Partitions, rooks, and symmetric functions in noncommuting variables

Mahir Bilen Can
Department of Mathematics, Tulane University
New Orleans, LA 70118, USA, mcan@tulane.edu
and
Bruce E. Sagan*
Department of Mathematics, Michigan State University,
East Lansing, MI 48824-1027, USA, sagan@math.msu.edu

August 17, 2010

This paper is dedicated to Doron Zeilberger on the occasion of his 60th birthday. His enthusiasm for combinatorics has been an inspiration to us all.

Key Words: noncommuting variables, rook, set partition, symmetric function

AMS subject classification (2000): Primary 05A18; Secondary 05E05.

Abstract

Let Π_n denote the set of all set partitions of $\{1,2,\ldots,n\}$. We consider two subsets of Π_n, one connected to rook theory and one associated with symmetric functions in noncommuting variables. Let $\mathcal{E}_n \subseteq \Pi_n$ be the subset of all partitions corresponding to an extendable rook (placement) on the upper-triangular board, $\mathcal{T}_{n-1}$. Given $\pi \in \Pi_m$ and $\sigma \in \Pi_n$, define their *slash product* to be $\pi|\sigma = \pi \cup (\sigma + m) \in \Pi_{m+n}$ where $\sigma + m$ is the partition obtained by adding m to every element of every block of σ. Call τ *atomic* if it can not be written as a nontrivial slash product and let $\mathcal{A}_n \subseteq \Pi_n$ denote the subset of atomic partitions. Atomic partitions were first defined by Bergeron, Hohlweg, Rosas, and Zabrocki during their study of $NCSym$, the symmetric

*Work partially done while a Program Officer at NSF. The views expressed are not necessarily those of the NSF.

functions in noncommuting variables. We show that, despite their very different definitions, $\mathcal{E}_n = \mathcal{A}_n$ for all $n \geq 0$. Furthermore, we put an algebra structure on the formal vector space generated by all rook placements on upper triangular boards which makes it isomorphic to $NCSym$. We end with some remarks and an open problem.

1 Extendable rooks and atomic partitions

For a nonnegative integer n, let $[n] = \{1, 2, \ldots, n\}$. Let Π_n denote the set of all set partitions π of $[n]$, i.e., $\pi = \{B_1, B_2, \ldots, B_k\}$ with $\uplus_i B_i = [n]$ (disjoint union). In this case we will write $\pi \vdash [n]$. The B_i are called *blocks*. We will often drop set parentheses and commas and just put slashes between blocks for readability's sake. Also, we will always write π is *standard form* which means that

$$\min B_1 < \min B_2 < \ldots < \min B_k \tag{1}$$

and the elements in each block are listed in increasing order. For example $\pi = 136|2459|78 \vdash [9]$. The *trivial partition* is the unique element of Π_0, while all other partitions are *nontrivial*.

The purpose of this note is to show that two subsets of Π_n, one connected with rook theory and the other associated to the Hopf algebra $NCSym$ of symmetric functions in noncommuting variables, are actually equal although they have very different definitions. After proving this result in the current section, we will devote the next to putting an algebra structure on certain rook placements which is isomorphic to $NCSym$. The final section contains some comments and open questions.

Let us first introduce the necessary rook theory. A *rook (placement)* is an $n \times n$ matrix, R, of 0's and 1's with at most one 1 in every row and column. So a permutation matrix, P, is just a rook of full rank. A *board* is $\mathcal{B} \subseteq [n] \times [n]$. We say that R is a *rook on* $\mathcal{B}$ if $R_{i,j} = 1$ implies $(i, j) \in \mathcal{B}$. In this case we write, by abuse of notation, $R \subseteq \mathcal{B}$. A rook $R \subseteq \mathcal{B}$ is *extendable in* $\mathcal{B}$ if there is a permutation matrix P such that $P_{i,j} = R_{i,j}$ for $(i, j) \in \mathcal{B}$. For example, consider the upper-triangular board $\mathcal{T}_n = \{(i, j) \ : \ i \leq j\}$. The $R \subseteq \mathcal{T}_2$ are displayed in Figure 1. Only the third and fifth rooks in Figure 1 are extendable, corresponding to the transposition and identity permutation matrices, respectively. Extendability is an important concept in rook theory because of its relation to the much-studied hit numbers of a board [5, page 163 and ff.].

There is a well-known bijection between $\pi \in \Pi_n$ and the rooks $R \subseteq \mathcal{T}_{n-1}$ [8, page 75]. Given R, define a partition π_R by putting i and j in the same block of π_R whenever $R_{i,j-1} = 1$. For each $R \subseteq \mathcal{T}_2$, the corresponding $\pi_R \in \Pi_3$ is shown in

$$R: \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\pi_R: \quad 1|2|3 \qquad 12|3 \qquad 13|2 \qquad 1|23 \qquad 123$$

Figure 1: The rooks on $\mathcal{T}_2$ and their associated partitions

Figure 1. Conversely, given π we define a rook R_π by letting $(R_\pi)_{i,j} = 1$ exactly when i and $j+1$ are adjacent elements in a block of π in standard form. It is easy to see that the maps $R \mapsto \pi_R$ and $\pi \mapsto R_\pi$ are inverses. If a matrix has a certain property then we will also say that the corresponding partition does, and vice-versa. Our first subset of Π_n will be the *extendable partitions* denoted by

$$\mathcal{E}_n = \{\pi \in \Pi_n \ : \ R_\pi \text{ is extendable in } \mathcal{T}_{n-1}\}.$$

So, from Figure 1, $\mathcal{E}_2 = \{13|2, 123\}$.

To define our second subset of Π_n, it is convenient to introduce an operation on partitions. For a set of integers $B = \{b_1, \ldots, b_j\}$ we let $B + m = \{b_1 + m, \ldots, b_j + m\}$. Similarly, for a partition $\pi = \{B_1, \ldots, B_k\}$ we use the notation $\pi + m = \{B_1 + m, \ldots, B_k + m\}$. If $\pi \in \Pi_m$ and $\sigma \in \Pi_n$ then define their *slash product* to be the partition in Π_{m+n} given by

$$\pi|\sigma = \pi \cup (\sigma + m).$$

Call a partition *atomic* if it can not be written as a slash product of two nontrivial partitions and let

$$\mathcal{A}_n = \{\pi \in \Pi_n \ : \ \pi \text{ is atomic}\}.$$

Atomic partitions were defined by Bergeron, Hohlweg, Rosas, and Zabrocki [2] because of their connection with symmetric functions in noncommuting variables. We will have more to say about this in Section 2.

Since $\mathcal{E}_n$ is defined in terms of rook placements, it will be convenient to have a rook interpretation of $\mathcal{A}_n$. Given any two matrices R and S, defined their *extended direct sum* to be

$$R \hat{\oplus} S = R \oplus (0) \oplus S$$

where $\oplus$ is ordinary matrix direct sum and (0) is the 1×1 zero matrix. To illustrate,

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \hat{\oplus} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b & c & 0 & 0 & 0 \\ d & e & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w & x \\ 0 & 0 & 0 & 0 & y & z \end{pmatrix}.$$

It is clear from the definitions that $\tau = \pi|\sigma$ if and only if $R_\tau = R_\pi \hat{\oplus} R_\sigma$. We now have everything we need to prove our first result.

Theorem 1.1. *For all $n \geq 0$ we have $\mathcal{E}_n = \mathcal{A}_n$.*

Proof. Suppose we have $\tau \in \mathcal{E}_n$. Assume, towards a contradiction, that τ is not atomic so that $\tau = \pi|\sigma$. On the matrix level we have $R_\tau = R_\pi \hat{\oplus} R_\sigma$ where R_π is $m \times m$ for some m. We are given that τ is extendable, so let P be a permutation matrix extending R_τ. Since P and R_τ agree above and including the diagonal, the first $m+1$ rows of P must be zero from column $m+1$ on. But P is a permutation matrix and so each of these $m+1$ rows must have a one in a different column, contradicting the fact that only m columns are available.

Now assume $\tau \in \mathcal{A}_n$. We will construct an extension P of R_τ. Let $i_1, \ldots, i_r$ be the indices of the zero rows of R_τ and similarly for $j_1, \ldots, j_r$ and the columns. If $i_k > j_k$ for all $k \in [r]$, then we can construct P by supplementing R_τ with ones in positions $(i_1, j_1), \ldots, (i_r, j_r)$.

So suppose, towards a contradiction, that there is some k with $i_k \leq j_k$. Now R_τ must contain $j_k - k$ ones in the columns to the left of column j_k. If $i_k < j_k$, then there are fewer than $j_k - k$ rows which could contain these ones since R_τ is upper triangular. This is a contradiction. If $i_k = j_k$, then the $j_k - k$ ones in the columns left of j_k must lie in the first $i_k - k = j_k - k$ rows. Furthermore, these ones together with the zero rows force the columns to the right of j_k to be zero up to and including row $i_k = j_k$. It follows that $R_\tau = R_\pi \hat{\oplus} R_\sigma$ for some π, σ with R_π being $(i_k - 1) \times (i_k - 1)$. This contradicts the fact that τ is atomic. $\square$

Having two descriptions of this set may make it easy to prove assertions about it from one definition which would be difficult to demonstrate if the other were used. Here is an example.

Corollary 1.2. *Let $R \subseteq \mathcal{T}_n$. If $R_{1,n} = 1$ then R is extendable in $\mathcal{T}_n$.*

Proof. If $R_{1,n} = 1$ then we can not have $R = R_\sigma \hat{\oplus} R_\tau$ for nontrivial σ, τ. So R is atomic and, by the previous theorem, R is extendable. $\square$

2 An algebra on rook placements and $NCSym$

The algebra of symmetric functions in noncommuting variables, $NCSym$, was first studied by Wolf [10] who proved a version of the Fundamental Theorem of Symmetric Functions in this context. The algebra was rediscovered by Gebhard and Sagan [4] who used it as a tool to make progress on Stanley's $(\mathbf{3}+\mathbf{1})$-free Conjecture for chromatic symmetric functions [7]. Rosas and Sagan [6] were the first to make a systematic study of the vector space properties of $NCSym$. Bergeron, Reutenauer, Rosas, and Zabrocki [3] introduced a Hopf algebra structure on $NCSym$ and described its invariants and covariants.

Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables which do not commute. Consider the corresponding ring of formal power series over the rationals $\mathbb{Q}\langle\langle X\rangle\rangle$. Let $\mathfrak{S}_m$ be the symmetric group on $[m]$. Then any $g \in \mathfrak{S}_n$ acts on a monomial $x = x_{i_1}x_{i_2}\cdots x_{i_n}$ by

$$g(x) = x_{g^{-1}(i_1)}x_{g^{-1}(i_2)}\cdots x_{g^{-1}(i_n)}$$

where $g(i) = i$ for $i > m$. Extend this action linearly to $\mathbb{Q}\langle\langle X\rangle\rangle$. The *symmetric functions in noncommuting variables*, $NCsym \subset \mathbb{Q}\langle\langle X\rangle\rangle$, are all power series which are of bounded degree and invariant under the action of $\mathfrak{S}_m$ for all $m \geq 0$.

The vector space bases of $NCSym$ are indexed by set partitions. We will be particularly interested in a basis which is the analogue of the power sum basis for ordinary symmetric functions. Given a monomial $x = x_{i_1}x_{i_2}\cdots x_{i_n}$, there is an associated set partition π_x where j and k are in the same block of π_x if and only if $i_j = i_k$ in x, i.e., the indices in the jth and kth positions are the same. For example, if $x = x_3x_5x_2x_3x_3x_2$ then $\pi_x = 145|2|36$. The *power sum symmetric functions in noncommuting variables* are defined by

$$p_\pi = \sum_{x\,:\,\pi_x \geq \pi} x,$$

where $\pi_x \geq \pi$ is the partial order in the lattice of partitions, so π_x is obtained by merging blocks of π. Equivalently, p_π is the sum of all monomials where the indices in the jth and kth places are equal if j and k are in the same block of π, but there may be other equalities as well. To illustrate,

$$p_{13|2} = x_1x_2x_1 + x_2x_1x_2 + \cdots + x_1^3 + x_2^3 + \cdots .$$

Note that, directly from the definitions,

$$p_{\pi|\sigma} = p_\pi p_\sigma. \tag{2}$$

Using this property, Bergeron, Hohlweg, Rosas, and Zabrocki [2] proved the following result which will be useful for our purposes.

Proposition 2.1 ([2]). *As an algebra, NCSym is freely generated by the p_π with π atomic.* □

Let
$$\mathcal{R} = \{R \subseteq \mathcal{T}_n \ : \ n \geq -1\},$$
where there is a single rook on $\mathcal{T}_{-1}$ called the *unit rook* and denoted $R = 1$ (not to be confused with the empty rook on $\mathcal{T}_0$). We extend the bijection between set partitions and rooks on upper triangular boards by letting the unit rook correspond to the empty partition. Consider the vector space $\mathbb{Q}\mathcal{R}$ of all formal linear combinations of rooks in $\mathcal{R}$. By both extending $\hat{\oplus}$ linearly and letting the unit rook act as an identity, the operation of extended direct sum can be considered as a product on this space. It is easy to verify that this turns $\mathbb{Q}\mathcal{R}$ into an algebra.

Proposition 2.2. *As an algebra, $\mathbb{Q}\mathcal{R}$ is freely generated by the R_π with π atomic.*

Proof. A simple induction on n shows that any $\tau \in \Pi_n$ can be uniquely factored as $\tau = \pi_1|\pi_2|\cdots|\pi_t$ with the π_i atomic. From the remark just before Theorem 1.1, it follows that each R_τ can be uniquely written as a product of atomic R_π's. Since the set of all R_τ forms a vector space basis, the atomic R_π form a free generating set. □

Comparing Propositions 2.1 and 2.2 as well as the remark before Theorem 1.1 and equation 2, we immediately get the desired isomorphism.

Theorem 2.3. *The map $p_\pi \mapsto R_\pi$ is an algebra isomorphism of NCSym with $\mathbb{Q}\mathcal{R}$.* □

3 Remarks and an open question

3.1 Unsplittable partitions

Bergeron, Reutenauer, Rosas, and Zabrocki [3] considered another free generating set for *NCSym* which we will now describe. A *restricted growth function of length n* is a sequence of positive integers $r = a_1a_2\ldots a_n$ such that

1. $a_1 = 1$, and
2. $a_i \leq 1 + \max\{a_1, \ldots, a_{i-1}\}$ for $2 \leq i \leq n$.

Let RG_n denoted the set of restricted growth functions of length n. There is a well-known bijection between Π_n and RG_n [8, page 34] as follows. Given $\pi \in \Pi_n$ we define r_π by $a_i = j$ if and only if $i \in B_j$ in π. For example, if $\pi = 124|36|5$ then

$r_\pi = 112132$. It is easy to see that having π in standard form makes the map well defined. And the reader should have no trouble constructing the inverse.

Define the *split product* of $\pi \in \Pi_m$ and $\sigma \in \Pi_n$ to be $\tau = \pi \circ \sigma \in \Pi_{m+n}$ where τ is the uniqe partition such that $r_\tau = r_\pi r_\sigma$ (concatenation). To illustrate, if π is as in the previous paragraph and $\sigma = 13|2$ then $r_\pi r_\sigma = 112132121$ and so $\pi \circ \sigma = 12479|368|5$. This is not Bergeron et al.'s original definition, but it is equivalent. Now define τ to be *unsplitable* if it can not be written as a split product of two nontrivial partitions. (Bergeron et al. used the term "nonsplitable" which is not a typical English word.) Let $\mathcal{US}_n \subseteq \Pi_n$ be the subset of unsplitable partitions. So $\mathcal{US}_2 = \{1|2|3, 1|23\}$.

Perhaps the simplest basis for $NCSym$ is the one gotten by symmetrizing a monomial. Define the *monomial symmetric functions in noncommuting variables* to be

$$m_\pi = \sum_{x \,:\, \pi_x = \pi} x.$$

So now indices in a term of m_π are equal precisely when their positions are in the same block of π. For example,

$$m_{13|2} = x_1 x_2 x_1 + x_2 x_1 x_2 + \cdots .$$

The following is a more explicit version of Wolf's original result [10].

Proposition 3.1 ([3])**.** *As an algebra, $NCSym$ is freely generated by the m_π with π unsplitable.* □

Comparing Propositions 2.1 and 3.1 we see that $|\mathcal{A}_n| = |\mathcal{US}_n|$ for all $n \geq 0$ where $|\cdot|$ denotes cardinality. (Although they are not the same set as can be seen by our computations when $n = 2$.) It would be interesting to find a bijective proof of this result.

3.2 Hopf structure

Thiem [9] found a connection between $NCSym$ and unipotent upper-triangular zero-one matrices using supercharacter theory. This work has very recently been extended using matrices over any field and a colored version of $NCSym$ during a workshop at the American Institute of Mathematics [1]. This approach gives an isomorphism even at the Hopf algebra level.

References

[1] Bergeron, N. personal communication.

[2] Bergeron, N., Hohlweg, C., Rosas, M., and Zabrocki, M. Grothendieck bialgebras, partition lattices, and symmetric functions in noncommutative variables. *Electron. J. Combin. 13*, 1 (2006), Research Paper 75, 19 pp. (electronic).

[3] Bergeron, N., Reutenauer, C., Rosas, M., and Zabrocki, M. Invariants and coinvariants of the symmetric groups in noncommuting variables. *Canad. J. Math. 60*, 2 (2008), 266–296.

[4] Gebhard, D. D., and Sagan, B. E. A chromatic symmetric function in noncommuting variables. *J. Algebraic Combin. 13*, 3 (2001), 227–255.

[5] Riordan, J. *An introduction to combinatorial analysis.* Dover Publications Inc., Mineola, NY, 2002. Reprint of the 1958 original [Wiley, New York; MR0096594 (20 #3077)].

[6] Rosas, M. H., and Sagan, B. E. Symmetric functions in noncommuting variables. *Trans. Amer. Math. Soc. 358*, 1 (2006), 215–232 (electronic).

[7] Stanley, R. P. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math. 111*, 1 (1995), 166–194.

[8] Stanley, R. P. *Enumerative Combinatorics. Vol. 1*, vol. 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[9] Thiem, N. Branching rules in the ring of superclass functions of unipotent upper-triangular matrices. *J. Algebraic Combin. 31*, 2 (2010), 267–298.

[10] Wolf, M. C. Symmetric functions of non-commutative elements. *Duke Math. J. 2*, 4 (1936), 626–637.