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Maximizing Möbius functions on subsets of Boolean algebras

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Abstract

Let $\mathscr{F}$ be a family of subsets of an n-set, considered as a subposet of the Boolean algebra B_n. Adjoin a minimum $\hat{0}$ and maximum $\hat{1}$ if necessary to form $\hat{\mathscr{F}}$. Let $\mu(\mathscr{F})$ denote the value of the Möbius function $\mu(\hat{0}, \hat{1})$ in $\hat{\mathscr{F}}$. We compute the maximum value of $|\mu(\mathscr{F})|$ as $\mathscr{F}$ ranges over three types of families in B_n: lower order ideals, intervals of rank levels, and arbitrary rank-selections. The maxima are obtained by taking the lower half, the middle third, and every other rank of B_n, respectively. The maximum for the first case was previously found by Eckhoff (1980) and Scheid (1979). It allows us to answer a question raised by Füredi based on his joint work with Chung, Graham and Seymour (1988). The third maximum was also previously given by Niven (1968) and de Bruijn (1970). Finally, we consider lower order ideal case for the lattice of subspaces of a vector space, the maximum being achieved by taking the whole poset.

1. Introduction

Let P be a partially ordered set (*poset*). We will follow Stanley [14] for any poset terminology not defined in this paper. Suppose P is *bounded*, that is, it has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. If μ is the Möbius function of P [10, 14], then we let $\mu(P)$ stand for the value of $\mu(\hat{0}, \hat{1})$ in P. We will be interested in maximizing $|\mu(P)|$ as P runs over certain subposets of a Boolean algebra.

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The problem of determining the maximum Möbius function of a set system (a subset of a Boolean algebra) seems to be difficult. In the case of a simplicial complex (ideal in a Boolean algebra) the extreme configuration is always a rank selection, see below. There is some evidence to suspect that the extreme configurations in the case of arbitrary set systems are rank selections as well.

This motivates the study of the Möbius function of a rank selection of a Boolean algebra. The principal tool there is a formula which expresses the Möbius function in terms of certain sets of permutations (Proposition 1.1). This allows us to give combinatorial proofs, and to solve the corresponding maximization problems completely (Theorem 1.2).

Given nonnegative integers n, k and l, we let

$$[n]=\{1,2,\ldots,n\} \quad \text{and} \quad [k,l]=\{k,k+1,\ldots,l\}.$$

In particular, these sets are empty if $n=0$ or $k>l$, respectively. Let B_n denote the Boolean algebra of all subsets of $[n]$ under the partial order of inclusion. Given any family of subsets $\mathscr{F} \subseteq B_n$, we let

$$\hat{\mathscr{F}}=\mathscr{F}\cup\{\emptyset\}\cup\{[n]\},$$

that is, we add $\hat{0}=\emptyset$ and $\hat{1}=[n]$ to $\mathscr{F}$ if necessary. Now $\hat{\mathscr{F}}$ is a bounded poset and we define the *Möbius function of* $\mathscr{F}$ by

$$\mu(\mathscr{F})=\mu(\hat{\mathscr{F}}).$$

If P is a bounded poset, with minimal element $\hat{0}$ and maximal element $\hat{1}$, then we let

$$\bar{P}=P\backslash\{\hat{0},\hat{1}\}$$

where the backslash denotes set difference. We will be concerned with the Möbius function of families $\mathscr{F}$ which are *lower order ideals* in that they satisfy

$$x\in\mathscr{F} \text{ and } y\leqslant x \;\Rightarrow\; y\in\mathscr{F}.$$

To describe our other two types of families, we need a little more notation.

The poset B_n is ranked with the *rank* of any $S\in B_n$ being $\operatorname{rk} S=|S|$, where $|\cdot|$ denotes cardinality. Thus the possible ranks of a set other than $\hat{0}$ and $\hat{1}$ are in $[n-1]$.

If $\mathscr{F}\subseteq B_n$ and $R\subseteq[n-1]$, then we will consider the corresponding *rank-selected* subposet

$$\mathscr{F}(R)=\{S\in\mathscr{F}\colon \operatorname{rk} S\in R\}.$$

In particular, the i-th *rank level* of $\mathscr{F}$ is given by

$$\mathscr{F}(i)=\mathscr{F}(\{i\})=\{A\in\mathscr{F}\colon |A|=i\},$$

and for intervals we use the shorthand

$$\mathscr{F}[k]=\mathscr{F}([k]) \quad \text{and} \quad \mathscr{F}[k,l]=\mathscr{F}([k,l]).$$

The Möbius function of rank-selected subposets of B_n can be described [14, pp. 131–132] in terms of descents of permutations. If $\pi = a_1 a_2 \cdots a_n$ is a permutation of $[n]$, then its *descent set* is the subset of $[n-1]$ defined by

$$D(\pi)=\{i\colon a_i>a_{i+1}\}.$$

Now, given $R\subseteq[n-1]$, we let

$$\mathscr{B}_n(R)=\{\pi=a_1\cdots a_n\colon D(\pi)=R\}$$

and

$$\beta_n(R)=|\mathscr{B}_n(R)|.$$

The following result is well known [14, Corollary 3.12.2].

Proposition 1.1. *If* $R\subseteq[n-1]$ *then*

$$\mu(B_n(R))=(-1)^{|R|-1}\beta_n(R).$$

For $\mathscr{B}_n$ and β_n we will use the same shorthand in the interval case, e.g.,

$$\mathscr{B}_n([k])=\mathscr{B}_n[k]$$

and similarly for $[k,l]$. Also the n-th *Euler number* [14, pp. 148–149] is defined as

$$E_n=\beta_n(1,3,5,\ldots)$$

and so counts permutations of the form $a_1>a_2<a_3>\cdots$ which are called *alternating*. These numbers $(E_0,E_1,E_2,\ldots)=(1,1,1,2,5,16,61,\ldots)$ have the generating function

$$\sum_{n\geqslant 0} E_n\frac{x^n}{n!}=\tan x+\sec x.$$

We are now in a position to state our main theorem. In it, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ stand for the floor (round down) and ceiling (round up) functions, respectively.

Theorem 1.2. *Suppose* $\mathscr{F}\subseteq\bar{B}_n$.

(1) *If* $\mathscr{F}$ *is a lower order ideal, then*

$$|\mu(\mathscr{F})|\leqslant\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}$$

with equality if and only if

$$\mathscr{F}=B_n[k]\quad\textit{with } k=\left\lfloor\frac{n-1}{2}\right\rfloor\quad\textit{or}\quad k=\left\lceil\frac{n-1}{2}\right\rceil.$$

(2) *If* $\mathscr{F}=B_n(R)$ *where* $R\subseteq[n-1]$ *is an arbitrary rank-selection of* B_n, *then*

$$|\mu(\mathscr{F})|\leqslant E_n$$

with equality if and only if

$$R=\{1,2,3,\ldots\} \quad or \quad R=\{2,4,6,\ldots\}.$$

(3) *If* $\mathscr{F}=B_n(R)$ *where* $R=[k,l]$ *is an interval of ranks, then*

$$|\mu(\mathscr{F})|\leqslant\begin{cases}\beta_n[\frac{n}{3},\frac{2n}{3}-1]=\beta_n[\frac{n}{3}+1,\frac{2n}{3}] & for\ n\equiv 0 \pmod 3,\\ \beta_n[\lceil\frac{n}{3}\rceil,\lfloor\frac{2n}{3}\rfloor] & for\ n\equiv\pm 1 \pmod 3)\end{cases}$$

with equality if and only if R is one of the intervals mentioned in the previous equations.

Part 1 of this theorem can be interpreted as giving the maximum absolute value of the Euler characteristic of a subcomplex of the $(n-1)$-simplex. It has been discovered independently by Eckhoff [8, Hilfssätze 2 und 3], and Scheid [1, p. 180]. In fact, Eckhoff solves the problem even with a restriction on the maximum size of a set in $\mathscr{F}$. A stronger result, bounding the sum of the Betti-numbers, was obtained by Björner and Kalai [5, Theorem 1.4, and note added in proof]. Noga Alon has also come up with a proof similar to ours [4].

In Section 2, after giving our very simple proof, we use the result to answer a question of Füredi [9] based on his joint work with Chung, Graham and Seymour [7]. Section 3 treats arbitrary rank-selections. This case has also been done independently by Niven [11] and de Bruijn [6]. Section 4 deals with interval rank-selections. This result is new, to our knowledge. Next, we consider lower order ideals in the lattice of subspaces of a vector space, Section 5, and show that μ is maximized by taking the whole poset (another original result). Finally, we end with some comments and open problems.

2. Lower order ideals

First we will need to review some general results about Möbius functions. If P is a bounded poset, then let

$$c_i=c_i(P)=\text{the number of chains of length } i \text{ in } \bar{P}$$

where a chain of length i is a totally ordered subset with $i+1$ elements. It is well known [13, Proposition 6] and [14, Proposition 3.8.5] that

$$\mu(P)=-1+c_0-c_1+c_2-c_3+\cdots\ . \tag{1}$$

(It is in this sense that μ can be interpreted as a reduced Euler characteristic.) We can use this equation to get Lemma 2.1, our primary tool for this section. Its systematic use for maximizing the Möbius function of posets was demonstrated before in [15, Proofs of Theorems 2.5, 3.2]. In what follows, $\max P$ denotes the set of maximal elements of the poset P.

Lemma 2.1 [14, Lemma 3.14.4]. *Let P be a bounded poset. If $x \in \bar{P}$ then*

$$\mu(P)=\mu(P\backslash x)+\mu(\hat{0},x)\mu(x,\hat{1}).$$

In particular, if $T \subseteq \max \bar{P}$ then

$$\mu(P)=\mu(P\backslash T)-\sum_{x \in T}\mu(\hat{0},x) \tag{2}$$

Proof. To prove the first equation, merely note that the right-hand side counts the chains in $\bar{P}$ according to whether x is not in the chain (the first term) or x is in the chain (the second term). The second equation follows by repeated application of the first, and the fact that $\mu(x,\hat{1})=-1$ if $x \in T$. □

As a corollary, we get the following useful result.

Lemma 2.2. *If $R \subseteq [n-1]$ and $k=\max R$ then*

$$\mu(B_n(R))=\mu(B_n(R\backslash k))-\binom{n}{k}\mu(B_k(R\backslash k)). \tag{3}$$

Thus for $k \in [n-1]$ we have

$$\mu(B_n[k])=(-1)^{k+1}\binom{n-1}{k}, \tag{4}$$

in particular

$$\mu(B_n)=(-1)^n. \tag{5}$$

Proof. The first equation follows immediately from taking $T=B_n(k)$ in Lemma 2.1. The second can be proved using the first by induction on k, and the third is a special case of the second. (These two equations also follow from Proposition 1.1.) □

We need one last lemma about the *shadow* of a subset S of rank r of B_n, which is defined to be

$$\Delta(S)=\{B \in B_n(r-1) \colon B \subseteq A \text{ for some } A \in S\}.$$

This result follows from Sperner's lemma [2, Lemma 2.1.1 and Corollary 2.1.2] except in the case where S is contained in one of two ranks of B_n having equal size. Since this case requires a proof which can also be used to show the whole lemma, we have included a complete demonstration. We thank Dennis Stanton for pointing out that edge-counting arguments could be used.

Lemma 2.3. *If $S \subseteq B_n(r)$ where $r>n/2$ then $|\Delta(S)| \geqslant |S|$ with equality only when n is odd and $S=B_n((n+1)/2)$.*

Proof. Consider the bipartite graph, G formed in the Hasse diagram of B_n by S and $\Delta(S)$. Since every vertex $A \in S$ has degree r, the graph has a total of $r|S|$ edges. Also, every vertex $B \in \Delta(S)$ has degree at most $n-r+1$ in G for an edge count of at most $(n-r+1)|\Delta(S)|$. Thus when $r>(n+1)/2$, the lemma follows.

The same reasoning works if n is odd and $r=(n+1)/2$ as long as some vertex in $\Delta(S)$ does not have degree $(n+1)/2$. If every $B \in \Delta(S)$ has this degree, then in B_n, vertices of $\Delta(S)$ are only adjacent to vertices of S (and, of course, vice versa). Thus having $S \subset B_n(r)$ (strict containment) would contradict, e.g., shellability of B_n [3]. □

We are now ready to prove part 1 of Theorem 1.2. Let $\mathscr{F} \subseteq \bar{B}_n$ be an ideal with maximum $|\mu(\mathscr{F})|$. Also, let k be the maximum size of a set in $\mathscr{F}$. We will compare $\mu(\mathscr{F})$ with $\mu(\mathscr{F}[k-1])$ and $\mu(\mathscr{F}[k-2])$. Applying equation (2) to $P=\hat{\mathscr{F}}$ and using (5), we obtain

$$\mu(\mathscr{F}[k-1])=\mu(\mathscr{F})+(-1)^k|\mathscr{F}(k)|. \tag{6}$$

Now applying (2) again to the $\mu(\mathscr{F}[k-1])$ term in previous equation, we get

$$\mu(\mathscr{F}[k-2])=\mu(\mathscr{F})-(-1)^k(|\mathscr{F}(k-1)|-|\mathscr{F}(k)|). \tag{7}$$

Note that since $\mathscr{F}$ is an ideal, $\mathscr{F}(k-1)$ contains the shadow $\Delta\mathscr{F}(k)$, and thus $|\mathscr{F}(k-1)|-|\Delta\mathscr{F}(k)|\geqslant 0$. If $k>\lceil n/2 \rceil$, then from Lemma 2.3 we know that the shadow of $\mathscr{F}(k)$ has size larger than that of $\mathscr{F}(k)$ itself. Thus $|\mathscr{F}(k-1)|-|\mathscr{F}(k)|>0$ and so one of equations (6), (7) implies that $|\mu(\mathscr{F})|$ is not maximum. Hence $k \leqslant \lceil n/2 \rceil$.

In fact, we claim that $k \leqslant \lfloor n/2 \rfloor$. This is immediate if n is even. If n is odd, then we must rule out the possibility that $k=\lceil n/2 \rceil=(n+1)/2$. But in that case there are two options. Either $\mathscr{F}(k) \subset B_n(k)$ and the argument above applies. Or $\mathscr{F}=B_n[k]$ and so, by equation (4),

$$|\mu(\mathscr{F})|=\binom{n-1}{\lceil n/2 \rceil}$$

which is not optimal.

We have shown that if $\mathscr{F}$ is lower order ideal whose Möbius function is maximum in absolute value, then

$$k=\max\{|A|\colon A \in \mathscr{F}\} \leqslant \left\lfloor \frac{n}{2} \right\rfloor.$$

Using a completely analogous argument, we see that

$$l=\min\{|B|\colon B \in \bar{B}_n \backslash \mathscr{F}\} \geqslant \left\lceil \frac{n}{2} \right\rceil.$$

From the definitions $k \geqslant l-1$. Because of the bounds on these two quantities, we see that $k=l-1$ or l. If n is odd, then this implies that $k=l-1=(n-1)/2$. So, in this case,

$$\mathscr{F}=B_n\left(\left[\frac{n-1}{2}\right]\right)$$

and we are done. For even n, we have

$$B_n\left(\left[\frac{n}{2}-1\right]\right)\subseteq\mathscr{F}\subseteq B_n\left(\left[\frac{n}{2}\right]\right).$$

Now applying equation (6) with $k=n/2$, we obtain

$$\mu\left(B_n\left[\frac{n}{2}-1\right]\right)=\mu(\mathscr{F})+(-1)^{n/2}\left|\mathscr{F}\left(\frac{n}{2}\right)\right|.$$

Furthermore, letting $P=B_n[n/2]$ and $T=B_n(n/2)\backslash\mathscr{F}(n/2)$ in Lemma 2.1, we get

$$\mu\left(B_n\left[\frac{n}{2}\right]\right)=\mu(\mathscr{F})-(-1)^{n/2}\left|B_n\left(\frac{n}{2}\right)\backslash\mathscr{F}\left(\frac{n}{2}\right)\right|.$$

This implies that either $\mathscr{F}(n/2)=\emptyset$ or $B_n(n/2)=\mathscr{F}(n/2)$. Thus we have $\mathscr{F}=(B_n[n/2-1])$ or $\mathscr{F}=B_n[n/2]$, and we are done with part 1 of Theorem 1.2. □

The same proof technique also yields Eckhoff's theorem [8] in the case of an upper bound for $k=\max\{|A|\colon A\in\mathscr{F}\}$.

We can immediately obtain an asymptotic estimate for max $|\mu(\mathscr{F})|$. Let $f(n)$, $g(n)$ be two functions of the integer variable n. We write $f(n)=\Theta(g(n))$ if there are positive constants c, d such that

$$cg(n)\leqslant f(n)\leqslant dg(n)$$

for all large n. A routine application of Stirlings formula yields the following result.

Proposition 2.4. *Let $f_O(n)$ denote the maximum value of $|\mu(\mathscr{F})|$ as $\mathscr{F}$ varies over all lower order ideals of B_n. Then*

$$f_O(n)=\Theta(2^n/\sqrt{n}).$$

It is also easy to obtain sharp upper bounds for the Möbius function itself (as opposed to its absolute value). Using the same techniques, but paying attention to the sign of μ, the following result can be derived. Its proof, being similar to what we have done, is omitted. An analogous statement can be made for lower bounds.

Corollary 2.5. *If $\mathscr{F}$ is an ideal in B_n, then we have the following sharp bounds.*

$$\mu(\mathscr{F})\leqslant\begin{cases}\dbinom{n-1}{\lfloor\frac{n-1}{2}\rfloor} & \text{for } n\equiv 0,2,3 \pmod 4,\\[2ex] \dbinom{n-1}{\frac{n+1}{2}} & \text{for } n\equiv 1 \pmod 4.\end{cases}$$

The maximum is only achieved when $\mathscr{F}=B_n([k])$, where k can be determined from part 1 of Theorem 1.2 and equation (4) in the first case. In the second case

$$k=\frac{n+1}{2}\quad\text{or}\quad\frac{n-3}{2}.$$

Finally, we come to an application related to the work of Chung, Füredi, Graham and Seymour [7]. For an arbitrary $\mathscr{F} \subseteq B_n$, they had defined

$$f(\mathscr{F}) = \sum_{\substack{A_i \in \mathscr{F},\\ |A_i|=n}} 1 - \sum_{\substack{A_i, A_j \in \mathscr{F},\\ |A_i \cup A_j|=n}} 1 + \sum_{\substack{A_i, A_j, A_k \in \mathscr{F},\\ |A_i \cup A_j \cup A_k|=n}} 1 - \cdots$$

and Füredi [9] asked for upper bounds of $f(\mathscr{F})$. Observations (i), (ii) and (iv) of their paper show that

$$f(\mathscr{F}) = \begin{cases} 0 & \text{if } \emptyset \in \mathscr{F}, \\ f(\min(\mathscr{F} \setminus [n])) & \text{otherwise,} \end{cases}$$

where $\min(\mathscr{F})$ is the set of minimal sets in $\mathscr{F}$. This reduces the problem to the case where $\min \mathscr{F} = \mathscr{F}$, i.e. $\mathscr{F}$ is an anti-chain (or clutter) in $\bar{B}_n$. Now by Rota's Cross-cut Theorem [13, Theorem 3] [14, Corollary 3.9.4], we obtain

$$f(\mathscr{F}) = \sum_{\substack{S \subseteq \mathscr{F},\\ \bigcup S=[n]}} (-1)^{|S|-1} = -\mu(\mathscr{F}^{\uparrow})$$

where $\mathscr{F}^{\uparrow}$ is the upper order ideal generated by $\mathscr{F}$, defined by

$$\mathscr{F}^{\uparrow} = \{A \in \bar{B}_n\colon A \supseteq B \text{ for some } B \in \mathscr{F}\}.$$

Since dualizing does not change the Möbius function, the minimum version of Corollary 2.5 implies

$$f(\mathscr{F}) \leqslant \begin{cases} \dbinom{n-1}{\lfloor \frac{n-1}{2} \rfloor} & \text{for } n \equiv 0, 1, 2 \pmod 4, \\[2ex] \dbinom{n-1}{\frac{n+1}{2}} & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

and these bounds are sharp.

3. Arbitrary rank-selections

Our major tool in this section and the next will be Proposition 1.1. Niven [11] was the first to show that the maximum size of a descent class is obtained by using the alternating permutations. He used an injection which differs from ours. It is slightly more complicated than our map (9), but the proof that it has the desired properties is somewhat easier. Later, de Bruijn [6] gave a very simple algorithmic demonstration.

We begin with some basic results about permutations. Give $A \subseteq [n]$, we write $A = \{a_1, a_2, \ldots, a_k\}_<$ if A is listed in increasing order: $a_1 < a_2 < \cdots < a_k$. If $a_i \in A$ then define the *A-complement of* a_i to be

$$a_i^A = a_{k-i+1}.$$

In other words, A-complementation is the permutation whose two-line form is given by

$$\begin{matrix} a_1 & a_2 & \cdots & a_k \\ a_k & a_{k-1} & \cdots & a_1. \end{matrix} \tag{8}$$

Complementation can be used to prove the following easy but useful lemma.

Lemma 3.1 ([11]). *Suppose* $R=\{r_1,\ldots,r_k\}\subseteq[n-1]$.

(1) *If* $R'=[n-1]\backslash R$, *then* $\beta_n(R')=\beta_n(R)$.

(2) *If* $R''=\{n-r_1,\ldots,n-r_k\}$, *then* $\beta_n(R'')=\beta_n(R)$.

Proof. Both of these can be seen bijectively. For the first part, use the function $f:\mathscr{B}_n(R')\to\mathscr{B}_n(R)$ defined by

$$f(a_1a_2\cdots a_n)=a_1^Aa_2^A\cdots a_n^A$$

where $A=[n]$. For the second, define $g:\mathscr{B}_n(R'')\to\mathscr{B}_n(R)$ by

$$g(a_1a_2\cdots a_n)=a_n^Aa_{n-1}^A\cdots a_1^A. \qquad \square$$

We will need to know how complementation behaves if we add an element to our set.

Lemma 3.2. *Consider* $A=\{a_1,\ldots,a_k\}_<$ *and fix* $a_i\in A$, *letting* $m=\min\{a_i,a_i^A\}$ *and* $M=\max\{a_i,a_i^A\}$. *Pick* $a\notin A$ *and let* $B=A\cup\{a\}$.

(1) *If* $a<m$ *then* $a_{k-i}\leqslant a_i^B<a_{k-i+1}$ *and* $a^B\geqslant M$.

(2) *If* $m<a<M$ *then* $a_i^B=a_{k-i+1}$ *and* $m<a^B<M$.

(3) *If* $a>M$ *then* $a_{k-i+1}<a_i^B\leqslant a_{k-i+2}$ *and* $a^B\leqslant m$.

Proof. All parts of this lemma follows easily from a case-by-case consideration of what happens when the element a is inserted in the diagram (8) for the A-complement to obtain the one for the B-complement. Right-hand portions of both rows are shifted to the right one place. Since

$$a_i^A=a_{k-i+1}$$

a_i^B must be one of the values a_{k-i}, a_{k-i+1}, a_{k-i+2} or a. A more detailed analysis yields the inequalities for a_i^B. The ones for a^B are obtained by similar considerations. $\square$

The situation if we subtract an element is even simpler, so we will merely state the result.

Lemma 3.3. *Consider* $A=\{a_1,\ldots,a_k\}_<$ *and fix* $a_i\in A$, *letting* $m=\min\{a_i,a_i^A\}$, $M=\max\{a_i,a_i^A\}$. *Pick* $a\in A$ *and let* $B=A\setminus\{a\}$.

(1) *If* $a<m$ *then* $a_i^B=a_{k-i+2}$.

(2) *If* $m<a<M$ *then* $a_i^B=a_{k-i+1}$.

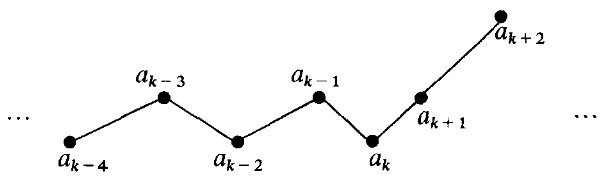
(3) *If* $a>M$ *then* $a_i^B=a_{k-i}$.

If $a=m$ *then the first case continues to hold if* $a_i=M$. *If* $a=M$ *then the third case continues to hold if* $a_i=m$.

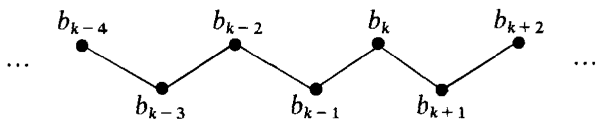
We are finally ready to begin the proof of Theorem 1.2, part 2. A *double ascent at* k occurs in $\pi=a_1\cdots a_n$ if $a_k<a_{k+1}<a_{k+2}$. Double descents are defined analogously. If R is a rank-selection with a double ascent, then it suffices to find a proper injective function (i.e., not also surjective)

$$f:\mathscr{B}_n(R)\to\mathscr{B}_n(R^+)$$

where R^+ has one fewer double ascents than R and the number of double descents stays the same. Because of part 1 of Lemma 3.1, we do not need another case with the roles of ascents and descents interchanged. By the same token, we can assume that the permutations $\pi\in\mathscr{B}_n(R)$ alternate (starting with either an ascent or a descent) until they reach their first double ascent at index k. That is to say we have the following picture.

$\pi=$ 

We will construct $\sigma=b_1\cdots b_n=f(\pi)$ so that σ looks as follows.

$\sigma=$ 

That is to say $R^+=([k]\setminus R)\cup(R\setminus[k+1])$.

It will be convenient to introduce one last bit of notation. Let $\pi=a_1\cdots a_n$ and consider a fixed index k. If $1\leqslant i\leqslant k$ then let a_i^K denote the complement in the set $K=\{a_1,\ldots,a_k\}$. Similarly, for $1\leqslant i\leqslant k+1$ we write a_i^{K+1} for the complement using $K+1=\{a_1,\ldots,a_{k+1}\}$. Now define $f(a_1\cdots a_n)=b_1\cdots b_n$ where

$$b_i=\begin{cases} a_i^K & \text{for } 1\leqslant i\leqslant k \quad \text{if } a_k^K>a_{k+1},\\ a_i^{K+1} & \text{for } 1\leqslant i\leqslant k+1 \quad \text{otherwise.}\end{cases} \tag{9}$$

and $a_i=b_i$ for all other indices i. We must first check that f is well defined, which amounts to showing that $b_{k+1}<b_{k+2}$. But in the first case of (9) we have $b_{k+1}=a_{k+1}<a_{k+2}=b_{k+2}$. In the second case we have assumed $a_k^K<a_{k+1}$ and we

always have $a_k < a_{k+1}$, so the third case of Lemma 3.2 applies with $A = \{a_1, \ldots, a_k\}$ and $a = a_{k+1}$. Thus

$$b_{k+1} = a_{k+1}^{K+1} \leqslant \min\{a_k, a_k^K\} < a_{k+2} = b_{k+2}.$$

We now need to prove that f is injective. But complementing in the first k elements of a permutation is an involution (as is complementation in the first $k+1$). Thus we need only show that there is some way of distinguishing the images of permutations belonging to the first and second cases of (9), so that we know which involution to apply. The following Lemma fulfills this need.

Lemma 3.4. *Suppose that* $f(a_1 \cdots a_n) = b_1 \cdots b_n$. *Then* $a_k^K > a_{k+1}$ *if and only if* $b_k^K < b_{k+1}$.

Proof. Of course, in b_k^K the complement is with respect to $\{b_1, \ldots, b_k\}$. For the forward direction, note that $b_k = a_k^K$ and the set K is the same for both the a's and b's. Thus $b_k^K = a_k < a_{k+1} = b_{k+1}$.

For the other direction, assume $a_k^K < a_{k+1}$. We first claim that

$$b_{k+1} \leqslant a_k. \tag{10}$$

Note that since complementation is order reversing, we have

$$b_{k+1} = a_{k+1}^{K+1} < a_k^{K+1}. \tag{11}$$

If $a_k^{K+1} \leqslant a_k$, then the claim follows immediately. Otherwise, we have $a_k^{K+1} > a_k$. Thus $a_k^K \geqslant a_k$ since this is true in all three cases of Lemma 3.3. Combining this with the overall assumption for this direction we obtain

$$a_{k+1} > a_k^K = \max\{a_k, a_k^K\}.$$

So Lemma 3.2, part 3, applies to yield $a_{k+1}^{K+1} \leqslant a_k$. This, along with the equality in (11), establishes the claim.

Now to finish the proof of our lemma. Note that $b_{k+1} < b_k$ and, from (10), $b_{k+1} \leqslant a_k = b_k^{K+1}$. Thus $b_{k+1} \leqslant \min\{b_k, b_k^{K+1}\}$ and $b_{k+1} \neq b_k$. Finally, part 1 of Lemma 3.3 gives us $b_k^K > b_k^{K+1} \geqslant b_{k+1}$. $\square$

We now have that the function f is an injection. To show it is strict, we must find $\sigma = b_1 \cdots b_n \in \mathscr{B}_n(R^+)$ which is not an image of f. But if $b_k = 3$, $b_{k+1} = 1$ and $b_{k+2} = 2$ then $b_k^K \geqslant 3 > b_{k+1}$ and $b_{k+1}^{K+1} \geqslant 3 \geqslant b_{k+2}$. Thus neither involution applied to σ gives an element of $\mathscr{B}_n(R)$. This finishes the proof of Theorem 1.2, part 2. $\square$

4. Interval rank-selections

We will first show that if we fix the length of the interval of descent, then one obtains the maximum number of permutations when the interval is centered. In fact we will prove a little more. A sequence $s_1, s_2, \ldots, s_n$ is called *symmetric* if $s_k = s_{n-k+1}$ for all k. A sequence is called *unimodal* if there is an index m such that

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m \geqslant a_{m+1} \geqslant \cdots \geqslant a_n.$$

It is *strictly unimodal* if all the preceding inequalities are strict, the only possible exception being that $a_m = a_{m+1}$.

Also, if $\pi = a_1 a_2 \cdots a_n$ is any strictly increasing (respectively strictly decreasing) sequence and $a \notin \pi$ is an integer then we let $\check{\pi}$ denote the sequence obtained by inserting a in the unique slot of π such that the sequence still increases (respectively decreases). Similarly, let $\hat{\pi}$ denote the sequence obtained by removing an element $a \in \pi$ while keeping the sequence monotone. Although the notation makes no mention of the element a, it will always be clear from context, what integer is meant.

Proposition 4.1. *For fixed $n > r \geqslant 0$, the sequence*

$$\beta_n[1, 1+r], \beta_n[2, 2+r], \ldots, \beta_n[n-r-1, n-1]$$

is symmetric and strictly unimodal. Thus its only maxima are at

$$\beta_n\left[\frac{n-r}{2}, \frac{n+r}{2}\right] \qquad \textit{for } n \equiv r \pmod 2,$$

$$\beta_n\left[\left\lfloor\frac{n-r}{2}\right\rfloor, \left\lfloor\frac{n+r}{2}\right\rfloor\right] = \beta_n\left[\left\lceil\frac{n-r}{2}\right\rceil, \left\lceil\frac{n+r}{2}\right\rceil\right] \qquad \textit{for } n \not\equiv r \pmod 2.$$

Proof. As far as symmetry, this is just a special case of the second part of Lemma 3.1. Thus for strict unimodality, it suffices to show that

$$\beta_n[i-1, i-1+r] < \beta_n[i, i+r] \quad \text{for } i \leqslant \left\lfloor\frac{n-r}{2}\right\rfloor \tag{12}$$

Construct a bipartite graph, G, as follows. The two vertex sets will be $V_1 = \mathscr{B}_n[i-1, i-1+r]$ and $V_2 = \mathscr{B}_n[i, i+r]$. Given $\pi \in V_1$, we draw an edge to every $\sigma \in V_2$ that can be obtained in the following manner. Write

$$\pi = a_1 < \cdots < a_{i-1} > \cdots > a_{i+r} < \cdots < a_n = \pi_1 \pi_2 \pi_3$$

where $\pi_1 = a_1 \cdots a_{i-1}$, $\pi_2 = a_i \cdots a_{i+r}$, and $\pi_3 = a_{i+r+1} \cdots a_n$. Pick any $a \in \pi_3$ and form the permutation

$$\sigma = \check{\pi}_1 \pi_2 \hat{\pi}_3.$$

Notice that we have $\sigma \in \mathscr{B}_n[i, i+r]$ since

$$\max \check{\pi}_1 \geqslant a_{i-1} > a_i = \max \pi_2 \quad \text{and} \quad \min \hat{\pi}_3 \geqslant a_{i+r+1} > a_{i+r} = \min \pi_2. \tag{13}$$

For example, if

$$\pi = 3\ 6\ 8\ 4\ 1\ 2\ 5\ 7\ 9 \in \mathscr{B}_9[3, 4]$$

then the $\sigma \in \mathscr{B}_9[4,5]$ adjacent to π are

$$\begin{array}{l} \sigma\colon 2\,3\,6\,8\,4\,1\,5\,7\,9 \\ \quad\; 3\,5\,6\,8\,4\,1\,2\,7\,9 \\ \quad\; 3\,6\,7\,8\,4\,1\,2\,5\,9 \\ \quad\; 3\,6\,8\,9\,4\,1\,2\,5\,7. \end{array}$$

Since there are $n-i-r$ choices for $a \in \pi_3$, every vertex $\pi \in V_1$ has degree $n-i-r$. However, each $\sigma \in V_2$ is adjacent to at most $i = |\check{\pi}_1|$ elements of V_1. This is because the crucial inequalities in (13) may not continue to hold if one attempts to invert the process of passing from π to σ. In fact, if the first element of σ is a 1, then it is impossible to change the 1's position and obtain a permutation in $\mathscr{B}_n[i-1, i-1+r]$. Thus there are some vertices in V_2 of degree smaller than i. Since $i \leqslant n-i-r$ for the range of i under consideration, equation (12) is proved and this finishes the proof of the lemma. □

Now we need to show that among all centered intervals, the one which contains the middle third of ranks results in the most permutations. This will finish our proof of Theorem 1.2, part 3.

Proposition 4.2. *For fixed $n>0$, the sequence*

$$\beta_n[\emptyset], \beta_n\left[\left\lfloor\frac{n}{2}\right\rfloor, \left\lfloor\frac{n}{2}\right\rfloor\right], \beta_n\left[\left\lfloor\frac{n-1}{2}\right\rfloor, \left\lfloor\frac{n+1}{2}\right\rfloor\right],$$

$$\beta_n\left[\left\lfloor\frac{n-2}{2}\right\rfloor, \left\lfloor\frac{n+2}{2}\right\rfloor\right], \ldots, \beta_n[1, n-1]$$

is strictly unimodal with its unique maximum at

$$\beta_n\left[\left\lceil\frac{n}{3}\right\rceil, \left\lfloor\frac{2n-1}{3}\right\rfloor\right].$$

In considering the sequence of

$$\beta_n\left[\left\lfloor\frac{n-i}{2}\right\rfloor, \left\lfloor\frac{n+i}{2}\right\rfloor\right]$$

there are four possibilities. They depend on the parity $n-i$ and whether we are before or after the maximum.

Let us consider the case where $i \equiv n \pmod 2$ and we are before the maximum. In other words, we wish to prove that

$$\beta_n\left[\frac{n-i}{2}, \frac{n+i}{2}\right] < \beta_n\left[\frac{n-i-2}{2}, \frac{n+i}{2}\right] \quad \text{for } i \leqslant \left\lfloor\frac{n-5}{3}\right\rfloor. \tag{14}$$

Again, we form a bipartite graph with V_1 and V_2 corresponding to the permutations counted by the left and right sides of (14). If $\pi \in V_1$ then write

$$\pi = a_1 < \cdots < a_{(n-i)/2} > \cdots > a_{(n+i+2)/2} < \cdots < a_n = \pi_1 \pi_2 \pi_3$$

where $\pi_1 = a_1 \cdots a_{(n-i-2)/2}$, $\pi_2 = a_{(n-i)/2} \cdots a_{(n+i+2)/2}$, and $\pi_3 = a_{(n+i+4)/2} \cdots a_n$. Pick any $a \in \pi_1$ and form the permutation

$$\sigma = \hat{\pi}_1 \check{\pi}_2 \pi_3.$$

Notice that we have $\sigma \in V_2$ since

$$\max \hat{\pi}_1 \leqslant a_{(n-i-2)/2} < a_{(n-i)/2} = \max \check{\pi}_2$$

and

$$\min \pi_3 = a_{(n+i+4)/2} > a_{(n+i+2)/2} \geqslant \min \check{\pi}_2.$$

Since $a \in \pi_1$ is arbitrary, every $\pi \in V_1$ has degree $|\pi_1| = (n-i-2)/2$. Every $\sigma \in V_2$ has degree at most $|\check{\pi}_2| - 1 = i+1$. (We must leave $a_{(n-i)/2}$ in $\check{\pi}_2$.). However the above steps will not always be reversible. For example, if $\min \pi_3 = 2$ then this forces $\min \check{\pi}_2 = 1$. But the 1 cannot be moved into $\hat{\pi}_1$ and still obtain a π in V_1. So some vertices in V_2 have degree smaller than $i+1$. From our assumption in (14), we have $i \leqslant (n-5)/3$ which implies $i+1 \leqslant (n-i-2)/2$. Thus $|V_1| < |V_2|$ and we are done with this case.

For the other cases, one moves elements of π_1 or π_3 (depending on the parity of $n-i$) either into or out of π_2 (depending on whether you are before or after the maximum). The proofs are similar and so we omit them. □

To find asymptotic estimates for this maximum, we will need a couple of lemmas.

Lemma 4.3. *For all* $1 \leqslant i \leqslant j < n$ *we have*

$$\beta_n[i,j] = \beta_{n-1}[i,j] + \beta_{n-1}[i-1,j-1] + \beta_{n-1}[i,j-1].$$

Proof. If $\pi = a_1 \cdots a_n \in \mathscr{B}_n[i,j]$ then either $a_i = n$ or $a_n = n$. Thus deleting the n from π gives us a bijection

$$\mathscr{B}_n[i,j] \leftrightarrow \mathscr{B}_{n-1}[i,j] \cup \mathscr{B}_{n-1}[i-1,j-1] \cup \mathscr{B}_{n-1}[i,j-1].$$

(Having $a_i = n$ gives rise to two of the three cases depending on the relative sizes of a_{i-1} and a_{i+1}.) □

The next result applies to any rank selection. It follows as an immediate corollary of Lemma 2.2 and Proposition 1.1. However, we choose to give a combinatorial proof.

Lemma 4.4. *If* $R \subseteq [n-1]$ *and* $k = \max R$ *then*

$$\beta_n(R) = \beta_k(R \backslash k) \binom{n}{k} - \beta_n(R \backslash k).$$

Proof. The term $\beta_k(R\backslash k)\binom{n}{k}$ counts all permutations $\pi=a_1\cdots a_n$ constructed in the following way. From $[n]$, pick the k elements to use as $a_1\cdots a_k$. Next arrange these elements so that they have descent set $R\backslash k$. Finally, put the remaining $n-k$ elements in increasing order as $a_{k+1}\cdots a_n$. Clearly such permutations are also counted by $\beta_n(R)+\beta_n(R\backslash k)$ depending on whether we have $a_k>a_{k+1}$ or $a_k<a_{k+1}$. □

Next we derive asymptotic bounds on our maximum.

Proposition 4.5. *Let $f_I(n)$ denote the maximum value of $\beta_n(R)$ where R is an interval of ranks then*

$$f_I(n)=\Theta(3^n/n).$$

Proof. By Theorem 1.2, part 3 and Lemma 4.3 we see that $f_I(n)$ is strictly increasing. Because of this and the fact that

$$\frac{3^{n+1}/(n+1)}{3^n/n}$$

is bounded, it suffices to prove the proposition for $n\equiv 0 \pmod 3$. If R is chosen so that $f_I(n)=\beta_n(R)$, then by maximality of $\beta_n(R)$ and Lemma 4.4

$$\frac{1}{2}\beta_k(R\backslash k)\binom{n}{k}\leqslant\beta_n(R)\leqslant\beta_k(R\backslash k)\binom{n}{k}.$$

Taking $R=[n/3+1,\ 2n/3]$, we have

$$\beta_k(R\backslash k)\binom{n}{k}=\binom{\frac{2n}{3}-1}{\frac{n}{3}}\binom{n}{\frac{2n}{3}}=\frac{1}{2}\frac{n!}{(n/3)!^3}\sim\frac{1}{2}\frac{3^{n+3/2}}{2\pi n}.$$

This completes the proof of the proposition. □

We can also characterize the maximum of the Möbius function itself rather than just its absolute value.

Corollary 4.6. *If $n\equiv\pm 2, 3 \pmod 6$, then the maximum value for $\mu(B_n(R))$, where R is an interval of ranks, is given by Theorem 1.2, part 3. Otherwise we have the following maxima.*

$$\mu(B_n(R))$$

$$\leqslant\begin{cases}\beta_n[\lceil\frac{n}{3}\rceil,\lfloor\frac{2n}{2}\rfloor-1]=\beta_n[\lceil\frac{n}{3}\rceil+1,\lfloor\frac{2n}{3}\rfloor] & \text{for } n\equiv -1 \pmod 6,\\ \beta_n[\frac{n}{3},\frac{2n}{3}]=\beta_n[\frac{n}{3}+1,\frac{2n}{3}-1] & \text{for } n\equiv 0 \pmod 6,\\ \beta_n[\lceil\frac{n}{3}\rceil-1,\lfloor\frac{2n}{3}\rfloor]=\beta_n[\lceil\frac{n}{3}\rceil,\lfloor\frac{2n}{3}\rfloor+1] & \text{for } n\equiv 1 \pmod 6.\end{cases}$$

with equality if and only if R is one of the intervals mentioned in the previous equations.

Proof. The only interesting cases are when the maximum of Theorem 1.2 is at a negative value of μ, that is, when $n\equiv 0, \pm 1 \pmod 6$. In these instances, all of the rank intervals having the same length as the one which maximizes $|\mu|$ also have negative Möbius functions. Thus, by Propositions 4.1 and 4.2, the largest value of μ must occur at the center of a sequence of rank intervals which is either one larger or one smaller than the one for maximizing $|\mu|$. Demonstrating which of the two choices is correct uses the same ideas as in the proofs of the previously cited propositions, thus we omit the details. □

A similar result exists for the minimum value of μ. The reader should have no trouble filling in the particulars.

5. The lattice of subspaces

Let us consider the q-analog of B_n, L_n, which is the lattice of all subspaces of an n-dimensional vector space V over the Galois field with q elements (ordered by inclusion). It is a ranked poset with $\operatorname{rk} W = \dim W$ for $W \leqslant V$. We will use the same techniques as in Section 2 to find the maximum value of $|\mu|$ as $\mathscr{F}$ ranges over all lower order ideals of L_n. Because of the similarity, we will only sketch the proofs, adapting our previous notation to this case. However, because of the appearance of extra powers of q, the answer will change radically. In fact, we will see that the maximum is achieved by taking the whole poset.

Of course, Lemma 2.1 continues to hold, so we can use it to prove the q-analog of Lemma 2.2. To state this result, we need the *q-binomial coefficients* (or *Gaussian polynomials*)

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1}.$$

If $\dim V = n$ then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ counts the number of $W \leqslant V$ with $\dim W = k$. Now for our first q-lemma.

Lemma 5.1. *If $R \subseteq [n-1]$ and $k = \max R$ then*

$$\mu(L_n(R)) = \mu(L_n(R \backslash k)) - \begin{bmatrix} n \\ k \end{bmatrix} \mu(L_k(R \backslash k)). \tag{15}$$

Thus for $k \in [n-1]$ we have

$$\mu(L_n[k]) = (-1)^{k+1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}, \tag{16}$$

in particular

$$\mu(L_n) = (-1)^n q^{\binom{n}{2}}. \tag{17}$$

Now if r is a given rank, we need a result about the *dual shadow* of $S \subseteq L_n(r)$ which is defined as

$$\nabla(S) = \{B \in L_n(r+1): B \supseteq A \text{ for some } A \in S\}.$$

Edge-counting yields the following analog of Lemma 2.3. Notice that because of the extra power of q, there is no restriction on the rank size.

Lemma 5.2. *If $S \subseteq L_n(r)$ for any $r \in [n-1]$, then $q^r |\nabla(S)| > |S|$.*

Finally, we can prove our main result about L_n.

Proposition 5.3. *If $\mathscr{F} \subseteq \bar{L}_n(q)$ is a lower order ideal then*

$$|\mu(\mathscr{F})| \leqslant q^{\binom{n}{2}}$$

with equality if and only if $\mathscr{F} = \bar{L}_n$.

Proof. Let $\mathscr{F}$ be a lower order ideal with maximum $|\mu|$, and consider

$$\mathscr{F}^c = \{W: W \in \bar{L}_n \setminus \mathscr{F}\}.$$

If

$$l = \min\{\dim W: W \in \mathscr{F}^c\},$$

then define two new families

$$\mathscr{G} = \mathscr{F} \cup \mathscr{F}^c(l)$$

and

$$\mathscr{H} = \mathscr{G} \cup \mathscr{F}^c(l+1).$$

Now use (2) and (17) applied first to $\mathscr{G}$ and then to $\mathscr{H}$. Thus we obtain the q-analog of (6):

$$\mu(\mathscr{G}) = \mu(\mathscr{F}) + (-1)^l q^{\binom{l}{2}} |\mathscr{F}^c(l)|,$$

and of (7):

$$\mu(\mathscr{H}) = \mu(\mathscr{F}) - (-)^l q^{\binom{l}{2}} (q^l |\mathscr{F}^c(l+1)| - |\mathscr{F}^c(l)|).$$

These two equations, together with Lemma 5.2 show that $|\mu(\mathscr{F})|$ will not be maximum unless $L_n[n-2] \subseteq \mathscr{F}$.

To show that $L_n(n-1) \subseteq \mathscr{F}$ as well, we use (2) and (17) one last time to get

$$\mu(\mathscr{F}) = (-1)^{n-1} q^{\binom{n-1}{2}} \begin{bmatrix} n-1 \\ n-2 \end{bmatrix} - (-1)^{n-1} q^{\binom{n-1}{2}} |\mathscr{F}(n-1)|.$$

The extremal values of the left side of this expression obviously occur for the cases $\mathscr{F}(n-1) = \emptyset$ or $\mathscr{F}(n-1) = L_n(n-1)$. It is easy to check that the latter gives a larger value than the former, so we are done with the proof. □

6. Comments and questions

Although equation (4) gives a nice formula for $\mu(B_n[k])$, there does not seem to be correspondingly simple one for $\mu(B_n[k,l])$. By iterating (3) together with (4) and dualization, one can obtain

$$\mu(B_n[k,l]) = -1 + \sum_{j=k}^{l} (-1)^j \binom{n}{j}\binom{j-1}{k-1}. \tag{18}$$

However, there seems little hope for a simpler expression. Using (18), one sees that asymptotically (n growing, with k, l fixed)

$$\mu(B_n[k,l]) \sim \frac{(-1)^{l-k}}{l!}\binom{l-1}{k-1} n^l.$$

It should also be possible to give proofs of Propositions 4.1 and 4.2 based on equation (4), but we have been unable to do so. There is an inductive proof utilizing Lemma 4.3, but we have chosen the combinatorial route as more illuminating. There is also a determinantal formula for the size, $\beta_n(R)$, of a descent class [14, p. 69]. However, we do not see how to use it to give alternative demonstrations of our results.

In all cases except that of an arbitrary rank-selection, we were able to also find the maximum positive value of the Möbius function. It is clear from what we have done in Section 3 that whenever μ and $|\mu|$ have different maxima, then the former is achieved using a set of permutations with exactly one double ascent (or, what comes to the same thing, exactly one double descent). Numerical evidence suggests that it is best to use permutations where the double ascent is as centered as possible, but we have no proof that this always works.

It is easy to show that constants c and d for the Θ estimate in Proposition 2.4 can be taken to be

$$c = \frac{1}{\sqrt{2\pi}} - \varepsilon \quad \text{and} \quad d = \frac{e}{\sqrt{2\pi}} + \varepsilon$$

for any small ε. What can be said about the best possible constants for Proposition 4.5?

We could consider other natural families $\mathscr{F} \subseteq B_n$. For example what if $\mathscr{F}$ ranges over all lattices or over all subsets of B_n? There is some evidence to suggest that the maxima in these two cases are the same as in the cases of an interval rank-selection and an arbitrary rank-selection, respectively. It would be good to have methods to show that the extremal configurations have to be very symmetric (rank selections) in these cases.

In this paper, we have studied the maximum Möbius function in B_n as a function of n. If one takes into account $|\mathscr{F}|$, the results obtained are quite different, see [10, 15].

Finally, we could consider other posets. We have already shown how our lower order ideal methods apply to the lattice of subspaces, L_n. For rank-selections, there is a q-analog of our fundamental tool, Proposition 1.1 [14, Theorem 3.12.3]. It is obtained by q-counting the permutations in $\mathscr{B}_n(R)$ using the inversion statistic.

Readdy [12] has recently shown that the maxima in both the arbitrary and interval rank-selection cases is still obtained by taking the whole lattice. She also has related results for the poset of faces of an n-dimensional cube.

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