On 021-Avoiding Ascent Sequences

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Abstract

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev in their study of $(\mathbf{2}+\mathbf{2})$-free posets. An ascent sequence of length n is a nonnegative integer sequence $x = x_1x_2\dots x_n$ such that $x_1 = 0$ and $x_i \leqslant \mathrm{asc}(x_1x_2\dots x_{i-1})+1$ for all $1 < i \leqslant n$, where $\mathrm{asc}(x_1x_2\dots x_{i-1})$ is the number of ascents in the sequence $x_1x_2\dots x_{i-1}$. We let $\mathcal{A}_n$ stand for the set of such sequences of legth n and use $\mathcal{A}_n(p)$ for the subset of sequences avoiding a pattern p. Similarly, we let $S_n(\tau)$ be the set of τ-avoiding permutations in the symmetric group S_n. Duncan and Steingrímsson have shown that the ascent statistic has the same distribution over $\mathcal{A}_n(021)$ as over $S_n(132)$. Furthermore, they conjectured that the pair $(\mathrm{asc}, \mathrm{rmin})$ is equidistributed over $\mathcal{A}_n(021)$ and $S_n(132)$, where rmin is the right-to-left minima statistic. We prove this conjecture by constructing a bistatistic-preserving bijection.

Keywords: 021-avoiding ascent sequence, 132-avoiding permutation, right-to-left minimum, number of ascents, bijection

1 Introduction

The objective of this paper is to establish a bijection which leads to the equidistribution of the pair of statistics $(\mathrm{asc}, \mathrm{rmin})$ over 021-avoiding ascent sequences and over 132-avoiding permutations. This confirms a conjecture posed by Duncan and Steingrímsson [5].

Let us give an overview of the notation and terminology. Let S_n denote the set of permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Given a permutation $\pi \in S_n$ and a permutation $\tau \in S_k$, we say that a subsequence $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, of π is of *pattern* τ if it is order isomorphic to τ, that is, this subsequence has the same relative order as τ. If π does not contain any subsequence of pattern τ, then we say that π *avoids* τ, or π is τ*-avoiding*. We denote by $S_n(\tau)$ the set of τ-avoiding permutations in S_n. For example, the permutation 763894512 contains the subsequence 3952 of pattern 2431, but it is 1234-avoiding. Pattern avoiding permutations have been intensively studied in recent years from many points of view, see [1, 6, 8].

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev [2]. For a sequence $x = x_1x_2\ldots x_n$ of nonnegative integers, we say that an index i $(1 \leqslant i < n)$ is an *ascent* if $x_i < x_{i+1}$. We denote by $\mathrm{asc}(x)$ or merely $\mathrm{asc}\, x$ the number of ascents of x. A sequence $x = x_1x_2\ldots x_n$ is called an *ascent sequence* if $x_1 = 0$ and

$$x_i \leqslant \mathrm{asc}(x_1x_2\ldots x_{i-1}) + 1$$

for all $1 < i \leqslant n$. For example, $x = 010122$ is an ascent sequence while $x = 010142$ is not since $x_5 = 4 > \mathrm{asc}(0101)+1 = 3$. We let $\mathcal{A}_n$ denote the set of ascent sequences of length n. For an ascent sequence, a *pattern* is a word on a nonnegative integers $\{0, 1, \ldots, k\}$, where each element i appears at least once. Containment and avoidance of patterns for ascent sequences are defined in the same way as for permutations. For example, the ascent sequence 01231234 has five occurrences of the pattern 001, namely, the subsequences $112, 113, 114, 223, 224$, and the ascent sequence 01012203 is 021-avoiding. We denote by $\mathcal{A}_n(p)$ the set of ascent sequences of length n avoiding pattern p.

In addition to the ascent statistic, we will be interested in the number of right-to-left minima. A *right-to-left minimum* of any sequence x of nonnegative integers is an index i such that $x_i < x_j$ for all $j > i$. The number of right-to-left minima of x is denoted by $\mathrm{rmin}(x)$ or $\mathrm{rmin}\, x$. For example, $\mathrm{rmin}(010122) = 3$.

Ascent sequences are closely connected to $(\mathbf{2}+\mathbf{2})$-free posets [2], upper-triangular matrices [4], Stoimenow's matchings [3], and the Catalan numbers C_n [5]. In particular, a poset is called $(\mathbf{2}+\mathbf{2})$*-free* if it does not contain an induced subposet which is isomorphic to the disjoint union of two 2-element chains. Bousquet-Mélou, Claesson, Dukes and Kitaev [2] found a bijection from $(\mathbf{2}+\mathbf{2})$-free posets to ascent sequences which maps the number of levels of the poset to the number of ascents of the sequence. Dukes and Parviainen [4] established a bijection between ascent sequences and nonnegative upper-triangular matrices. Duncan and Steingrímsson [5] have shown that $\#\mathcal{A}_n(p) = C_n$ for any of the patterns $p = 101$, 0101, or 021. Mansour and Shattuck [7] have shown that $\#\mathcal{A}_{0123}(n)$ equals the number of Dyck paths of semilength n and height at most 5, as conjectured by Duncan and Steingrímsson. It is well known that $\#S_n(132) = C_n$. Duncan and Steingrímsson also proved that the ascent statistic is equidistributed over $\mathcal{A}_n(021)$ and $S_n(132)$. Furthermore, they proposed the following conjecture.

Conjecture 1. The bistatistic $(\mathrm{asc}, \mathrm{rmin})$ has the same distribution over $\mathcal{A}_n(021)$ and $S_n(132)$.

The objective of this paper is to give a bijective proof of above conjecture.

2 Proof of the conjecture

In order to construct our bijection, we will need the concept of the tight maximum value of an ascent sequence $x = x_1x_2\dots x_n$. Call x_i *tight* if

$$x_i = \mathrm{asc}(x_1x_2\dots x_{i-1}) + 1 \tag{1}$$

giving us equality in the defining relation for an ascent sequence. The *tight maximum value* of x is the largest integer M such that there is a tight x_i with $x_i = M$. To illustrate the notion, if $x = 01013312434$ then $M = 3$. Note that, except for the zero sequence, M will always exist since the first 1 in any nonzero sequence satisfies (1). So we define $M = 0$ for a zero sequence.

Also define a *tight maximum index* as an index i where x_i satisfies $x_i = M$ as well as condition (1). The first index i with $x_i = M$ is always a tight maximum index. In fact, if $[i, j]$ is the largest interval of indices starting with the first tight maximum index and satisfying $x_i = x_{i+1} = \cdots = x_j = M$ then we claim that these are exactly the tight maximum indices. To see this, first note that if $k \in [i, j]$ then k is a tight maximum index because

$$x_k = x_i = \mathrm{asc}(x_1x_2\dots x_{i-1}) + 1 = \mathrm{asc}(x_1x_2\dots x_{k-1}) + 1$$

since there are no ascents between x_i and x_k. To see that no other index can be tight maximum, suppose $x_k = M$ with $k \geqslant j+2$. Now $x_j > x_{j+1}$ because if $x_j < x_{j+1}$ then the tight maximum value would be at least $M + 1$. Thus there must be an ascent between x_{j+1} and x_k so that (1) is no longer an equality when $i = k$. Call the tight maximum value *unique* if there is only one tight maximum index and *repeated* otherwise.

Theorem 2. *The bistatistic* $(\mathrm{asc}, \mathrm{rmin})$ *has the same distribution over* $\mathcal{A}_n(021)$ *and* $S_n(132)$*.*

Proof. We will inductively build a bijection $\phi_n \colon \mathcal{A}_n(021) \to S_n(132)$ preserving the bistatistic. To do so, we will need decompositions of $\mathcal{A}_n(021)$ and $S_n(132)$ into pieces indexed by smaller subscripts. We will start on the ascent side.

A simple but important observation for what follows is that $p \in \mathcal{A}_n(021)$ if and only if the nonzero entries of p are weakly increasing. We will use this fact to construct a bijection $f = f_n$ between $\mathcal{A}_n(021)$ and the set of pairs

$$\bigcup_{i=1}^{n} \mathcal{A}_{i-1}(021) \times \mathcal{A}_{n-i}(021).$$

Consider $x \in \mathcal{A}_n(021)$ and suppose first that x has a repeated tight maximum value M. Let k be any of the tight maximum indices and define

$$f(x) = (\epsilon, z)$$

where ϵ is the empty sequence and z is x with x_k removed. For example, if $x = 01013300304$ then $z = 0101300304$. Clearly z still avoids 021 since its nonzero entries still increase and, since x's tight maximum value was repeated, z still has M as its tight maximum value. Since the tight maximum value does not change, one can construct an inverse map from $\mathcal{A}_0(021) \times \mathcal{A}_{n-1}(021)$ back to the elements of $\mathcal{A}_n$ which have a repeated tight maximum in the obvious way. Finally note that in this case

$$\operatorname{asc} x = \operatorname{asc} z \quad \text{and} \quad \operatorname{rmin} x = \operatorname{rmin} z.$$

Now suppose that x has a unique tight maximum value $x_i = M$. Here we let

$$f(x) = (y, z)$$

where $y = x_1 \ldots x_{i-1}$ and z is obtained from $x' = x_{i+1} \ldots x_n$ by subtracting $M - 1$ from all the nonzero entries. To illustrate, if $x = 0101300304$ then $y = 0101$ and $z = 00102$. It is clear that $y \in \mathcal{A}_{i-1}(021)$. To show that $z \in \mathcal{A}_{n-i}(021)$, we first note that z still has weakly increasing nonzero elements and so avoids 012. We must also demonstrate that z is an ascent sequence. Since the defining condition for an ascent sequence is trivial for zero elements, we need only consider $z_r \neq 0$. But since we have subtracted the same amount from all nonzero entries of x', the index r is an ascent of z if and only if the index $r + i$ is an ascent of x'. Also, since M was the tight maximum value of x, we have $\operatorname{asc}(x_1 \ldots x_i) = M$, $x_i > x_{i+1}$ and $x_{r+i} \leqslant \operatorname{asc}(x_1 x_2 \ldots x_{r+i-1})$ for any $r \geqslant 1$. It follows that for any $z_r \neq 0$ we have

$$\begin{array}{rcl} z_r & = & x_{r+i} - M + 1 \\ & \leqslant & \operatorname{asc}(x_1 \ldots x_{r+i-1}) - M + 1 \\ & = & \operatorname{asc}(x_1 \ldots x_i) + \operatorname{asc}(x_{i+1} \ldots x_{r+i-1}) - M + 1 \\ & = & \operatorname{asc}(z_1 \ldots z_{r-1}) + 1 \end{array}$$

which is what we wished to prove. Constructing the inverse of this part of the map is similar to what was done in the first case.

It will be useful to record what happens to our two statistics in the second case defining f. For the ascent statistic, we have everything in place from the previous paragraph and the fact that, by definition of M, $x_i > x_{i-1}$. Thus

$$\begin{array}{rcl} \operatorname{asc} x & = & \operatorname{asc}(x_1 \ldots x_i) + \operatorname{asc}(x_{i+1} \ldots x_n) \\ & = & \operatorname{asc}(x_1 \ldots x_{i-1}) + 1 + \operatorname{asc}(x_{i+1} \ldots x_n) \\ & = & \operatorname{asc} y + \operatorname{asc} z + 1. \end{array}$$

In terms of right-to-left minima, we distinguish two subcases. If $i \leqslant n-1$ then, since $x_{i+1} = 0$, the right-to-left minima of x must occur in the sequence $x_{i+1} \dots x_n$. Since the subtraction of $M-1$ does not change the positions of these minima, we have

$$\operatorname{rmin} x = \operatorname{rmin} z.$$

On the other hand, if $i = n$ then $z = \epsilon$ and x_n is a right-to-left minimum, giving

$$\operatorname{rmin} x = \operatorname{rmin} y + 1.$$

We will now review the standard decomposition of $S_n(132)$ which gives a bijection g from this set to

$$\bigcup_{i=1}^{n} S_{i-1}(132) \times S_{n-i}(132).$$

If $\pi \in S_n(132)$ then we write $\pi = \pi_L n \pi_R$ where π_L, π_R are the elements to the left and right of n, respectively. Define the index i by $\pi_i = n$. Then it is well known that $\pi \in S_n(132)$ if and only if π_L, π_R avoid 132 and every element of π_L is bigger than every element of π_R. So we let

$$g(\pi) = (\rho, \sigma)$$

where $\rho \in S_{i-1}(132)$ and $\sigma \in S_{n-i}(132)$ are order isomorphic to π_L and π_R, respectively.

As with the ascent sequence decomposition, we have to consider what happens to our statistics in two separate cases. The first is when $\rho = \epsilon$, equivalently, $i = 1$. So $\pi = n\pi_R$ and so n makes no contribution either to the ascents or right-to-left minimum. It follows that

$$\operatorname{asc} \pi = \operatorname{asc} \sigma \quad \text{and} \quad \operatorname{rmin} \pi = \operatorname{rmin} \sigma$$

just as in the corresponding case for ascent sequences.

Now suppose $1 < i \leqslant n$. So there will be an ascent ending at n and all other ascents of π correspond to ascents of ρ or ascents of σ. It follows that

$$\operatorname{asc} \pi = \operatorname{asc} \rho + \operatorname{asc} \sigma + 1.$$

For the right-to-left minima we again break into two subcases depending on whether the second component of our bijection is ϵ or not. If $\sigma \neq \epsilon$ then $i < n$ and the right-to-left minima of π are all in π_R because of the relative sizes of the elements of π_R and π_L. This gives

$$\operatorname{rmin} \pi = \operatorname{rmin} \sigma.$$

Now consider $\sigma = \epsilon$ so that $\pi = \pi_L n$. Thus n is a right-to-left minimum of π as is every right-to-left minimum of π_L. So in this subcase

$$\operatorname{rmin} \pi = \operatorname{rmin} \rho + 1.$$

Finally, we construct $\phi_n \colon \mathcal{A}_n(021) \to S_n(132)$ as follows. Start with $\phi_0(\epsilon) = \epsilon$. Assuming that ϕ_i has been defined for all $i < n$, we define ϕ_n to be the composition

$$\mathcal{A}_n(021) \xrightarrow{f} \bigcup_{i=1}^{n} \mathcal{A}_{i-1}(021) \times \mathcal{A}_{n-i}(021) \xrightarrow{h} \bigcup_{i=1}^{n} S_{i-1}(132) \times S_{n-i}(132) \xrightarrow{g^{-1}} S_n(132)$$

where the restriction of h to $\mathcal{A}_{i-1}(021) \times \mathcal{A}_{n-i}(021)$ is $\phi_{i-1} \times \phi_{n-i}$. It should be clear from the equations derived for asc and rmin when defining f and g that this bijection preserves the bistatistic. $\square$

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