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Perspective

Descent polynomials

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ABSTRACT

Let n be a nonnegative integer and I be a finite set of positive integers. In 1915, MacMahon proved that the number of permutations in the symmetric group $\mathfrak{S}_n$ with descent set I is a polynomial in n. We call this the descent polynomial. However, basic properties of these polynomials such as a description of their coefficients and roots do not seem to have been studied in the literature. Much more recently, in 2013, Billey, Burdzy, and Sagan showed that the number of elements of $\mathfrak{S}_n$ with peak set I is a polynomial in n times a certain power of two. Since then, there have been a flurry of papers investigating properties of this peak polynomial. The purpose of the present paper is to study the descent polynomial. We will see that it displays some interesting parallels with its peak relative. Conjectures and questions for future research are scattered throughout.

1. Introduction

For the rest of this paper, we let n be a nonnegative integer and I be a finite set of positive integers. (In Section 5 we will permit I to contain 0.) We will also use the notation

$$m = \max(I \cup \{0\}), \tag{1}$$

where the presence of zero ensures that m is well defined even when I is empty. We also use the standard notation $[n] = \{1, 2, \ldots, n\}$. More generally, given integers ℓ, n we set

$$[\ell, n] = \{\ell, \ell + 1, \ldots, n\},$$

and similarly for other interval notations.

Denote by $\mathfrak{S}_n$ the symmetric group of permutations $\pi = \pi_1\pi_2 \ldots \pi_n$ of $[n]$ written in one-line notation. Note that we will sometimes insert commas into such sequences for clarity in distinguishing adjacent elements. The *descent set* of π is

$$\operatorname{Des} \pi = \{i \mid \pi_i > \pi_{i+1}\} \subseteq [n-1].$$

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Note that a similar definition can be given for any sequence π of integers and we will have occasion to use that level of generality. Given I and $n > m$, where m is defined by (1), we wish to study the set

$$D(I; n) = \{\pi \in \mathfrak{S}_n \mid \operatorname{Des} \pi = I\},$$

and its cardinality

$$d(I; n) = \#D(I; n).$$

We call $d(I; n)$ the *descent polynomial* of I. Initially we will be assuming $n > m$ since we will be counting permutations. As an example, if $I = \{1, 2\}$ then

$$D(\{1, 2\}; n) = \{\pi \in \mathfrak{S}_n \mid \pi_1 > \pi_2 > \pi_3 < \pi_4 < \cdots < \pi_n\}. \tag{2}$$

It follows that $\pi_3 = 1$. Furthermore, one can pick any two integers from $[2, n]$ to be to the left of π_3. Placing the integers to the left of π_3 in decreasing order and the remaining ones to the right of π_3 in increasing order completely determines π. Thus

$$d(\{1, 2\}; n) = \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}, \tag{3}$$

which is a polynomial in n. Using the Principle of Inclusion and Exclusion, MacMahon [17, Art. 157] proved that this is always the case.

Theorem 1.1 ([17]). *For any I and all $n > m$ we have that $d(I; n)$ is a polynomial in n.* □

Although this result was proved in 1915, very little work has been done in the intervening years to study these polynomials in more detail. The purpose of this work is to rectify this oversight. We also note that since $d(I; n)$ is a polynomial, we can extend its domain of definition to all complex n, which will be a useful viewpoint in the sequel.

Another well-studied statistic on permutations is the *peak set* defined by

$$\operatorname{Peak} \pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [2, n-1].$$

It is not true that any set of integers $I \subseteq [2, \infty)$ is the peak set of some permutation. For example, clearly I cannot contain two consecutive indices. Say that I is *admissible* if there is some permutation π with $\operatorname{Peak} \pi = I$. For I admissible and $n > m$, consider the set

$$P(I; n) = \{\pi \in \mathfrak{S}_n \mid \operatorname{Peak} \pi = I\}.$$

To illustrate, if $I = \emptyset$, then

$$P(\emptyset; n) = \{\pi \in \mathfrak{S}_n \mid \pi_1 > \cdots > \pi_i < \cdots < \pi_n \text{ for some } 1 \le i \le n\}.$$

Noting that π_i must be 1, such a permutation is determined by picking some subset of $[2, n]$ to be to the left of π_i, then arranging those elements in decreasing order, and finally making the rest an increasing sequence to the right of π_i. It follows that

$$\#P(\emptyset; n) = 2^{n-1},$$

which is certainly not a polynomial in n. However recently, Billey, Burdzy, and Sagan [4] proved the following result.

Theorem 1.2 ([4]). *For any admissible I and all $n > m$ we have that*

$$\#P(I; n) = p(I; n)2^{n-\#I-1},$$

where $p(I; n)$ is a polynomial in n taking on integer values in the range (m, ∞). □

As might be expected, $p(I; n)$ is called the *peak polynomial* of I. Inspired by this theorem, a number of papers have been written about properties of peak and related polynomials [3,5,11–14,16]. It turns out that many of our results about descent polynomials have analogues for peak polynomials.

The rest of this paper is organized as follows. In the next section we derive two recursions for $d(I; n)$ that prove useful in the sequel. Section 3 is devoted to the study of the coefficients of $d(I; n)$ when expanded in an appropriately centered binomial coefficient basis for the polynomial ring $\mathbb{Q}[n]$. In particular, we give a combinatorial interpretation for these constants which permits us to prove a log-concavity result. We also explore a conjecture that the coefficients of $d(I; n)$ when expanded in a differently centered basis alternate in sign. In Section 4, we study the roots of the descent polynomial, including those which are complex. It will be shown that the elements of I are always integral zeros, and a conjecture about the location of the full set of roots in the complex plane will be given. Analogues of $d(I; n)$ in Coxeter groups of types B and D are considered in Section 5. We end with a section containing comments and open questions. There we present a result that unifies Theorems 1.1 and 1.2 using the concept of consecutive pattern avoidance.

2. Two recursions

In this section we derive two recursions for $d(I;n)$. The first will be useful in a number of ways, for example in determining the degree of $d(I;n)$ and in finding some of its roots.

If $I \neq \emptyset$, then we let

$$I^- = I - \{m\}.$$

We first express $d(I;n)$ in terms of $d(I^-;n)$ which will permit latter proofs by induction on m or on $\#I$, the cardinality of the set I.

Proposition 2.1. *If $I \neq \emptyset$, then*

$$d(I;n) = \binom{n}{m} d(I^-;m) - d(I^-;n). \tag{4}$$

Proof. Consider the set P of permutations $\pi \in \mathfrak{S}_n$ that can be written as a concatenation $\pi = \pi'\pi''$ satisfying

1. $\#\pi' = m$ and $\#\pi'' = n-m$, where $\#\pi$ is the number of elements in the sequence π, and
2. $\operatorname{Des}\pi' = I^-$ and π'' is increasing.

We can write P as the disjoint union of those π where $\pi'_m > \pi''_1$ and those where the reverse inequality holds. So $\#P = d(I;n) + d(I^-;n)$.

On the other hand, the elements of P can be constructed as follows. Pick m elements of $[n]$ to be in π' which can be done in $\binom{n}{m}$ ways. Arrange those elements to have descent set I^- which can be done in $d(I^-;m)$ ways. Finally, put the remaining elements in π'' in increasing order which can only be done in one way. If follows that $\#P = \binom{n}{m} d(I^-;m)$, completing the proof. □

One can easily use the previous result to provide a new proof of MacMahon's theorem and to also obtain the degree of $d(I;n)$. So we leave the proof of this corollary to the reader.

Corollary 2.2. *For all I we have that $d(I;n)$ is a polynomial in n with degree m.* □

MacMahon also gave an explicit formula for $d(I;n)$ using the Principle of Inclusion and Exclusion. As a further application of (4), we will now rederive this expression. Before doing so, we set the following notation. Recall that a *composition of n* is a sequence of positive integers summing to n. Given a set of positive integers $I = \{i_1 < \cdots < i_k\}$ and $n > i_k$ it will be convenient to let $i_0 = 0$ and $i_{k+1} = n$. Now we can form the *difference composition*

$$\delta(I) = (i_1 - i_0,\ i_2 - i_1,\ \ldots,\ i_{k+1} - i_k). \tag{5}$$

To any composition $\delta = (\delta_1, \ldots, \delta_k)$ of n we associate the multinomial coefficient

$$\binom{n}{\delta} = \frac{n!}{\delta_1! \ldots \delta_k!}.$$

Finally, we let 2^I be the set of all subsets of I.

Theorem 2.3 ([17]). *If I is a set of k positive integers, then*

$$d(I;n) = \sum_{J \in 2^I} (-1)^{k-\#J} \binom{n}{\delta(J)}. \tag{6}$$

Proof. We proceed by induction on $\#I$. If $I = \emptyset$, then $d(I;n) = 1$. In this case the right-hand side of (6) is $\binom{n}{\delta(\emptyset)} = 1$. We assume that the result holds for all sets I with $\#I \le k$. Consider $\#I = k+1$ and $m = \max(I)$. Note that if δ^- is a composition of m then $\binom{n}{m}\binom{m}{\delta^-} = \binom{n}{\delta}$ where δ is δ^- with $n-m$ appended. Now using this fact, Eq. (4), and the induction hypothesis we have

$$\begin{aligned} d(I;n) &= \binom{n}{m} \sum_{J \in 2^{I^-}} (-1)^{k-\#J} \binom{m}{\delta(J)} - \sum_{J \in 2^{I^-}} (-1)^{k-\#J} \binom{n}{\delta(J)} \\ &= \sum_{J \in 2^I,\ m \in J} (-1)^{k+1-\#J} \binom{n}{\delta(J)} + \sum_{J \in 2^I,\ m \notin J} (-1)^{k+1-\#J} \binom{n}{\delta(J)} \\ &= \sum_{J \in 2^I} (-1)^{k+1-\#J} \binom{n}{\delta(J)}, \end{aligned}$$

as desired. □

It will be useful to have a recursion that does not contain any negative terms. We will see an application of this recursion when we investigate the expansion of $d(I; n)$ in a certain binomial basis. A similar recursion was used by Diaz-Lopez, Harris, Insko and Omar [13] when they proved the peak polynomial positivity conjecture of Billey, Burdzy, and Sagan [4]. To state our recursion, we need some notation.

Suppose $I = \{i_1, \ldots, i_\ell\}$ where the integers are listed in increasing order. We define two related sets of positive integers. Specifically, for $1 \le k \le \ell$ we let

$$I_k = \{i_1, \ldots, i_{k-1}, i_k - 1, \ldots, i_\ell - 1\} - \{0\},$$

and

$$\hat{I}_k = \{i_1, \ldots, i_{k-1}, i_{k+1} - 1, \ldots, i_\ell - 1\}$$

where repeated elements in these lists are considered as a single element. Note that subtracting $\{0\}$ in I_k is only necessary when $k = 1$ and $i_1 = 1$ so that I_k is still a set of positive integers. The reason these sets are interesting is that if one removes $n+1$ from a $\pi \in D(I; n+1)$ then the resulting π' has $\mathrm{Des}\,\pi' = I$, $\mathrm{Des}\,\pi' = I_k$, or $\mathrm{Des}\,\pi' = \hat{I}_k$ for some k. Also note that $n+1$ can only appear at the end of π or at a position i_k where $i_k - 1 \notin I$. So define

$$I' = \{i_k \mid i_k - 1 \notin I\}$$

and $I'' = I' - \{1\}$.

Theorem 2.4. *If $I \neq \emptyset$, then*

$$d(I; n+1) = d(I; n) + \sum_{i_k \in I''} d(I_k; n) + \sum_{i_k \in I'} d(\hat{I}_k; n).$$

Proof. We partition $D(I; n+1)$ according to the position of $n+1$. If $\pi \in D(I; n+1)$ then we let π' be the permutation obtained from π by deleting $n+1$. If $\pi_{n+1} = n+1$ then the corresponding π' are the elements of $D(I; n)$ which gives the first term in the sum for $d(I; n+1)$. Now suppose $\pi_{i_k} = n+1$ where $i_k > 1$ and $\pi_{i_k - 1} > \pi_{i_k + 1}$. Then the possible i_k where this could occur are exactly the elements of I'', and the π' which result form the set $D(I_k; n)$. This explains the first summation. Similarly, suppose $\pi_{i_k} = n+1$ where either $i_k = 1$, or $i_k > 1$ and $\pi_{i_k - 1} < \pi_{i_k + 1}$. Then the corresponding π' are counted by the second sum and we are done. □

3. Coefficients

In this section we show that the coefficients of descent polynomials, written in a certain polynomial basis, are nonnegative by providing a combinatorial interpretation for them. Based on a partial result and computer evidence, we then conjecture that these coefficients form a log-concave sequence. We also make a conjecture that the coefficients in another polynomial basis alternate in sign and prove it in a special case.

The study of coefficients of polynomials has a rich history and many important examples. For instance, Ehrhart polynomials [21] and chromatic polynomials [8] can be written in certain polynomial bases using nonnegative coefficients. In 2013 Billey, Burdzy, and Sagan conjectured that peak polynomials could be written with non-negative coefficients in a binomial basis [4]. This conjecture was proved in 2017 by Diaz-Lopez et al. [13]. We restate their result here and then prove a similar, but stronger, result for descent polynomials in Theorem 3.3.

Theorem 3.1 ([13]). *For any non-empty admissible set I we have*

$$p(I; n) = b_1(I)\binom{n-m}{1} + \cdots + b_{m-1}(I)\binom{n-m}{m-1},$$

where the constant $b_k(I)$ is positive for all $1 \le k \le m-1$. □

Before proving our main result of this section, we need a lemma which is of interest in its own right. For integers ℓ, n we use the notation $[\ell, n]$ for both the set and the sequence $\ell, \ell+1, \ldots, n$. Context should make it clear which interpretation is meant. The next result is easily proved so we omit the demonstration.

Lemma 3.2. *For any finite set of positive integers I and $n > m$ we have $D(I; n) \neq \emptyset$.* □

We can now state the main result of this section for descent polynomials. We remark that Oğuz [19] has recently found a similar result for peak polynomials and gave a related combinatorial interpretation for coefficients when expanding in a binomial basis. Key points in her arguments include our Theorem 3.3, the study of a geometric flip of a permutation, and an expansion of descent polynomials in terms of peak polynomials. This expansion can be obtained by adapting the combinatorial argument in Proposition 3.2 of [1] to Proposition 5.8 of the same paper.

Theorem 3.3. *For any finite set of positive integers I we have that*

$$d(I;n) = a_1(I)\binom{n-m}{1} + \cdots + a_m(I)\binom{n-m}{m}, \tag{7}$$

where, for $k \geq 1$, the constant $a_k(I)$ is the number of $\pi \in D(I;2m)$ such that

$$\{\pi_1, \ldots, \pi_m\} \cap [m+1, 2m] = [m+1, m+k]. \tag{8}$$

Moreover, $a_k(I) > 0$ for $1 \leq k \leq m$.

Proof. By Corollary 2.2, $d(I;n)$ is a polynomial in n of degree m, so we can write it uniquely as a linear combination of the polynomial basis

$$\left\{\binom{n-m}{0}, \binom{n-m}{1}, \ldots, \binom{n-m}{m}\right\}.$$

For ease of notation, given $\pi \in D(I;n)$ we let

$$\pi[m] = \{\pi_1, \ldots, \pi_m\} \cap [m+1, n].$$

Now consider

$$D_k(I;n) = \{\pi \in D(I;n) \mid \#\pi[m] = k\}.$$

Clearly $D(I;n)$ is the disjoint union of the sets $D_k(I;n)$ for $k \geq 0$. So to prove the summation formula in (7), it suffices to demonstrate that $\#D_k(I;n) = a_k(I)\binom{n-m}{k}$. We also claim that $D_0(I;n) = \emptyset$ which forces $a_0(I) = 0$. Indeed, if there is an element $\pi \in D_0(I;n)$ then $\pi[m] = \emptyset$. This implies that $\{\pi_1, \ldots, \pi_m\} = [m]$. Thus $\pi_m \leq m$ and $\pi_{m+1} > m$ which contradicts the fact that m is a descent.

For the rest of the proof we will assume $n \geq 2m$. This assumption is without loss of generality since if we can show that the polynomials on both sides of Eq. (7) agree for an infinite number of values, then they must agree everywhere. For $k \geq 1$, consider the elements $\pi \in D_k(I;n)$. There are $\binom{n-m}{k}$ ways to pick the k elements of $\pi[m]$. Furthermore, given any two k-element subsets X and Y of $[m+1,n]$, there is an order preserving bijection $f : X \to Y$. This induces a one-to-one correspondence between $\pi \in D_k(I;n)$ with $\pi[m] = X$ and $\sigma \in D_k(I;n)$ with $\sigma[m] = Y$ by applying f to the elements of $\pi[m]$, leaving the elements in the first m positions from $[m]$ unchanged, and then listing the remaining elements in increasing order. Note that all the elements of $[m]$ remain unchanged as f is only applied to elements of $[m+1,n]$. This bijection clearly preserves the descent set everywhere except possibly at position m. To see that the descent at m is preserved, note that $\pi_{m+1} \in [m]$ since the subsequence $\pi_{m+1}\cdots\pi_n$ is increasing and there is at least one element of $[m]$ not in $\{\pi_1, \ldots, \pi_m\}$ because of the assumption $k \geq 1$. But then in $\sigma = f(\pi)$ we have $\sigma_{m+1} = \pi_{m+1}$ since elements of $[m]$ are unchanged. So if $\pi_m \in [m]$ then $\sigma_m = \pi_m > \pi_{m+1} = \sigma_{m+1}$ and if $\pi_m > m$ then $\sigma_m > m \geq \sigma_{m+1}$ as desired.

Letting $X = [m+1, m+k]$ we have shown that

$$\#D_k(I;n) = \#X \cdot \binom{n-m}{k}.$$

Furthermore $k = \#X$ is less than or equal to m, which means that the largest interval we need to consider is $[m+1, 2m]$ and this is contained in $[m+1,n]$ by our assumption that $n \geq 2m$. Thus $\#X = a_k(I)$ which is clearly a constant independent of n. This completes the proof of the summation formula (7).

To prove the last statement of the theorem, suppose $1 \leq k \leq m$. It is enough to show that $D_k(I;2m) \neq \emptyset$. By Lemma 3.2 there is $\pi \in D(I^-;m)$. Thus the concatenation $\sigma = \pi'[1,k][m+k+1,2m]$ is in $D_k(I;2m)$ where π' is π with every element increased by k. □

To illustrate this result, let $I = \{1,2\}$. Then $a_1(I)$ is the number of $\pi = \pi_1\pi_2\pi_3\pi_4 \in D(I;4)$ such that $\{\pi_1,\pi_2\} \cap [3,4] = [3]$. Similarly, $a_2(I)$ is the number of $\pi \in D(I;4)$ such that $\{\pi_1,\pi_2\} \cap [3,4] = [3,4]$. Out of the three elements in $D(I;4)$ one can quickly check that only $\pi = 3214$ satisfies the condition for $a_1(I)$, thus $a_1(I) = 1$. Similarly, only $\pi = 4312$ satisfies the condition for $a_2(I)$, so $a_2(I) = 1$. Theorem 3.3 states that

$$d(I;n) = \binom{n-2}{1} + \binom{n-2}{2},$$

which agrees with (3).

Many coefficient sequences of combinatorial polynomials have interesting properties, one of which we will investigate in the context of the previous theorem. A sequence of real numbers $(a_k) = (a_k)_{k\geq 0}$ is *log-concave* if, for every $k \geq 1$, we have $a_{k-1}a_{k+1} \leq a_k^2$. Log-concave sequences appear naturally in combinatorics, algebra, and geometry; we refer the reader to [7,9,22] for important examples and results. We make the following conjecture about the sequence $(a_k(I))$.

Conjecture 3.4. *For any finite set of positive integers I, the sequence $(a_k(I))$ is log-concave.*

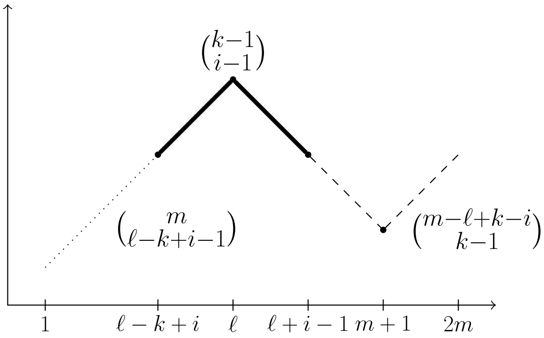

Fig. 1. The diagram of a $\pi \in D(I; 2m)$. The binomial numbers correspond to the possible of ways of choosing each of the three highlighted segments.

We are able to prove this conjecture for certain I, but first we need a lemma. In it, the sequence (a_k) is said to have a certain property, such as nonnegativity, if all the individual a_k do. Also, the sequence has *no internal zeros* if the elements between any two nonzero elements of the sequence are also nonzero.

Lemma 3.5.

(i) If (a_k) and (b_k) are log-concave sequences, then so is (a_kb_k).
(ii) Let (a_k) be a nonnegative log-concave sequence with no internal zeros and let ℓ be a positive integer. Then the sequence $(a_k + a_{k+1} + \cdots + a_{k+\ell})$ is log-concave.

Proof. Statement (i) follows easily from the definition of log-concavity. For statement (ii), note that both sides of the desired inequality contain the terms in the expansion of

$$(a_k + a_{k+1} + \cdots + a_{k+\ell-1})(a_{k+1} + a_{k+2} + \cdots + a_{k+\ell}).$$

Subtracting these from both sides leaves us with proving

$$\begin{aligned} &a_{k-1}a_{k+1} + a_{k-1}a_{k+2} + \cdots + a_{k-1}a_{k+\ell+1} + a_ka_{k+\ell+1} + \cdots + a_{k+\ell-1}a_{k+\ell+1} \\ &\quad \le a_k^2 + a_ka_{k+1} + \cdots + a_ka_{k+\ell} + a_{k+1}a_{k+\ell} + \cdots + a_{k+\ell}^2. \end{aligned} \tag{9}$$

It is well known that for any sequence of positive reals, log concavity is equivalent to the condition that $a_ia_j \le a_{i+1}a_{j-1}$ for all $i < j$. Comparing corresponding terms in (9) finishes the proof. □

The next result shows that the sequence $(a_k(I))$ is log-concave in a special case.

Proposition 3.6. *Let $\ell \le m$ be positive integers and let $I = \{\ell, \ell+1, \ldots, m\}$. Then $(a_k(I))$ is log-concave.*

Proof. We first use the combinatorial description of $a_k(I)$ in Theorem 3.3 to derive an explicit formula for this quantity. Let $\pi \in D(I; 2m)$ satisfy Eq. (8). In Fig. 1 we create a diagram of the permutation π by plotting the points (i, π_i) and connecting them by, possibly dotted or dashed, segments. Note that the form of I implies that $\pi_1 \ldots \pi_m$ has a single local maximum at π_ℓ. Combining this with (8) we see that $\pi_\ell = m + k$ and the elements of $[m + 1, m + k]$ are $\pi_{\ell-k+i}, \pi_{\ell-k+i+1}, \ldots, \pi_{\ell+i-1}$ for some i with $1 \le i \le m - \ell + 1$. Now there are $\binom{k-1}{i-1}$ ways of selecting the elements $\pi_{\ell+1}, \ldots, \pi_{\ell+i-1}$. Once these elements are put in a decreasing sequence just after π_ℓ, the rest of the elements of $[m + 1, m + k]$ must form an increasing sequence just before π_ℓ. Next we choose the elements of the increasing sequence $\pi_1, \ldots, \pi_{\ell-k+i-1}$ from $[m]$ in $\binom{m}{\ell-k+i-1}$ ways. The remaining $m - \ell + k - i + 1$ elements of $[m]$ must be arranged as the elements $\pi_{\ell+i}, \ldots, \pi_{m+k}$ with unique local minimum at π_{m+1}. So the number of ways to choose $\pi_{m+2}, \ldots, \pi_{m+k}$ is $\binom{m-\ell+k-i}{k-1}$. And once these elements are chosen there is only one way to arrange them and the remaining elements since they are all in increasing or decreasing order. So

$$a_k(I) = \sum_{i=1}^{m-\ell+1} \binom{k-1}{i-1}\binom{m}{\ell-k+i-1}\binom{m-\ell+k-i}{k-1}. \tag{10}$$

Now for any fixed c, the binomial coefficient sequences $\left(\binom{k}{c}\right)_{k\ge0}$ and $\left(\binom{c}{k}\right)_{k\ge0}$ are well known to be log-concave. It follows from Lemma 3.5(i) that, for fixed i and varying k, each triple product appearing in (10) defines a log-concave sequence. Now Lemma 3.5 (ii) shows that the sum itself is log concave. □

If we expand $d(I; n)$ in the binomial basis centered at -1 then these coefficients also seem to be well behaved.

Conjecture 3.7. *For any I we have*

$$d(I; n) = \sum_{k=0}^{m} (-1)^{m-k} c_k(I) \binom{n+1}{k},$$

where $c_k(I)$ is a nonnegative integer for all $0 \le k \le m$.

We are able to prove this conjecture for $c_0(I)$. To do so, we need a couple of lemmas. Recall that since $d(I; n)$ is a polynomial in n, it is defined for all complex numbers. The following result follows immediately from Theorem 2.3.

Lemma 3.8. *We have*

$$d(I; 0) = (-1)^{\#I}.$$

for all I. □

Keeping the notation of Conjecture 3.7, we note that

$$d(I; -1) = (-1)^m c_0(I). \tag{11}$$

This is why our next result will be useful.

Proposition 3.9. *For any I and any $n \ge m+2$ we have*

$$d(I; n) \ge |d(I; -1)|.$$

Proof. Note that $d(I; n)$ is an increasing function of n for integral $n > m$ because any permutation $\pi \in D(I; n)$ can be extended to one in $D(I; n+1)$ by merely appending $n+1$. So it suffices to prove the result when $n = m+2$.

We proceed by induction on m. If $m = 0$ then $I = \emptyset$ and $d(I; n) = 1$ and the result follows. For the induction step, we first note that by Lemmas 3.2 and 3.8

$$|d(I; 0)| = 1 \le d(I; m+1).$$

We now apply Theorem 2.4, keeping the notation therein, as well as induction and the previous displayed equation to obtain

$$\begin{aligned} d(I; m+2) &= d(I; m+1) + \sum_{i_k \in I''} d(I_k; m+1) + \sum_{i_k \in I'} d(\hat{I}_k; m+1) \\ &\ge d(I; m+1) + \sum_{i_k \in I''} |d(I_k; -1)| + \sum_{i_k \in I'} |d(\hat{I}_k; -1)| \\ &\ge |d(I; 0)| + \sum_{i_k \in I''} |d(I_k; -1)| + \sum_{i_k \in I'} |d(\hat{I}_k; -1)| \\ &\ge \left| d(I; 0) - \sum_{i_k \in I''} d(I_k; -1) - \sum_{i_k \in I'} d(\hat{I}_k; -1) \right| \\ &= |d(I; -1)|, \end{aligned}$$

as desired. □

Proposition 3.10. *For any I we have $c_0(I) \ge 0$.*

Proof. By Eq. (11), it suffices to show that the sign of $d(I; -1)$ is $(-1)^m$. We will proceed by induction on $\#I$. As usual, the case $I = \emptyset$ is trivial. For $I \neq \emptyset$, applying recursion (4) yields

$$d(I; -1) = \binom{-1}{m} d(I^-; m) - d(I^-; -1) = (-1)^m d(I^-; m) - d(I^-; -1). \tag{12}$$

By Lemma 3.2 we have $d(I^-; m) > 0$. And by induction, the sign of $d(I^-; -1)$ is $(-1)^{m^-}$ where $m^- = \max(I^- \cup \{0\})$. So if m and m^- have opposite parity, then the result follows from (12). If they have the same parity, then $m \ge m^- + 2$. Applying Proposition 3.9 to I^- we get $d(I^-; m) \ge |d(I^-; -1)|$. So, using Eq. (12) again, the sign of $d(I; -1)$ is $(-1)^m$ in this case as well. □

4. Roots

Although we defined $d(I; n)$ only for $n > m$ because we wished to count a nonempty set of permutations, by Theorem 1.1, $d(I; n)$ is a polynomial in n so we can extend the definition to $d(I; z)$ for any complex number z. In this context, it makes sense to talk about the roots of $d(I; z)$ and we study them in this section. We start by showing that elements of I are roots of $d(I; z)$, a result analogous to one for peak polynomials [5].

Proposition 4.1. *If I is a set of positive integers and $i \in I$ then $d(I; i) = 0$.*

Proof. We induct on $\#I$ using the recursion (4). The result is vacuously true when I is empty. If $i \in I^-$ then, by the induction hypothesis, $d(I^-; i) = 0$. Also $\binom{i}{m} = 0$ since $i < m$. Substituting these values into (4) shows that $d(I; i) = 0$. The only other case is $i = m$. But then, using Eq. (4) again, we have that

$$d(I; m) = \binom{m}{m} d(I^-; m) - d(I^-; m) = 0,$$

as desired. □

Now that we have established that the elements of I are themselves roots of $d(I; z)$, the remainder of this section focuses on understanding the remaining roots of this polynomial lying in the complex plane. Throughout we denote by $|z_0|$, $\Re(z_0)$ and $\Im(z_0)$ the norm, real and imaginary parts, respectively, of the complex number z_0.

We begin by commenting on the analogous problem for peak polynomials. Billey, Fahrbach and Talmage [5] extensively studied the roots of peak polynomials. Their observations led to the following conjecture regarding the position of the roots in the complex plane.

Conjecture 4.2 ([5]). *For any admissible I and $z_0 \in \mathbb{C}$ which is a root of $p(I; z)$, we have*

(i) $|z_0| \le m$, and
(ii) $\Re(z_0) \ge -3$.

In fact, in Section 2 of their paper, Billey, Fahrbach and Talmage establish that Theorem 3.1 for peak polynomials was implied by this conjecture. They verified Conjecture 4.2 computationally for all polynomials $p(I; z)$ where $m \le 15$. We have a similar, but more restrictive, conjecture.

Conjecture 4.3. *For any I and $z_0 \in \mathbb{C}$ which is a root of $d(I; z)$ we have*

(i) $|z_0| \le m$, and
(ii) $\Re(z_0) \ge -1$.

Since the current article was written, this conjecture has been proved by Jiradilok and McConnville [15]. Also, a very different proof of (i) was given by Bencs [2]. So here we will content ourselves with some results concerning the case $\#I = 1$. This special case is assumed for the proof of the full conjecture in [15].

Theorem 4.4. *If $I = \{m\}$ and $d(I; z_0) = 0$ then*

(i) $|z_0| \le m$, and
(ii) $\Re(z_0) \ge -1$.

Proof. Consider the equation

$$0 = d(I; z) = \binom{z}{m} - 1.$$

First suppose that $|z| > m$. Then, by the triangle inequality, $|z - k| \ge |z| - k > m - k$ and it follows that

$$\left|\binom{z}{m}\right| = \frac{|z| \cdot |z-1| \cdots |z-m+1|}{m!} > 1.$$

So such z cannot be a root of $d(I; z)$ and the first statement in the theorem is proved.

Now suppose $\Re(z) < -1$. Then $|z - k| \ge |\Re(z - k)| > k + 1$ and the previous displayed equation still holds. This finishes the proof of the second statement. □

For $\#I = 1$ we can also improve over some of the bounds implicit in Conjecture 4.3. We will need some background on bounding the moduli of roots of polynomials. Given a polynomial $f(z) = \sum_{i=0}^{d} c_i z^i$, with $c_d \neq 0$, the maximum modulus of a root of $f(z)$ is bounded above by the *Cauchy bound* of f, denoted $\rho(f)$, which is the unique positive real solution to the equation

$$|c_0| + |c_1| z + \cdots + |c_{d-1}| z^{d-1} = |c_d| z^d, \tag{13}$$

when f is not a monomial, and zero otherwise [20, Theorem 8.1.3]. It is possible to obtain bounds for polynomials expressed in other bases, such as Newton bases, which we now define. Given a sequence of complex numbers $\xi_1, \xi_2, \ldots, \xi_d$ the set of polynomials

$$P_k(z) = \prod_{i=1}^{k}(z - \xi_i),$$

$0 \leq k \leq d$, is called the *Newton basis* with respect to the nodes $\xi_1, \xi_2, \ldots, \xi_d$. Since $\deg(P_k(z)) = k$ we have that $\{P_0(z), P_1(z), \ldots, P_d(z)\}$ forms a basis for the vector space of polynomials of degree at most d.

Theorem 4.5 (*Theorem 8.6.3 in [20]*). *Let $f(z) = \sum_{k=0}^{d} c_k P_k(z)$ be a polynomial of degree d where the P_k's form the Newton basis with respect to the nodes $\xi_1, \ldots, \xi_d$. Then f has all of its zeros in the union of the discs*

$$\mathcal{D}_k := \{z \in \mathbb{C} \mid |z - \xi_k| \leq \rho\}, \tag{14}$$

where $k = 1, \ldots, d$ and ρ is the Cauchy bound of $\sum_{k=0}^{d} c_k z^k$. □

Theorem 4.5 played an important role in the work of Brown and Erey that improved known bounds for the moduli of the roots of chromatic polynomials for dense graphs [10]. Because of recursion (4) we consider the Newton bases with respect to the nodes 0, 1, 2, 3, . . ., which is

$$z\downarrow_k := z(z-1)\cdots(z-k+1),$$

$k \geq 0$. This is known as the *falling factorial basis*. Expanding $d(I; z)$ in terms of this basis and using the previous theorem immediately gives us the following bounds on the roots of $d(I; z)$.

Lemma 4.6. *Suppose $d(I; z) = \sum_{k=0}^{m} c_k z\downarrow_k$. Then the roots of $d(I; z)$ lie in the union of the discs*

$$\mathcal{D}_k = \{z \in \mathbb{C} \mid |z - k| \leq \rho(I)\},$$

where $k = 0, \ldots, m-1$ and $\rho(I)$ is the Cauchy bound of the polynomial $\sum_{k=0}^{m} c_k z^k$. □

We now specialize to the case $\#I = 1$.

Theorem 4.7. *Let $I = \{m\}$ and*

$$\rho_m = \frac{m}{e}\sqrt[m]{me}.$$

Then the roots of $d(I; z)$ lie in the union of the discs

$$\mathcal{D}_k = \{z \in \mathbb{C} \mid |z - k| \leq \rho_m\},$$

where $k = 0, \ldots, m-1$.

Proof. By Lemma 4.6, it suffices to show that $\rho(I) \leq \rho_m$. Since $d(I; z) = \binom{z}{m} - 1$ which has the same roots as $z\downarrow_m - m!$, it suffices to show that ρ_m is an upper bound for the unique positive real solution to the equation $z^m = m!$. This solution is $\sqrt[m]{m!}$, and using lower Riemann sums to estimate the function $\ln m!$ from $\int \ln x\, dx$ establishes that $m! \leq m^{m+1}/e^{m-1}$. The result follows. □

To emphasize how the region in the previous theorem is in some ways an improvement over that given in Conjecture 4.3, we prove the following result.

Corollary 4.8. *If $I = \{m\}$ and $d(I; z_0) = 0$ then*

$$|\Im(z_0)| \leq \rho_m.$$

Furthermore, for all $m \geq 1$, we have

$$\frac{m}{e} < \rho_m \leq m.$$

Proof. The first assertion follows immediately from the description of the discs in Theorem 4.7. To obtain the bounds on ρ_m, consider the function $f(m) = \sqrt[m]{me}$. Taking the derivative gives

$$f'(m) = \sqrt[m]{me} \cdot \frac{-\ln m}{m^2} \leq 0,$$

for $m \geq 1$. So $f(m)$ is decreasing on the interval $[1, \infty)$ and thus is bounded above by $f(1) = e$. Applying l'Hôpital's Rule shows that $\lim_{m\to\infty} f(m) = 1$ and this limit is a lower bound. The desired inequalities follow from observing $\rho_m = mf(m)/e$. □

We should mention that Bencs has proposed another conjecture about the roots of $d(I; n)$ which substantially improves on Conjecture 4.3.

Conjecture 4.9 ([2]). *For any I and $z_0 \in \mathbb{C}$ which is a root of $d(I; z)$ we have*

$$\left| z_0 - \frac{m-1}{2} \right| \le \frac{m+1}{2}.$$

5. Other Coxeter groups

For information about Coxeter groups, see the book of Björner and Brenti [6]. Recall that for any finite Coxeter system (W, S), the *(right) descent set* of $w \in W$ is

$$\operatorname{Des} w = \{s \in S \mid \ell(ws) < \ell(w)\}, \tag{15}$$

where ℓ is the length function. In this section we will consider the Coxeter groups B_n and D_n. We will use symbols near the beginning of the Greek alphabet for elements of B_n and D_n to distinguish them from the permutations in $A_{n-1} = \mathfrak{S}_n$.

We view B_n as the group of signed permutations $\beta = \beta_1 \dots \beta_n$ where $\beta_i \in \{\pm 1, \dots, \pm n\}$ for all $i \in \mathbb{Z}$ and the sequence $|\beta_1| \dots |\beta_n|$ is a permutation in A_{n-1}, and we view D_n as the subgroup of B_n consisting of all $\beta = \beta_1 \dots \beta_n$ where there are an even number of β_i in $\{-1, -2, \dots, -n\}$. Since D_n is a subgroup of B_n, the notation defined below in terms of B_n also applies to D_n. We will use the common convention that $-b$ will be written as $\bar{b}$. For example two elements of B_6 are $\beta = \bar{3}4\bar{1}\bar{5}62$ and $\gamma = \bar{3}\bar{4}\bar{1}\bar{5}62$, and the second element is also an element of the subgroup D_6, whereas the first is not.

The simple reflections in B_n are $S_B = S_A \cup \{s_0\}$ where $s_0 = (1, \bar{1})$ and S_A denotes the set of adjacent transpositions generating the Coxeter group of type A_{n-1}. Identifying reflections and subscripts as we have done in the symmetric group, we see that for $\beta \in B_n$ we have $\operatorname{Des} \beta \subseteq [n-1] \cup \{0\}$. Because of this, it will be convenient to extend permutations in B_n by writing $\beta = \beta_0\beta_1 \dots \beta_n$ where $\beta_0 = 0$. In this notation, our previous examples would be written $\beta = 0\bar{3}4\bar{1}\bar{5}62$ and $\gamma = 0\bar{3}\bar{4}\bar{1}\bar{5}62$. Translating definition (15) using our conventions, we see that if $\beta = \beta_0\beta_1 \dots \beta_n \in B_n$ then

$$\operatorname{Des} \beta = \{i \ge 0 \mid \beta_i > \beta_{i+1}\}, \tag{16}$$

where we are using the usual order on the integers for the inequalities. To continue our examples in B_6, we have $\operatorname{Des} \beta = \{0, 2, 3, 5\}$ and $\operatorname{Des} \gamma = \{0, 1, 3, 5\}$.

Now given a finite set of nonnegative integers I and $n > m$ where m continues to be defined by Eq. (1), we let

$$D_B(I; n) = \{\beta \in B_n \mid \operatorname{Des}(\beta) = I\} \quad \text{and} \quad d_B(I; n) = \#D_B(I; n). \tag{17}$$

We will first derive a recursive formula for $d_B(I; n)$ analogous to the one for $d(I; n)$ in Proposition 2.1.

Theorem 5.1. *Let I be a nonempty, finite set of nonnegative integers. Then we have*

$$d_B(I; n) = \binom{n}{m} 2^{n-m} d_B(I^-; m) - d_B(I^-; n). \tag{18}$$

Proof. Consider the set P of signed permutations $\beta \in B_n$ which can be written as a concatenation $\beta = 0\beta'\beta''$ satisfying

1. $\#\beta' = m$ and $\#\beta'' = n - m$, and
2. $\operatorname{Des} \beta' = I^-$ and β'' is increasing.

We can write P as the disjoint union of those β where $\beta'_m > \beta''_1$ and those where the reverse inequality holds. So $\#P = d_B(I; n) + d_B(I^-; n)$.

On the other hand, the elements of P can be constructed as follows. Pick a subset T of m elements of $[n]$ which can be done in $\binom{n}{m}$ ways. Form a signed permutation from the elements of T whose descent set is I^- which can be done in $d_B(I^-; m)$ ways. Next choose the sign of the $n-m$ elements in $[n] - T$ which can be done in 2^{n-m} ways. Then arrange them in increasing order to form β'' which can be done in only one way. It follows that $\#P = \binom{n}{m} 2^{n-m} d_B(I^-; m)$. Comparing this with the expression for $\#P$ at the end of the previous paragraph completes the proof. □

Next we prove the type B analogue of Theorem 2.3. To state it, we let

$$I^+ = I - \{0\}.$$

Also, if J is a set of positive integers then we will let $\delta_1(J)$ denote the first component of the composition $\delta(J)$. Note that

$$\delta_1(J) = \begin{cases} \min J & \text{if } J \neq \emptyset, \\ n & \text{if } J = \emptyset. \end{cases}$$

Theorem 5.2. *If I is a set of nonnegative integers with $\#I^+ = k$, then*

$$d_B(I;n) = \begin{cases} \sum_{J\in 2^{I^+}} (-1)^{k-\#J} \binom{n}{\delta(J)} \cdot 2^{n-\delta_1(J)} & \text{if } 0 \notin I, \\ \sum_{J\in 2^{I^+}} (-1)^{k-\#J} \binom{n}{\delta(J)} \cdot (2^n - 2^{n-\delta_1(J)}) & \text{if } 0 \in I. \end{cases} \tag{19}$$

Proof. We first consider the case where $0 \notin I$ so that $I = I^+$, and proceed by induction on $\#I$. If $I = \emptyset$, then $d_B(I;n) = 1$. In this case, the right-hand side of Eq. (19) also gives $\binom{n}{\delta(\emptyset)} = 1$. We assume that the result holds for all sets I not containing 0 with $\#I \le k$. Consider $\#I = k+1$ and $m = \max(I)$. Using recursion (18), and the induction hypothesis we have

$$\begin{aligned} d_B(I;n) &= \binom{n}{m} 2^{n-m} \sum_{J\in 2^{I^-}} (-1)^{k-\#J} \binom{m}{\delta(J)} 2^{m-\delta_1(J)} - \sum_{J\in 2^{I^-}} (-1)^{k-\#J} \binom{n}{\delta(J)} 2^{n-\delta_1(J)} \\ &= \sum_{J\in 2^I,\, m\in J} (-1)^{k-\#J+1} \binom{n}{\delta(J)} 2^{n-\delta_1(J)} + \sum_{J\in 2^I,\, m\notin J} (-1)^{k-\#J+1} \binom{n}{\delta(J)} 2^{n-\delta_1(J)} \\ &= \sum_{J\in 2^I} (-1)^{k-\#J+1} \binom{n}{\delta(J)} 2^{n-\delta_1(J)}. \end{aligned}$$

Since $I = I^+$ when $0 \notin I$, this completes the proof for this case.

Next we consider when $0 \in I$. If $I = \{0\}$ then Theorem 5.1 shows $d_B(I;n) = 2^n - 1$, and the right hand of Eq. (19) gives $\binom{n}{\delta(\emptyset)}(2^n - 2^{n-n})$. So Eq. (19) holds in this case. The induction argument is exactly the same as that of the case when $0 \notin I$, but one replaces $2^{m-\delta_1(J)}$ with $2^m - 2^{m-\delta_1(J)}$ and $2^{n-\delta_1(J)}$ with $2^n - 2^{n-\delta_1(J)}$. $\square$

Using Theorems 2.3 and 5.2, we can also give a simple numerical relationship between the descent formulas in types A and B.

Corollary 5.3. *Let I be a finite set of positive integers and $I_0 = I \cup \{0\}$. Then*

$$d_B(I;n) + d_B(I_0;n) = 2^n d(I;n). \quad \square$$

Since the right-hand side of Eq. (19) is well defined for all real numbers n, we use it to extend the definition $d_B(I;n)$ to $\mathbb{R}$ and talk about its roots. The proof of the following theorem is similar to that of Proposition 4.1 and so is omitted.

Proposition 5.4. *If I is a set of nonnegative integers and $i \in I$ then $d_B(I;i) = 0$.* $\square$

The remaining results of this section pertain to the Coxeter group D_n. We continue to use all the conventions for B_n with this subgroup. In particular, we will use the same definition of Des β as in Eq. (16), and the notation $D_D(I;n)$ and $d_D(I;n)$ is defined exactly as in Eq. (17) except that β runs over D_n rather than B_n. Our results in type D_n are very similar to those in type B_n except with some changes imposed by using a different power of two and the intermingling of d_D and d_B in the same formula.

Theorem 5.5. *Let I be a nonempty, finite set of nonnegative integers. Then*

$$d_D(I;n) = \binom{n}{m} 2^{n-m-1} d_B(I^-;m) - d_D(I^-;n). \tag{20}$$

Proof. Consider the set P of signed permutations $\beta \in D_n$ satisfying the same two conditions as in the proof of Theorem 5.1. As before, $\#P = d_D(I;n) + d_D(I^-;n)$.

An alternative construction of the elements of P is as follows. Pick m elements from $[n]$ which can be done in $\binom{n}{m}$ ways. Use those elements to create a type B signed permutation β' with descent set I^- which can be done in $d_B(I^-;m)$ ways. Since a type D_n permutation must have an even number of negative signs, of the remaining $n-m$ elements choose the sign of the first $n-m-1$ of them; the sign of the last element in the set of numbers appearing in β'' is then determined by the number of negative signs assigned previously. Thus choosing the signs of the elements appearing in β'' can be done in 2^{n-m-1} ways. Now form the unique increasing arrangement of these signed integers to form β''. It follows that $\#P = \binom{n}{m}2^{n-m-1}d_B(I^-;m)$ and we are done as in the proof of Theorem 5.1. $\square$

Next we can use Theorem 5.5 to prove a Type D_n analogue of Theorems 2.3 and 5.2. As the proofs are similar to those we have seen before, we omit them.

Theorem 5.6. *If I is a set of nonnegative integers with $\#I^+ = k$, then*

$$d_D(I;n) = \begin{cases} (-1)^k + \sum_{J \in 2^{I^+}} (-1)^{k-\#J} \binom{n}{\delta(J)} \cdot 2^{n-\delta_1(J)-1} & \text{if } 0 \notin I, \\ (-1)^k(2^{n-1}-1) + \sum_{J \in 2^{I^+}} (-1)^{k-\#J} \binom{n}{\delta(J)} \cdot (2^{n-1} - 2^{n-\delta_1(J)-1}) & \text{if } 0 \in I. \end{cases}$$

for all $n > m$ □

Finally we present the analogues of Corollary 5.3, and Proposition 5.4 for type D_n.

Corollary 5.7. *Let I be a nonempty set of positive integers and $I_0 = I \cup \{0\}$. Then*

1. $d_D(I; n) + d_D(I_0; n) = 2^{n-1} d(I; n)$*, and*
2. $d_D(I; i) = d_D(I_0; i) = 0$ *whenever $i \in I^-$.* □

It would be interesting to discover properties of the roots of $d_B(I; n)$ and $d_D(I; n)$ other than the fact that we have zeros at the elements of I. However, this may be difficult because these quantities are no longer polynomials and so there are fewer techniques available to analyze their roots.

6. Comments and open questions

We end with some comments about our results. These include avenues for future research and more conjectures.

(1) Consecutive pattern avoidance. One way to unify Theorems 1.1 and 1.2 is through the theory of consecutive pattern avoidance. Call two sequences of integers $a_1a_2 \dots a_k$ and $b_1b_2 \dots b_k$ *order isomorphic* provided $a_i < a_j$ if and only if $b_i < b_j$ for all pairs of indices $1 \le i, j \le k$. Given $\sigma \in \mathfrak{S}_k$ called the *pattern*, we say that $\pi \in \mathfrak{S}_n$ *contains a consecutive copy of σ at index i* if the factor $\pi_i\pi_{i+1} \dots \pi_{i+k-1}$ is order isomorphic to σ. If π contains no consecutive copies of σ then we say that π *consecutively avoids* σ. Note that a consecutive copy of 21 is just a descent while a peak is a consecutive copy of 132 or 231.

Given any finite set of patterns Π and a finite set of positive integers I define

$$\Pi(I; n) = \{\pi \in \mathfrak{S}_n \mid \pi \text{ has a consecutive copy of some } \sigma \in \Pi \text{ precisely at the indices in } I\}.$$

Also define the function

$$\mathrm{av}_\Pi(n) = \#\Pi(\emptyset; n),$$

the number of permutations in $\mathfrak{S}_n$ consecutively avoiding all permutations in Π. Given $\Pi \subseteq \mathfrak{S}_k$ say that Π is *nonoverlapping* if for any (not necessarily distinct) $\sigma, \tau \in \Pi$ and any l with $1 < l < k$ the prefix of σ of length l is not order isomorphic to the suffix of τ of length l. We will now prove our analogue of Theorems 1.1 and 1.2 in this setting.

Theorem 6.1. *Let $\Pi \subseteq \mathfrak{S}_k$ be a nonoverlapping set of patterns and let I be a finite set of positive integers. Then for all $n \ge m+k-1$ we have $\#\Pi(I; n) \in V_\Pi$ where V_Π is the vector space of all $\mathbb{Q}$-linear combinations of functions in the set*

$$\{n^k \, \mathrm{av}_\Pi(n+l) \mid k \in \mathbb{Z}_{\ge 0},\ l \in \mathbb{Z}\}.$$

Proof. We induct on m. We have $\#\Pi(\emptyset; n) = \mathrm{av}_\Pi(n)$ and so the result clearly holds when $m = 0$. For $m \ge 1$, consider the set P of permutations $\pi \in \mathfrak{S}_n$ which can be written as a concatenation $\pi = \pi'\pi''$ such that $\pi' \in \Pi(I^-; m)$ and $\pi'' \in \Pi(\emptyset; n-m)$. Since Π is nonoverlapping, copies of consecutive patterns from Π in π occur at the positions in I^- and possibly also at exactly one of the indices $m, m-1, \dots, m-k+2$. It follows that

$$\#P = \#\Pi(I^-; n) + \#\Pi(I; n) + \sum_{i=1}^{k-2} \#\Pi(I^- \cup \{m-i\}; n).$$

We can also construct the elements of P as follows. Pick the m elements of $[n]$ to be in π' which can be done in $\binom{n}{m}$ ways. Arrange those elements to have consecutive copies of elements of Π at the indices of I^- which can be done in $\#\Pi(I^-; m)$ ways. Finally, put the remaining elements in π'' so that it avoids consecutive copies of elements of Π which can be done in $\mathrm{av}_\Pi(n-m)$ ways. Equating the two counts for P and rearranging terms we get

$$\#\Pi(I; n) = \binom{n}{m} \mathrm{av}_\Pi(n-m)\#\Pi(I^-; m) - \#\Pi(I^-; n) - \sum_{i=1}^{k-2} \#\Pi(I^- \cup \{m-i\}; n),$$

from which the theorem follows by induction. □

Note that if $\Pi = \{21\}$ then $\mathrm{av}_{\Pi}(n) = 1$ for all n. So $V_{\Pi} = \mathbb{Q}[n]$ and thus Theorem 1.1 is a special case of the previous result. On the other hand, if $\Pi = \{132, 231\}$ then $\mathrm{av}_{\Pi}(n) = 2^{n-1}$ which explains the appearance of the power of 2 in Theorem 1.2. Theorem 6.1 suggests that there might be other sets of patterns which would yield interesting enumerative results, and that such sets could be found by looking at Π such that the numbers $\mathrm{av}_{\Pi}(n)$ have nice combinatorial properties.

(2) The sequence $(a_k(I))$. On reading a version of this paper on the arXiv, Ferenc Bencs [2] has found a proof of Conjecture 3.4. But there is a stronger condition which could also be investigated. Consider a finite, real sequence $(a_k)_{0\le k\le n}$ and the corresponding generating function $f(x) = \sum_{k\ge 0} a_k x^k$. It is well known that if the a_k are positive and $f(x)$ has only real roots then the original sequence is log-concave. However, if one takes $I = \{1, 3\}$ then the corresponding generating function is $f(x) = 2x^3 + 6x^2 + 5x$ which has complex roots. So this stronger condition does not always apply to the $(a_k(I))$ sequence.

(3) Remarks on Conjecture 3.7. Bencs [2] has proved this conjecture as well. His argument is inductive, using the recursions we derived in Section 2 as well as Proposition 3.10 as the base case. It would be very interesting to prove nonnegativity by finding a combinatorial interpretation of the $c_k(I)$.

(4) Limiting behavior of roots. Bencs [2] has proved a result about the limiting behavior of the roots of $d(I; n)$ for certain sets I. Given I, consider the set $I^k = I \cup \{m+1, m+2, \ldots, m+k\}$. Using Neumaier's Gershgorin-type results on location of polynomial roots [18], Bencs has demonstrated the following.

Theorem 6.2. *Suppose I is a finite set of positive integers with $m - 1 \notin I$. Then as $k \to \infty$ the roots of $d(I^k; n)$ converge to $[0, m+k] - \{m-1\}$.* □

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