(a,b) \xrightarrow{p} (c,d)$. Let

$$\mathbf{x}^p := \prod_{s_i = E \in p} x_{N(s_i)} \quad \Rightarrow \quad h_n = \sum_{(a,b) \xrightarrow{p} (a+n,\infty)} \mathbf{x}^p.$$

Fix $(u_1, \ldots, u_l), (v_1, \ldots, v_l)$ & form $\mathcal{P} = (p_1, \ldots, p_l)$ where for all i : $u_i \xrightarrow{p_i} v_{\pi i}$ for some $\pi \in \mathfrak{S}_l$. Let

$$\mathbf{x}^{\mathcal{P}} := \prod_i \mathbf{x}^{p_i} \quad \text{and} \quad \operatorname{sgn} \mathcal{P} := \operatorname{sgn} \pi.$$

$\mathcal{P} =$ 

$$\begin{aligned} \mathbf{x}^{\mathcal{P}} &= x_2^4 x_3^2 x_4, \\ \operatorname{sgn} \mathcal{P} &= \operatorname{sgn}(1,2,3)(4) \\ &= +1. \end{aligned}$$

Given $\lambda = (\lambda_1, \ldots, \lambda_l)$ pick

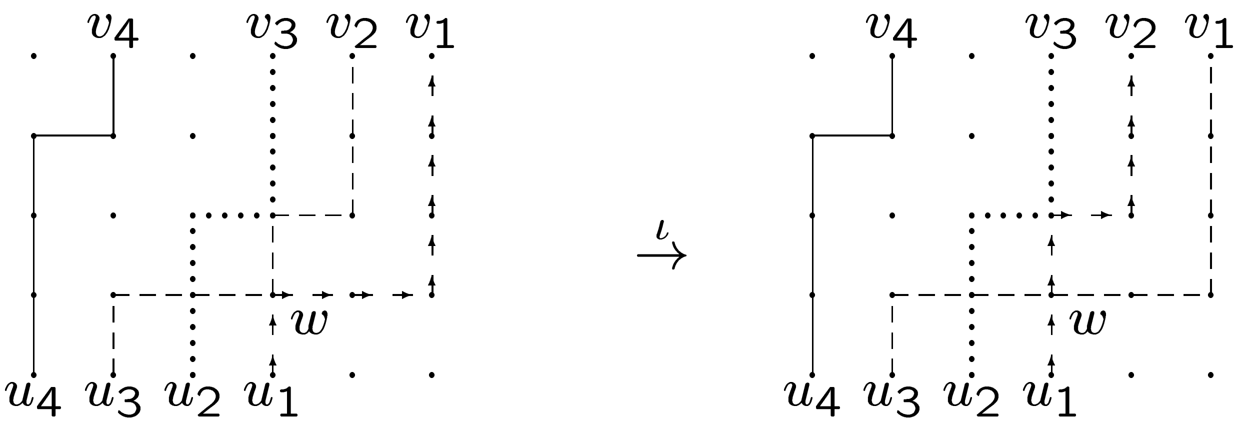
$$u_i := (1-i, 0) \quad \text{and} \quad v_i := (\lambda_i - i + 1, \infty) \Rightarrow$$
$$h_{\lambda_i - i + j} = \sum_{u_j \xrightarrow{p} v_i} \mathbf{x}^p \quad \text{and} \quad |h_{\lambda_i - i + j}| = \sum_{\mathcal{P}} (\operatorname{sgn} \mathcal{P}) \mathbf{x}^{\mathcal{P}}.$$

Define a sign-reversing involution $\mathcal{P} \stackrel{\iota}{\longleftrightarrow} \mathcal{P}'$ by

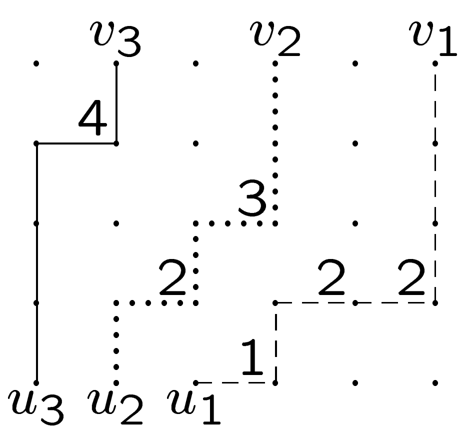
1. If $\mathcal{P}$ is non-$\cap$ then $\mathcal{P}' = \mathcal{P}$.

2. Else, let (i, j) be the lex least pair s.t. $p_i \cap p_j \neq \emptyset$, and $w \in p_i \cap p_j$ be SW-most, so $\mathcal{P}' = (\mathcal{P} \setminus p_i, p_j) \cup p_i', p_j'$.

$$p_i' := u_i \stackrel{p_i}{\to} w \stackrel{p_j}{\to} v_{\pi j} \quad \text{and} \quad p_j' := u_j \stackrel{p_j}{\to} w \stackrel{p_i}{\to} v_{\pi i}.$$



All terms in the det cancel except $\mathcal{P}$ for non-$\cap$ paths which correspond to semistandard λ-tableaux T.

$\mathcal{P} =$ 

$$T = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 3 & \\ 4 & & \end{array}.$$

C. Knuth's algorithm

Theorem 31 (Littlewood) *If* $\mathbf{y} = \{y_1, y_2, \ldots\}$ *then*

$$\sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) = \prod_{i,j \ge 1} 1/(1 - x_i y_j).$$

Proof (Knuth). Want a wt-preserving bijection

$$\pi \stackrel{\text{R-S-K}}{\longleftrightarrow} (T, U)$$

where T, U are semistandard of the same shape,

$$\text{wt}(T, U) = \mathbf{x}^T \mathbf{y}^U.$$

Furthermore, π is a *generalized permutation*: a 2-line array with entries in $\mathbb{Z}^+$ in lex order, and

$$\text{wt}\, \pi = \prod x_j y_i$$

where the product is over all col $\begin{pmatrix} i \\ j \end{pmatrix} \in \pi$.

<u>Ex.</u> $\pi = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 1 & 2 \end{array}$ with $\text{wt}\, \pi = x_1 x_2^2 x_3^2 y_1^3 y_2^2$.

The bijection is now the same as R-S.

<u>Ex. (cont)</u>

$$\begin{array}{lcccccccccccc} T_i: & \phi & , & 2 & , & 2\,3 & , & 2\,3\,3 & , & \begin{array}{l}1\,3\,3\\2\end{array} & , & \begin{array}{l}1\,2\,3\\2\,3\end{array} & = T, \\ U_i: & \phi & , & 1 & , & 1\,1 & , & 1\,1\,1 & , & \begin{array}{l}1\,1\,1\\2\end{array} & , & \begin{array}{l}1\,1\,1\\2\,2\end{array} & = U. \end{array}$$

D. The characteristic map

Let $R^n = R(\mathfrak{S}_n)$ (class functions) and $R = \oplus_{n\geq 0} R^n$. The *characteristic map*, ch : $R \to \Lambda$, linearly extends

$$\mathrm{ch}(\chi) := \frac{1}{n!} \sum_{\mu \vdash n} k_\mu \ \chi_\mu \ p_\mu \quad \text{where} \quad \chi \in R^n.$$

If χ^λ is an irr character then $\mathrm{ch}(\chi^\lambda) = s_\lambda$ so ch is a v.s. iso which becomes an isometry if we define

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

Finally for χ, ψ chars of $\mathfrak{S}_n, \mathfrak{S}_m$ let

$$\chi \cdot \psi = (\chi \otimes \psi) \uparrow^{\mathfrak{S}_{n+m}}$$

and extend linearly. Then we have

$$\begin{aligned}
\mathrm{ch}(\chi \cdot \psi) &= \langle \chi \cdot \psi, p \rangle \\
&= \langle (\chi \otimes \psi) \uparrow^{\mathfrak{S}_{n+m}}, p \rangle \\
&= \langle (\chi \otimes \psi), p \downarrow_{\mathfrak{S}_n \times \mathfrak{S}_m} \rangle \\
&= \tfrac{1}{n!m!} \sum_{\lambda \vdash n, \ \mu \vdash m} k_\lambda k_\mu \chi_\lambda \psi_\mu p_\lambda p_\mu \\
&= \mathrm{ch}(\chi)\, \mathrm{ch}(\psi).
\end{aligned}$$

Theorem 32 *The map* ch : $R \to \Lambda$ *is an isomorphism of algebras.* ■

E. The Littlewood-Richardson Rule

Word $R = r_1 \ldots r_n \in (\mathbb{Z}^+)^n$ is a *lattice permutation (lp)* if for all $R_i = r_1 \ldots r_i$ and all $j \in \mathbb{Z}^+$

$$\text{number of } j\text{'s} \geq \text{number of } j+1\text{'s in } R_i.$$

Such R corresponds to a standard tableau P by

$$\text{if } r_i = j \text{ then put } i \text{ in row } j \text{ of } P.$$

<u>Ex.</u> $R = 1\ 1\ 2\ 3\ 2\ 1\ 3 \longleftrightarrow P = \begin{array}{lll} 1 & 2 & 6 \\ 3 & 5 & \\ 4 & 7 & \end{array}.$

Theorem 33 (Littlewood-Richardson, L-R) *If*

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

then $c^\nu_{\lambda\mu}$ *is the number of semistandard* T *such that*

1. $\operatorname{sh} T = \nu/\lambda$ *and* $\operatorname{ct} T = \mu$*,*

2. the reverse row word $\rho(T)^r$ *is an lp.* ■

<u>Ex.</u> For $s_{(2)}s_{(2,1)}$

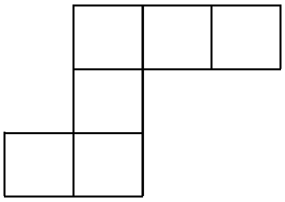
$$T: \begin{array}{llll} \bullet & \bullet & 1 & 1 \\ 2 & & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & 1 \\ 1 & 2 & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & 1 \\ 1 & & \\ 2 & & \end{array}, \quad \begin{array}{ll} \bullet & \bullet \\ 1 & 1 \\ 2 & \end{array}$$

$$s_{(2)}s_{(2,1)} = s_{(4,1)} + s_{(3,2)} + s_{(3,1^2)} + s_{(2^2,1)}$$

The L-R rule generalizes both the Branching Rule (for $s_\lambda s_{(1)}$) and Young's Rule (for $s_{(l)}s_{(m)}$).

F. The Murnagham-Nakayama Rule

A *rim hook*, H, is a skew shape that's a lattice path. A *rim hook tableau* T has rows and cols weakly increasing and all i's in a rim hook for each $i \in T$.

<u>Ex.</u> $H =$  and $T = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 2 & 2 & 4 \\ 3 & 3 & 3 & 4 & 4 \end{array}$.

Rim hook H has *leg length*

$$l(H) = (\text{number of rows of } H) - 1$$

and a rim hook tableau T has sign

$$\operatorname{sgn} T = \prod_{H \in T} (-1)^{l(H)}.$$

<u>Ex. (cont)</u> $l(H) = 2$, $\operatorname{sgn} T = (-1)^{0+1+0+2} = -1$.

Theorem 34 (Murnagham-Nakayama) *We have*

$$\chi^\lambda_\mu = \sum_T \operatorname{sgn} T$$

sum over all rim hook tableaux, $\operatorname{sh} T = \lambda$, $\operatorname{ct} T = \mu$.■

Note $\chi^\lambda_{(1^n)} = f^\lambda$ is a special case.

<u>Ex.</u> For $\chi = \chi^{(2,1)}$

μ	(1^3)	$(2,1)$	(3)
T	$\begin{array}{cc}1&2\\3\end{array}$, $\begin{array}{cc}1&3\\2\end{array}$	$\begin{array}{cc}1&1\\2\end{array}$, $\begin{array}{cc}1&2\\1\end{array}$	$\begin{array}{cc}1&1\\1\end{array}$
χ_μ	$1+1=2$	$1-1=0$	-1

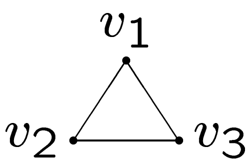
G. Chromatic symmetric functions

A *proper coloring* of $G = (V, E)$ is $c : V \to \{1, \ldots, t\}$

$$uv \in E \;\Rightarrow\; c(u) \neq c(v).$$

The *chromatic polynomial* of G is

$$P(G) = P(G, t) := \# \text{ of proper } c : V \to \{1, \ldots, t\}.$$

Ex. If $G =$ 

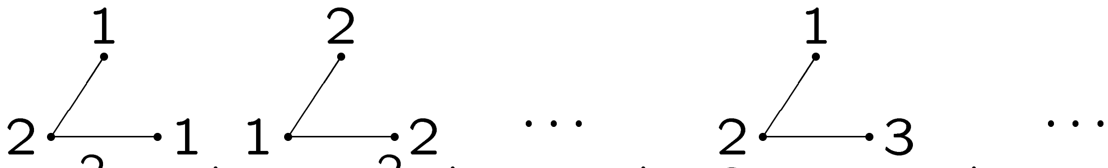
then

$$P(G) = \prod_i (\# \text{ of } c(v_i)) = t(t-1)(t-2).$$

The *chromatic symmetric function* of G is

$$X(G) = X(G, \mathbf{x}) = \sum_{\text{proper } c: V \to \mathbb{Z}^+} x_{c(v_1)} \cdots x_{c(v_n)}.$$

Ex.

G : 

$$X(G) = x_1^2 x_2 + x_1 x_2^2 + \cdots + 6 x_1 x_2 x_3 + \cdots$$

Poset P has *incomparability graph* $G = \operatorname{inc} P$ with

$$V = P, \quad E = \{uv \mid u, v \text{ incomparable in } P\}$$

and is $\mathbf{3+1}$-*free* if it has no induced $\{a < b < c, d\}$.

Conjecture 35 (Stanley-Stembridge) *If poset P is $\mathbf{3+1}$-free and $X(\operatorname{inc} P) = \sum_\lambda c_\lambda e_\lambda \Rightarrow c_\lambda \in \mathbb{Z}^+ \cup \{0\}$.*

Gasharov has proved this with e_λ replaced by s_λ.

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