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Shellability of Exponential Structures

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Abstract. We show that the poset of all partitions of an *nd*-set with block size divisible by *d* is shellable. Using similar techniques, it also follows that various other examples of exponential structures cited by Stanley are also shellable. The method used involves the notion of recursive atom orderings introduced by Björner and Wachs.

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1. The Problem

In a private communication [7], Richard Stanley asked whether the poset, $\Pi_n^{(d)}$, of all partitions of an *nd*-set with block size divisible by d ($\hat{0}$ adjourned) was 'Cohen–Macaulay or even shellable?' The motivation for this question stemmed from research concerning group actions on the associated homology complex, the subject of a forthcoming paper of Calderbank, Hanlon and Robinson [3]. Michelle Wachs [9] was the first to prove that $\Pi_n^{(d)}$ is indeed shellable and an identical proof was subsequently rediscovered by the author.

$\Pi_n^{(d)}$ is only one example of an exponential structure as introduced by Stanley [5]. Exponential posets are so-called because they give rise to identities similar to the well known exponential formula for $\Pi_n = \Pi_n^{(1)}$ (see Foata [4]). Although it is not true that all exponential structures are shellable, most of the 'natural' examples cited by Stanley can be shelled using techniques similar to those for $\Pi_n^{(d)}$. The purpose of the present work is to give explicit shellings of these exponential posets. I am grateful to Michelle Wachs for letting me include her original result.

2. Preliminaries Concerning Shellability

We assume that most readers are already familiar with the general theory of shellability and so only review here the particular definitions and theorems needed for our purposes. Those wishing more background should consult the survey article of Björner *et al.* [2].

A poset P is said to be *graded* if it is finite, has a $\hat{0}$ and $\hat{1}$ (i.e., $\hat{0} \leqslant x \leqslant \hat{1}$ for all $x \in P$)

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and all maximal chains have the same length. In this section, all our posets will be graded. A poset is *shellable* if there is an ordering of its maximal chains $m_1, m_2, \ldots, m_s$, such that for all m_j and all m_i, $i<j$, there is a $k<j$ with $m_j \cap m_i \subseteq m_j \cap m_k = m_j - \{x\}$ for some $x \in m_j$.

Many different methods have been developed for showing that a poset is shellable. The one that we will use employs atom orderings. Given $x, y \in P$ then x *is covered by* y, written $x \to y$ to indicate that xy is an arc of the Hasse diagram of P, if $x<y$ and there is no z with $x<z<y$. An *atom* of P is an element covering $\hat{0}$ and the set of atoms of P will be denoted $A(P)$. We say P admits a *recursive atom ordering* or *RAO* if there is a total ordering of $A(P), a_1, a_2, \ldots, a_h$, satisfying the two conditions.

(R1) For all j, the interval $[a_j, \hat{1}]$ admits an RAO where the atoms that come first are those covering some a_i, $i<j$.

(R2) For all a_j and all a_i, $i<j$, if $a_i, a_j \leqslant y$ then there exists a_k with $k<j$ and $a_j, a_k \to z \leqslant y$.

Björner and Wachs [1] were the first to define RAO's and prove the following two results:

THEOREM 1 [1, Theorem 3.2]. *If P admits an RAO then P is shellable.* □

THEOREM 2 [1, Theorem 5.1]. *If P is a semimodular lattice then P admits an RAO. In fact every total ordering of $A(P)$ extends to an RAO.* □

For our application we will also need the following rather technical lemma.

LEMMA 3. *Suppose P is such that $[a, \hat{1}]$ is a semimodular lattice for all $a \in A(P)$. Then P admits an RAO if and only if some ordering of the atoms of P satisfies condition* (R2) *above.*

Proof. The 'only if' implication is a triviality. For the other direction let $a_1, a_2, \ldots, a_h$ be an ordering of $A(P)$ such that (R2) holds. For each i order the atoms of $[a_i, \hat{1}]$ in any way so long as those covering $a_1, \ldots, a_{i-1}$ come first. Now use Theorem 2 to extend atom ordering of each $[a_i, \hat{1}]$ to an RAO. □

3. Exponential Structures

Let Π_n be the lattice of partitions of the set $[n] = \{1, 2, \ldots, n\}$ ordered by refinement. Given a partition $\pi \in \Pi_n$, i.e., $\pi = \{B_1, B_2, \ldots, B_k\}$ with the disjoint union $\uplus_i B_i = [n]$, we call the sets B_i *the blocks of* π and write for convenience $\pi = B_1/B_2/\ldots/B_k$. If the cardinality of each B_i is $|B_i| = \lambda_i$ then we say π has type $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ and write $|\pi| = (\lambda_1, \ldots, \lambda_k)$. Now take any function $f\colon \mathbf{P} \to \mathbf{Z}$, where $\mathbf{Z}$ and $\mathbf{P}$ denote the integers and positive integers respectively, and define $f(|\pi|) = f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k)$ if $|\pi| = (\lambda_1, \ldots, \lambda_k)$. Furthermore, f has a corresponding *exponential generating function*

$$f_\Pi(x) = \sum_{n=1}^{\infty} f(n) \frac{x^n}{n!}$$

The reason for the subscript Π referring to Π_n will become clear shortly.

One of the most useful tools of enumerative combinatorics is the exponential formula which shows that if two functions are simply related via the lattice Π_n then there is a simple relationship between the corresponding generating functions.

THEOREM 4 (The exponential formula). *If $f, g\colon \mathbf{P} \to \mathbf{Z}$ satisfy*

$$g(n) = \sum_{\pi \in \Pi_n} f(|\pi|) \quad \textit{for all } n,$$

then $1 + g_\Pi(x) = e^{f_\Pi(x)}$. □

Generalizations, applications, and other information about the exponential formula can be found in Foata's monograph [4].

If $\pi \in \Pi_n$, $|\pi| = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then it is well known and easy to prove that

(1) the upper order ideal $U_\pi = \{\sigma \mid \sigma \geqslant \pi\}$ is isomorphic to Π_k, and
(2) the lower order ideal $L_\pi = \{\sigma \mid \sigma \leqslant \pi\}$ is isomorphic to $\Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_k}$.

It is these two properties that are crucial to Stanley's generalization [5]. Specifically, define an exponential structure Q to be a sequence $Q = (Q_1, Q_2, \ldots)$ of posets satisfying the following three axioms.

(E1) For each n, Q_n has a $\hat{1}$ and all maximal chains have length $n-1$,
(E2) For each minimal element $\rho \in Q_n$ we have $U_\rho \simeq \Pi_n$,
(E3) For any $\pi \in Q_n$, π has the same type $|\pi| = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ in all copies of Π_n in which it lies via the isomorphism in (E2) and $L_\pi = Q_{\lambda_1} \times Q_{\lambda_2} \times \cdots \times Q_{\lambda_k}$.

Associated with any function $f\colon \mathbf{P} \to \mathbf{Z}$ and any exponential structure $Q = (Q_1, Q_2, \ldots)$ we have the generating function

$$f_Q(x) = \sum_{n=1}^{\infty} f(n) \frac{x^n}{n!M(n)}$$

where $M(n)$ is the number of minimal elements in Q_n. Note that if $Q = (\Pi_1, \Pi_2, \ldots)$ then $M(n) = 1$ for all n and $f_Q(x) = f_\Pi(x)$. The analog of Theorem 4 in this setting is:

THEOREM 5 [5, Corollary 3.3]. *If $f, g\colon \mathbf{P} \to \mathbf{Z}$ satisfy*

$$g(n) = \sum_{\pi \in Q_n} f(|\pi|) \quad \textit{for all } n$$

then

$$1 + g_Q(x) = e^{f_Q(x)}.$$ □

The examples of exponential structures given by Stanley are the following.

EXAMPLE 1: *d-Divisible Partitions*. Let $\Pi_n^{(d)}$ be the set of all partitions of $[nd]$ with blocks of size divisible by d. Ordering $\Pi_n^{(d)}$ by refinement we see that $Q^{(d)} = (\Pi_1^{(d)}, \Pi_2^{(d)}, \ldots)$ is an exponential structure. Garrett Sylvester [8] employed $Q^{(2)}$ in his study of Ising ferromagnets. The Hasse diagram of $\Pi_2^{(2)}$ is shown in Figure 1.

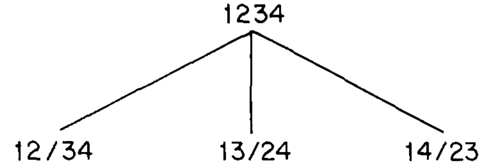

Fig. 1. $\Pi_2^{(2)}$.

EXAMPLE 2: *Vector Partitions.* Consider the Cartesian product of sets $[n]^r = ([n], [n], \ldots, [n])$. Assuming all operations are component-wise, we can define a vector partition $\pi = B_1/B_2/\ldots/B_k$ where each block B_i is an r-vector of sets $B_i = (B_{i1}, B_{i2}, \ldots, B_{ir})$ by

(1) $\uplus_i B_i = [n]^r$, i.e., $\uplus_i B_{is} = [n]$ for $s = 1, 2, \ldots, r$, and
(2) For all indices i, we have $|B_{i1}| = |B_{i2}| = \cdots = |B_{ir}|$.

The reader should be careful to note that the components of the blocks are ordered: $(12, 13) \neq (13, 12)$, but the blocks themselves are not: $(12, 13)/(3, 2) = (3, 2)/(12, 13)$.

The set $\Pi_{n,r}$ of all such partitions has an obvious refinement order which yields, for fixed r, an exponential structure. An application of vector partitions to matrices with equal row and column sums is found in Stanley [6, Example 6.11]. Figure 2 displays $\Pi_{3,2}$.

EXAMPLE 3: *Colored Graphs.* Given a graph G on the vertex set $V(G) = [n]$ then a *coloring* of G is a partition $\pi = B_1/\ldots/B_k$ of $V(G)$ such that if $x, y \in B_i$ then xy is not an edge in $E(G)$. Partially order the set of colored graphs $\chi_n = \{(G, \pi) \mid \pi$ is a coloring of $G\}$ by setting $(G, \pi) \leqslant (H, \pi)$ if and only if

(1) $\pi \leqslant \sigma$ in Π_n, and
(2) given x and y in different blocks of σ then $xy \in E(G)$ if and only if $xy \in E(H)$.

It is easy to verify that $\chi = (\chi_1, \chi_2, \ldots)$ is an exponential structure and χ_3 is illustrated in Figure 3. To simplify the diagram we have suppressed the labels on the vertices. Also the pair (G, π) is represented by taking the graph G and encircling the vertices in each block of π.

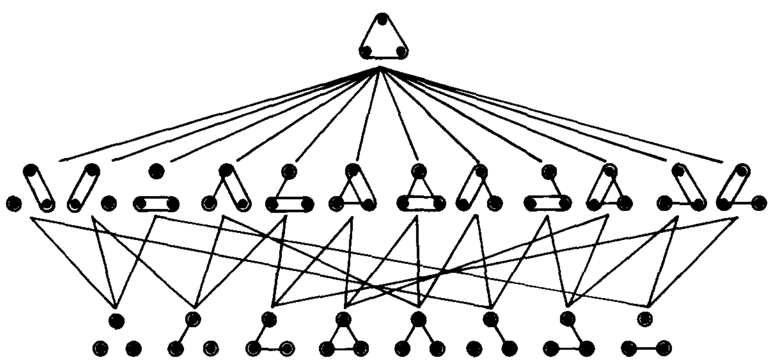

Fig. 3. χ_3.

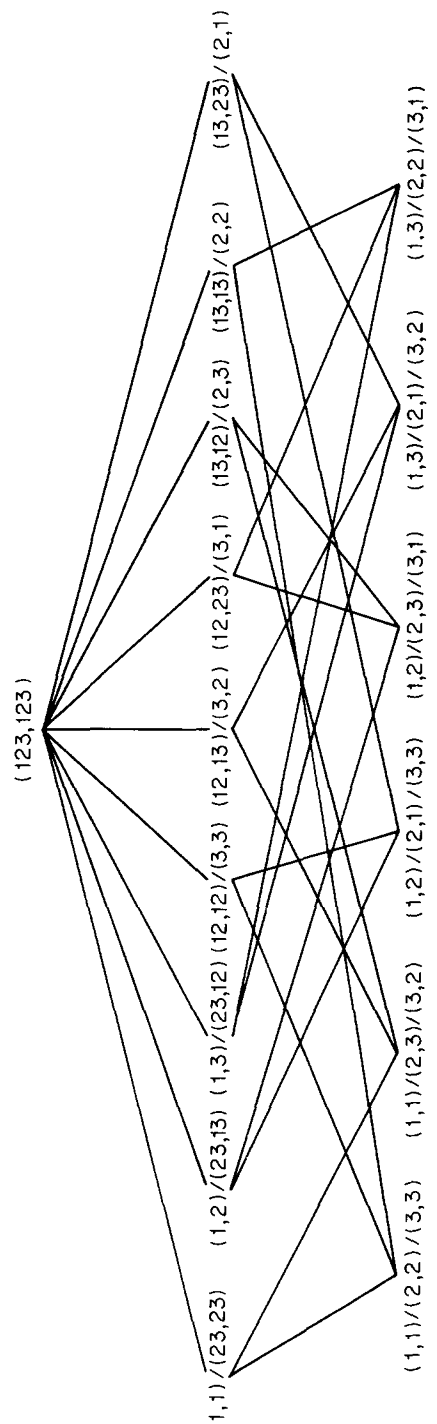
(123,123)
(1,1)/(23,23)
(1,2)/(23,13)
(1,3)/(23,12)
(12,12)/(3,3)
(12,13)/(3,2)
(12,23)/(3,1)
(13,12)/(2,3)
(13,13)/(2,2)
(13,23)/(2,1)
(1,1)/(2,2)/(3,3)
(1,1)/(2,3)/(3,2)
(1,2)/(2,1)/(3,3)
(1,2)/(2,3)/(3,1)
(1,3)/(2,1)/(3,2)
(1,3)/(2,2)/(3,1)

Fig. 2. $\Pi_{3,2}$.

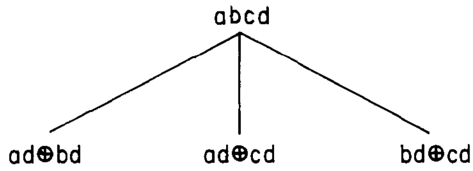

Fig. 4. P_2 over GF(2).

EXAMPLE 4: *Vector Space Partitions.* Let V_n be an n-dimensional vector space over the galois field GF(q) and let P_n consist of all direct-sum decompositions $V_n = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ into nonzero subspaces W_i. Again, refinement provides a partial order making $V = (P_1, P_2, \ldots)$ into an exponential structure. The poset for P_2 over GF(2) is shown in Figure 4. For simplicity, we represent the vectors in V_2 by the letters $a = (1, 0), b = (0, 1), c = (1, 1)$ and $d = (0, 0)$.

4. The Shellings

First of all we must make the examples of the previous section graded by adding a unique minimal element, cf. condition (E1). Hence, given any exponential structure $Q = (Q_1, Q_2, \ldots)$ let $\hat{Q} = (\hat{Q}_1, \hat{Q}_2, \ldots)$ where $\hat{Q}_i$ is Q_i with a $\hat{0}$ adjoined. This alone is not enough to guarantee shellability, as illustrated by the following example.

Let $Q_n = \Pi_n$ for $n = 1, 2$ and for $n \geqslant 3$ the elements of Q_n will be partially colored partitions of $[n]$. A *colored partition of* $[n]$ is a partition where some of the integers have been distinguished, i.e., colored. A *partially colored partition* is a colored partition where only elements in blocks of size at most two can be colored. If, given an uncolored integer i, we let i' be the corresponding colored integer then some partially colored partitions of [4] are $1'/2/3/4'$, $1/2'3/4'$, $1'3'/24'$ and $134/2'$. Note that $13'4/2 \notin Q_4$ since a block of size three contains a colored element.

To complete the description of Q_n, $n \geqslant 3$, we need only define the covering relation. Given a partially colored partition $\pi = B_1/B_2/\ldots/B_k$ then a cover of π is obtained by replacing some pair of blocks B_i, B_j by their union $B_i \cup B_j$ and either perserving the coloring of the elements of B_i and B_j in the case $|B_i \cup B_j| = 2$, or removing the coloring from all colored elements in the case $|B_i \cup B_j| \geqslant 3$. For example $1/2'3/4'$ is covered by $14'/2'3$, $123/4'$ and $1/234$ but not by $14/2'3$ or $123/4$. Now Q_3 consists of 2^3 copies of Π_3 with their maximal elements pasted together so $\hat{Q}_3$ is clearly not shellable. However, Examples 1 to 3 do become shellable and, in fact, have RAO's that arise from the lexicographic ordering of their atoms.

Let $(S, \leqslant)$ be any totally ordered set and let the set of words on S be $W(S) = \{t_1 t_2 \ldots t_n \mid t_i \in S\}$. The *lexicographic order,* $\leqslant_L$, on $W(S)$ is defined by $t_1 t_2 \ldots t_n <_L s_1 s_2 \ldots s_m$ if and only if $t_i < s_i$ in the first position i where the two words differ (to apply this definition in all cases, we must pad both words out to the same length with a symbol less than every element of S). Now consider the power set of S, $P(S) = \{T \mid T \subseteq S\}$. $P(S)$ is also ordered by $\leqslant_L$ since we can associate with each $T \in P(S)$ the word $t_1 t_2 \ldots t_n \in W(S)$ where $T = \{t_1, t_2, \ldots, t_n\}$ and $t_1 < t_2 < \cdots < t_n$. Hence we can also lexicographically order $W(P(S))$, $P(W(S))$, etc.

THEOREM 6. *$\hat{\Pi}_n^{(d)}$, $\hat{\Pi}_{n,r}$, and $\hat{\chi}_n$ all admit recursive atom orderings.*

Proof. It is well-known that Π_n is a semi-modular lattice and by condition (E2) in the definition of an exponential structure, given any atom $a \in Q_n$ we must have $[a, \hat{1}] \simeq \Pi_n$. Thus, Lemma 3 applies so to finish the proof we need only show that in each of the three examples we may order the atoms so as to satisfy condition (R2). In each case the lexicographic order will suffice.

$\hat{\Pi}_n^{(d)}$:
Every $\pi \in \hat{\Pi}_n^{(d)}$ is in $P(P([nd]))$. Thus, the atoms of $\hat{\Pi}_n^{(d)}$ are ordered by $\leqslant_L$ in a sequence $\pi_1 \pi_2 \ldots \pi_h$. Given $i < j$ and $\pi_i = B_1/\ldots/B_n$, $\pi_j = C_1/\ldots/C_n$ it is convenient to distinguish two cases.

(i) $B_1 \neq C_1$. Since $\pi_i <_L \pi_j$ and $B_1 \neq C_1$ we must have $B_1 <_L C_1$. Consider $l = \min(B_1 - C_1)$, $m = \max C_1$, and note that $l < m$ (since $B_1 <_L C_1$). Find the block C_t in π_j such that $l \in C_t$ and construct the d-sets

$$C_1' = C_1 - \{m\} + \{l\}, C_t' = C_t - \{l\} + \{m\}$$

and the atom

$$\pi_k = \pi_j - \{C_1, C_t\} + \{C_1', C_t'\}.$$

Now $l < m$ so $C_1' <_L C_1$ and $\pi_k <_L \pi_j$ which, in turn, implies $k < j$. Furthermore,

$$\pi_j \vee \pi_k = C_1 \cup C_t/C_2/C_3/\ldots \leftarrow \pi_j, \pi_k.$$

Finally, if $\pi > \pi_i, \pi_j$ then l and m are in the same block of π (since $l \in B_1$, $m \in C_1$ and $B_1 \cap C_1 \supseteq \{1\} \neq \emptyset$). Hence, $C_1 \cup C_t$ is contained in a single block of π and $\pi \geqslant \pi_j \vee \pi_k$ showing that (R2) is indeed satisfied.

(ii) $B_1 = C_1$. Find the first index s such that $B_s \neq C_s$ and apply the same construction as in (i) to B_s and C_s. The only additional verification required is that $l \in C_t >_L C_s$, which is true since $B_q = C_q$ for $q < s$.

$\hat{\Pi}_{n,r}$:
Elements of $\hat{\Pi}_{n,r}$ are in $P(W(P([n])))$ so $\leqslant_L$ applies again to give an atom ordering $\pi_1 \pi_2 \ldots \pi_h$ in $\hat{\Pi}_{n,r}$. Given $i < j$ with

$$\pi_i = B_1/\ldots/B_n = (1, b_{12}, \ldots, b_{1r})/\ldots/(n, b_{n2}, \ldots, b_{nr})$$

and

$$\pi_j = C_1/\ldots/C_n = (1, c_{12}, \ldots, c_{1r})/\ldots/(n, c_{n2}, \ldots, c_{nr})$$

there are, as before, two cases.

(i) $B_1 \neq C_1$. Hence $B_1 <_L C_1$ so consider the smallest index m such that $b_{1m} < c_{1m}$.

Now find the unique element c_{lm}, $2 \leq l \leq r$, satisfying $c_{lm} = b_{1m}$ and construct the atom $\pi_k = \pi_j$ with c_{1m} and c_{lm} interchanged.

Now $c_{lm} = b_{1m} < c_{1m}$ so $\pi_k <_L \pi_j$ and $k < j$. Also

$$\pi_j \vee \pi_k = (1l, \ldots, c_{1m}c_{lm}, \ldots, c_{1r}c_{lr})/(2, \ldots, c_{2r})/\ldots \leftarrow \pi_j, \pi_k.$$

To finish things off, note that if $\pi > \pi_i, \pi_j$ then c_{1m} and $b_{1m} = c_{lm}$ are in the same block of π, forcing $\pi \geq \pi_j \vee \pi_k$.

(ii) $B_1 = C_1$. Find the smallest s such that $B_s \neq C_s$ and apply the construction in (i). The verification that $l > s$ holds as before.

$\hat{\chi}_n$:

The atoms of $\hat{\chi}_n$ are of the form $(G_i, \hat{0})$ where $\hat{0} = 1/2/\ldots/n$ and G_i is a graph on the set $[n]$. Each such graph can be viewed as a set of edges $\{u, v\} = uv$, hence $G_i \in P(P([n]))$ and $\leq_L$ orders the $(G_i, \hat{0})$. Given atoms $(G_i, \hat{0}), (G_j, \hat{0})$ with $i < j$ there are two cases.

(i) $G_i \subset G_j$. In this case consider the edge $uv = \max E(G_j)$ and the graph $G_k = G_j - uv$.

(ii) $G_i \not\subset G_j$. In this case take instead the edge $uv = \min(E(G_i) - E(G_j))$ and the graph $G_k = G_j + uv$.

By construction in both cases $(G_k, \hat{0}) <_L (G_j, \hat{0})$ and also

$$(G_k, \hat{0}) \vee (G_j, \hat{0}) = (G, 1/\ldots/uv/\ldots/n) \leftarrow (G_k, \hat{0}), (G_j, \hat{0})$$

where $G = G_k$ (in case (i)) or $G = G_j$ (in case (ii)). Now if $(H, \pi) \geq (G_i, \hat{0}), (G_j, \hat{0})$ then $H \subseteq G_i \cap G_j \subseteq G$ in both cases. Also, u and v are both in the same block of π since uv is an edge in exactly one of the two graphs G_i, G_j. Hence, $(H, \pi) \geq (G_k, \hat{0}) \vee (G_j, \hat{0})$. □

By combining Theorems 1 and 6 we obtain our desired result.

THEOREM 7. *$\hat{\Pi}_n^{(d)}$, $\hat{\Pi}_{n,r}$, and $\hat{\chi}_n$ are all shellable.* □

The reader will note that we have not found a recursive atom ordering for $\hat{P}_n$, the vector space partition poset. It would be interesting to resolve the shellability question in this last case.

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