The Multiplicities of a Dual-thin Q-polynomial Association Scheme

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Abstract

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme, and assume that Y is Q-polynomial with respect to an ordering $E_0, ..., E_D$ of the primitive idempotents. Bannai and Ito conjectured that the associated sequence of multiplicities m_i $(0 \leq i \leq D)$ of Y is unimodal. Talking to Terwilliger, Stanton made the related conjecture that $m_i \leq m_{i+1}$ and $m_i \leq m_{D-i}$ for $i < D/2$. We prove that if Y is dual-thin in the sense of Terwilliger, then the Stanton conjecture is true.

1 Introduction

For a general introduction to association schemes, we refer to [1], [2], [5], or [9]. Our notation follows that found in [3].

Throughout this article, $Y = (X, \{R_i\}_{0 \leq i \leq D})$ will denote a symmetric, D-class association scheme. Our point of departure is the following well-known result of Taylor and Levingston.

1.1 Theorem. *[7] If Y is P-polynomial with respect to an ordering $R_0, ..., R_D$ of the associate classes, then the corresponding sequence of valencies*

$$k_0, k_1, \ldots, k_D$$

is unimodal. Furthermore,

$$k_i \leq k_{i+1} \quad \text{and} \quad k_i \leq k_{D-i} \quad \text{for } i < D/2. \qquad \blacksquare$$

Indeed, the sequence is log-concave, as is easily derived from the inequalities $b_{i-1} \geq b_i$ and $c_i \leq c_{i+1}$ $(0 < i < D)$, which are satisfied by the intersection numbers of any P-polynomial scheme (cf. [5, p. 199]).

In their book on association schemes, Bannai and Ito made the dual conjecture.

1.2 Conjecture. *[1, p. 205] If Y is Q-polynomial with respect to an ordering $E_0, ..., E_D$ of the primitive idempotents, then the corresponding sequence of multiplicites*

$$m_0, m_1, \ldots, m_D$$

is unimodal.

Bannai and Ito further remark that although unimodality of the multiplicities follows easily whenever the dual intersection numbers satisfy the inequalities $b^*_{i-1} \geq b^*_i$ and $c^*_i \leq c^*_{i+1}$ $(0 < i < D)$, unfortunately these inequalities do not always hold. For example, in the Johnson scheme $J(k^2, k)$ we find that $c^*_{k-1} > c^*_k$ whenever $k > 3$.

Talking to Terwilliger, Stanton made the following related conjecture. **1.3 Conjecture.** *[8] If Y is Q-polynomial with respect to an ordering $E_0, ..., E_D$ of the primitive idempotents, then the corresponding multiplicites satisfy*

$$m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for } i < D/2.$$

Our main result shows that under a suitable restriction on Y, these last inequalities are satisfied.

To state our result more precisely, we first review a few definitions. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices with entries in $\mathbb{C}$, where the rows and columns are indexed by X, and let A_0,...,A_D denote the associate matrices for Y. Now fix any $x \in X$, and for each integer i $(0 \leq i \leq D)$, let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{if } xy \notin R_i. \end{cases} \qquad (y \in X). \tag{1}$$

The *Terwilliger algebra* for Y with respect to x is the subalgebra $T = T(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by A_0,...,A_D and E^*_0,...,E^*_D. The Terwilliger algebra was first introduced in [9] as an aid to the study of association schemes. For any $x \in X$, $T = T(x)$ is a finite dimensional, semisimple $\mathbb{C}$-algebra, and is noncommutative in general. We refer to [3] or [9] for more details. T acts faithfully on the vector space $V := \mathbb{C}^X$ by matrix multiplication. V is endowed with the inner product $\langle\ ,\ \rangle$ defined by $\langle u, v\rangle := u^t\overline{v}$ for all $u, v \in V$. Since T is semisimple, V decomposes into a direct sum of irreducible T-modules.

Let W denote an irreducible T-module. Observe that $W = \sum E^*_i W$ (orthogonal direct sum), where the sum is taken over all the indices i $(0 \leq i \leq D)$ such that $E^*_i W \neq 0$. We set

$$d := |\{i \ :\ E^*_i W \neq 0\}| - 1,$$

and note that the dimension of W is at least $d+1$. We refer to d as the *diameter* of W. The module W is said to be *thin* whenever $\dim(E_i^*W) \leq 1$ $(0 \leq i \leq D)$. Note that W is thin if and only if the diameter of W equals $\dim(W)-1$. We say Y is *thin* if every irreducible $T(x)$-module is thin for every $x \in X$.

Similarly, note that $W = \sum E_iW$ (orthogonal direct sum), where the sum is over all i $(0 \leq i \leq D)$ such that $E_iW \neq 0$. We define the *dual diameter* of W to be

$$d^* := |\{i \ : \ E_iW \neq 0\}| - 1,$$

and note that $\dim W \geq d^*+1$. A *dual thin* module W satisfies $\dim(E_iW) \leq 1$ $(0 \leq i \leq D)$. So W is dual thin if and only if $\dim(W) = d^*+1$. Finally, Y is *dual thin* if every irreducible $T(x)$-module is dual thin for every vertex $x \in X$.

Many of the known examples of Q-polynomial schemes are dual thin. (See [10] for a list.) Our main theorem is as follows.

1.4 Theorem. *Let Y denote a symmetric association scheme which is Q-polynomial with respect to an ordering $E_0, ..., E_D$ of the primitive idempotents. If Y is dual-thin, then the multiplicities satisfy*

$$m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for } i < D/2.$$

The proof of Theorem 1.4 is contained in the next section.

We remark that if Y is bipartite P- and Q-polynomial, then it must be dual-thin and $m_i = m_{D-i}$ for $i < D/2$. So Theorem 1.4 implies the following corollary. (cf. [4, Theorem 9.6]).

1.5 Corollary. *Let Y denote a symmetric association scheme which is bipartite P- and Q-polynomial with respect to an ordering $E_0, ..., E_D$ of the primitive idempotents. Then the corresponding sequence of multiplicites*

$$m_0, m_1, \ldots, m_D$$

is unimodal. ∎

1.6 Remark. By recent work of Ito, Tanabe, and Terwilliger [6], the Stanton inequalities (Conjecture 1.3) have been shown to hold for any Q-polynomial scheme which is also P-polynomial. In other words, our Theorem 1.4 remains true if the words "dual-thin" are replaced by "P-polynomial".

2 Proof of the Theorem

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme which is Q-polynomial with respect to the ordering $E_0, ..., E_D$ of the primitive idempotents. Fix any $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra for Y with respect to x. Let W denote any irreducible T-module. We define the *dual endpoint* of W to be the integer t given by

$$t \ := \ \min\{i \ : \ 0 \leq i \leq D, \ E_iW \neq 0\}. \tag{2}$$

We observe that $0 \leq t \leq D - d^*$, where d^* denotes the dual diameter of W.

2.1 Lemma. *[9, p.385] Let Y be a symmetric association scheme which is Q-polynomial with respect to the ordering $E_0, ..., E_D$ of the primitive idempotents. Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, $T = T(x)$. Let W denote an irreducible T-module with dual endpoint t. Then*

(i) $E_i W \neq 0$ *iff* $t \leq i \leq t + d^*$ $(0 \leq i \leq D)$.

(ii) Suppose W is dual-thin. Then W is thin, and $d = d^*$. ■

2.2 Lemma. *[3, Lemma 4.1] Under the assumptions of the previous lemma, the dual endpoint t and diameter d of any irreducible T-module satisfy*

$$2t + d \geq D. \quad ■$$

Proof of Theorem 1.4. Fix any $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra for Y with respect to x. Since T is semisimple, there exists a positive integer s and irreducible T-modules W_1, W_2,...,W_s such that

$$V = W_1 + W_2 + \cdots + W_s \qquad \text{(orthogonal direct sum).} \tag{3}$$

For each integer j, $1 \leq j \leq s$, let t_j (respectively, d_j^*) denote the dual endpoint (respectively, dual diameter) of W_j. Now fix any nonnegative integer $i < D/2$. Then for any j, $1 \leq j \leq s$,

$$\begin{array}{rcll} E_i W_j \neq 0 & \Rightarrow & t_j \leq i & \text{(by Lemma 2.1(i))} \\ & \Rightarrow & t_j < i + 1 \leq D - i \leq D - t_j & \text{(since } i < D/2) \\ & \Rightarrow & t_j < i + 1 \leq D - i \leq t_j + d_j^* & \text{(by Lemmas 2.1(ii), 2.2)} \\ & \Rightarrow & E_{i+1} W_j \neq 0 \text{ and } E_{D-i} W_j \neq 0 & \text{(by Lemma 2.1(i)).} \end{array}$$

So we can now argue that, since Y is dual thin,

$$\begin{array}{rcl} \dim(E_i V) & = & |\{j \ : \ 0 \leq j \leq s, \, E_i W_j \neq 0\}| \\ & \leq & |\{j \ : ; 0 \leq j \leq s, \, E_{i+1} W_j \neq 0\}| \\ & = & \dim(E_{i+1} V). \end{array}$$

In other words, $m_i \leq m_{i+1}$. Similarly,

$$\begin{array}{rcl} \dim(E_i V) & = & |\{j \ : \ 0 \leq j \leq s, \, E_i W_j \neq 0\}| \\ & \leq & |\{j \ : \ 0 \leq j \leq s, \, E_{D-i} W_j \neq 0\}| \\ & = & \dim(E_{D-i} V) \end{array}$$

This yields $m_i \leq m_{D-i}$. ■

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