# Bijective Proofs of Two Broken Circuit Theorems

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## ABSTRACT

We prove, by means of explicit bijections, theorems of Whitney and Stanley that express the coefficients of the chromatic polynomial of a graph $G$ and the number of acyclic orientations of $G$ in terms of numbers of sets of edges that contain no broken circuits of $G$.

Let $G$ be a graph, and let a total ordering of its edge set $E(G)$ be fixed. Words like "first" or "later," when applied to edges, will always refer to this ordering. We adopt, unless the contrary is explicitly stated, the graph-theoretic notation and terminology of [1]; in particular, $G$ is undirected and has no loops or multiple edges, $p$ is the cardinality of the vertex set $V(G)$, and $f(G, \lambda)$ is the chromatic polynomial. Following Whitney [3], we define a *broken circuit* to be a set of edges obtained by removing from some circuit in $G$ its last edge. We shall give bijective proofs of the following two results.

**Whitney's Theorem** [3]. Let $\mathcal{A}_i(G)$ be the collection of all sets $A$ that consist of exactly $i$ edges of $G$ and contain no broken circuit; then

$$f(G, \lambda) = \sum_{i=0}^{p} (-1)^i |\mathcal{A}_i(G)| \lambda^{p-i}.$$

**Stanley's Theorem** [2]. The number of acyclic orientations of $G$ equals the number of sets $A \subseteq E(G)$ that contain no broken circuit.

(Actually, Stanley expressed the number of acyclic orientations as $(-1)^p f(G, -1)$. This result is equivalent to what we have called Stanley's theorem, by virtue of Whitney's theorem.)

Some comments about the notion of a "bijective proof" seem to be in order here. In the case of Stanley's theorem, it is fairly clear that what is required is

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an explicit bijection between the family of all acyclic orientations and the family of sets $A$ containing no broken circuit. The case of Whitney's theorem is less clear, for two reasons. The minor reason is the presence of negative terms in the equation to be proved. This difficulty is easily solved by transposing the troublesome terms; we shall give a bijective proof of

$$f(G,\lambda) + \sum_{i \text{ odd}} |\mathcal{A}_i(G)| \cdot \lambda^{p-i} = \sum_{i \text{ even}} |\mathcal{A}_i(G)| \cdot \lambda^{p-i}.$$

The second, more serious, difficulty is that not all constituents of this equation have natural interpretations as cardinalities. Of course, once we fix a set of $\lambda$ colors, $f(G,\lambda)$ is the cardinality of the set $\mathcal{P}$ of all proper colorings of $G$, and $|\mathcal{A}_i(G)|$ is explicitly given as a cardinality, but what about $\lambda^{p-i}$? It is the number of functions, into our set of $\lambda$ colors, from a set of size $p - i$, but there is no natural set of size $p - i$ in sight. We shall circumvent this difficulty by not interpreting $|\mathcal{A}_i(G)|$ and $\lambda^{p-i}$ individually as cardinalities but rather interpreting the products $|\mathcal{A}_i(G)| \cdot \lambda^{p-i}$ that occur in the equation to be proved.

Each $A \in \mathcal{A}_i(G)$ can be viewed as a graph, with the same vertices as $G$ but only the edges in $A$. Since $A$ contains no broken circuit, and therefore certainly no circuit, and since it has $p$ vertices and $i$ edges, it has $p - i$ components. Thus, for each fixed $A \in \mathcal{A}_i(G)$, we can interpret $\lambda^{p-i}$ as the number of functions from the components of $A$ to our set of $\lambda$ colors. Equivalently, $\lambda^{p-i}$ is the number of ways of coloring the vertices of $G$ so that each edge of $A$ joins two vertices of the same color; let us call such colorings *A-improper* (because a proper coloring is then one that is not $A$-improper for any nonempty $A$). The preceding discussion can be summarized by the statement that $|\mathcal{A}_i(G)| \cdot \lambda^{p-i}$ is the cardinality of

$$\mathcal{C}_i = \{(A, C) \mid A \in \mathcal{A}_i(G), \quad C \text{ is an } A\text{-improper coloring}\}.$$

Recalling that $\mathcal{P}$ is the set of proper colorings of $G$, we can finally state Whitney's theorem in a form amenable to bijective proof: The sets

$$\mathcal{P} \cup \bigcup_{i \text{ odd}} \mathcal{C}_i \quad \text{and} \quad \bigcup_{i \text{ even}} \mathcal{C}_i$$

have the same cardinality. (Note that all these unions are disjoint ones.)

*Proof of Whitney's Theorem.* We construct an explicit bijection between the sets indicated above. We begin by sending each proper coloring $C \in \mathcal{P}$ to $(\emptyset, C) \in \mathcal{C}_0$. This is clearly a bijection between the elements of $\mathcal{P}$ and those elements of $\mathcal{C}_0$ whose second component is a proper coloring. Note that there are no such elements in $\mathcal{C}_i$ for $i > 0$, by definition of $\mathcal{C}_i$, so from now on we may confine our attention to pairs $(A, C)$ with $C$ improper. For each fixed improper coloring, we shall give an explicit bijection between $\{A \mid (A, C) \in \bigcup_{i \text{ odd}} \mathcal{C}_i\}$ and $\{A \mid (A, C) \in \bigcup_{i \text{ even}} \mathcal{C}_i\}$; this will clearly suffice to complete the proof.

Since $C$ is improper, there are edges joining two vertices of the same color. Let $e$ be the last such edge, and define a map $\phi$ from sets of edges to sets of edges by

$$\phi(A) = \begin{cases} A - \{e\} & \text{if } e \in A \\ A \cup \{e\} & \text{if } e \notin A . \end{cases}$$

Clearly, $\phi$ is a bijection (it is its own inverse) and $|\phi(A)|$ is even if and only if $|A|$ is odd. Also, the fact that $e$ joins two vertices of the same color implies that $C$ is $\phi(A)$-improper if and only if $C$ is $A$-improper. All that remains is to check that, if $A$ contains no broken circuit, then the same is true of $\phi(A)$.

Suppose, for a contradiction, that $\phi(A)$ contains a broken circuit $B$ but $A$ does not. Then of course $\phi(A) = A \cup \{e\}$ and $B$ contains $e$. Let $B \cup \{l\}$ be the circuit from which $B$ is obtained by deletion of the last edge $l$. So $l$ is later than $e$, and, by our choice of $e$, the endpoints of $l$ have different colors in the coloring $C$. But this is impossible because these endpoints are joined in $A \cup \{e\}$ by the path $B$ and $C$ is $A \cup \{e\}$-improper. This contradiction completes the verification that $\phi$ is, for each fixed $C$, a bijection; it thus also completes the proof of Whitney's theorem. ∎

*Remark.* Whitney's original proof of his theorem [3] involves an operation similar to the $\phi$ in our proof in that it adds or deletes a single edge. Unlike us, Whitney applies this operation to sets $A$ that contain broken circuits, the edge to be added or deleted being the one that completes, to a circuit, the first (in a fixed ordering) broken circuit in $A$. He uses this pairing to show that the contributions of such $A$'s cancel in a formula for the chromatic polynomial that he obtains by an inclusion-exclusion argument; thus only the sets that do not contain broken circuits contribute to the chromatic polynomial. Since inclusion-exclusion arguments can be presented in bijective form, it is conceivable that a combination of such a presentation with Whitney's bijection could be made to yield our bijective proof. This seems unlikely, however, because of the dependence of Whitney's bijection on a somewhat arbitrary ordering of the broken circuits.*

*Proof of Stanley's Theorem.* We shall present an algorithm which, given an acyclic orientation of $G$, examines each edge in turn, in the fixed order, and either "un-orients" it or deletes it. After each edge has been considered, what remains is a certain set of (unoriented) edges. We shall show that this set contains no broken circuit, and we shall show that every set that contains no broken circuit arises from a unique acyclic orientation of $G$.

Since we shall have to deal, during the algorithm, with graphs in which some but not all edges are oriented, it will be convenient to adopt the convention that an unoriented edge is to be viewed as a pair of oppositely directed arcs. Thus, to "un-orient" an arc in a digraph is simply to adjoin the reverse arc.

It will also be convenient to adopt the unorthodox convention that a digraph is acyclic if it has no cycles of length $\geq 3$. Thus, a symmetric pair of arcs, $(u, v)$ and $(v, u)$, is permissible in an acyclic digraph; in particular, an unoriented forest,

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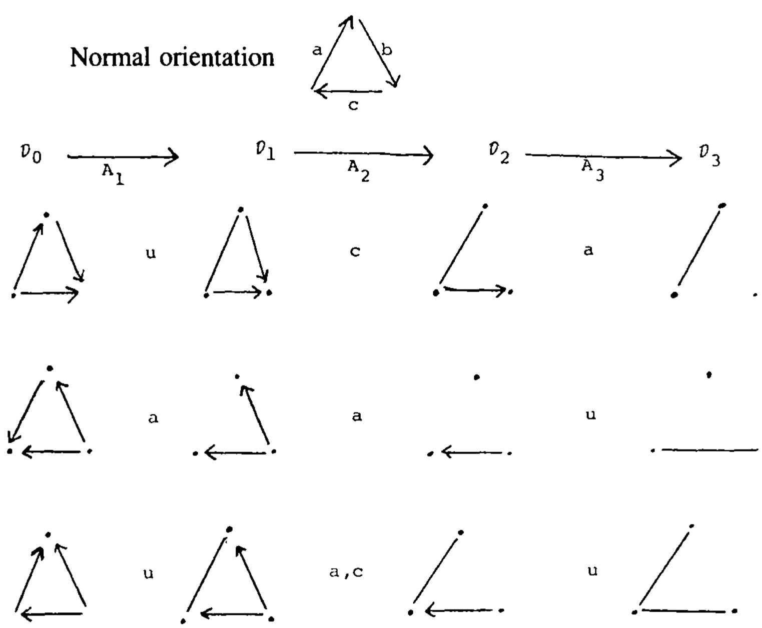
**Added in proof:* D. Stanton and D. White have shown that our proof of Whitney's Theorem can be obtained from Whitney's proof by applying the general involution principle of A. Garsia and S. Milne (*Proc. Nat. Acad. Sci. U.S.A.* 78 (1981) 2026-2028).

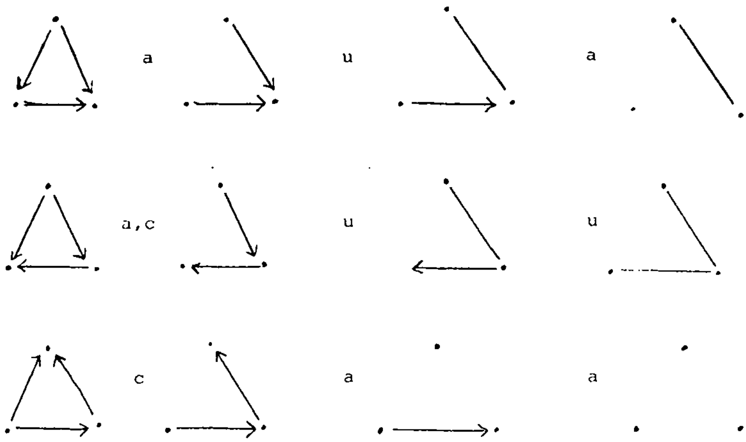
when viewed as a digraph, is acyclic. It is important to note that a digraph without symmetric pairs, i.e., an orientation of a graph, is acyclic in our sense if and only if it is acyclic in the usual sense, so we have not tampered with the meaning of "acyclic orientation" in the statement of the theorem.

Let us fix, once and for all, some orientation (not necessarily acyclic) of $G$. The orientation thus assigned to an edge will be called its normal orientation, and the other orientation will be called abnormal.

With these conventions, our algorithm can be simply described. As indicated at the beginning of the proof, the algorithm accepts as input an arbitrary acyclic orientation of $G$. It treats each arc in turn, in the fixed order used to define broken circuits. At the stage where an arc $(u, v)$ of the given acyclic digraph is being examined, un-orient it by adding the arc $(v, u)$ if and only if both of the following conditions are met: $(u, v)$ is normally oriented, and the addition of $(v, u)$ yields an acyclic digraph. Otherwise, delete $(u, v)$.

The figure below shows how the algorithm operates when $G$ is a triangle, with the edges ordered so that $a < b < c$ and with the normal orientation shown. The labels $\mathcal{D}_i$ and $A_i$ in the figure will be explained shortly. Each row of the figure shows the three steps in the algorithm, applied to the acyclic orientation on the left, producing the set of edges on the right, a set that does not include the (unique) broken circuit $\{a, b\}$. The letters $a, c, u$ in the figure indicate steps of the algorithm where an edge was (respectively) deleted because of abnormal orientation, deleted because unorienting it would produce a cycle, or unoriented.






To show that this algorithm has the desired properties, we shall first introduce a sequence of sets $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_q$ such that $\mathcal{D}_0$ is the set of all acyclic orientations of $G$ and $\mathcal{D}_q$, where $q$ is the number of edges of $G$, is the set of subsets of $E(G)$ that contain no broken circuit. Then we shall show that the $k$th step in the algorithm produces a bijection $A_k$ from $\mathcal{D}_{k-1}$ to $\mathcal{D}_k$; in fact, we will explicitly describe, in the proof of Lemma 3, the inverse of each step. This will clearly suffice to complete the proof.

The set $\mathcal{D}_k$, which should be viewed as the set of possible results of the first $k$ steps of the algorithm, is defined to be the set of all sub-digraphs $D$ of $G$ having the same vertices as $G$ and satisfying:

(a) Each of the first $k$ edges of $G$ is either present in $D$ (as a symmetric pair of arcs) or absent from $D$, but each of the remaining $q - k$ edges is oriented in $D$ (i.e., $D$ contains exactly one arc from the symmetric pair).

(b) $D$ is acyclic.

(c) The unoriented part of $D$ (as in the first part of (a)) contains no broken circuit of $G$.

In the figure above, each column exhibits the members of the $\mathcal{D}_i$ named at the top of the column. The following three lemmas show that $A_k$, the function defined by the $k$th step of the algorithm, is a bijection from $\mathcal{D}_{k-1}$ to $\mathcal{D}_k$. They thus establish Stanley's theorem.

**Lemma 1.** $A_k$ maps $\mathcal{D}_{k-1}$ into $\mathcal{D}_k$.

*Proof.* If $D \in \mathcal{D}_{k-1}$, then $A_k$ either deletes or unorients the $k$th edge, so condition (a) for $A_k(D)$ follows immediately from the corresponding condition (with $k - 1$ in place of $k$) for $D$. Condition (b) for $A_k(D)$ follows from the fact that the algorithm never adds an arc without first checking that acyclicity is preserved. To see that $A_k(D)$ also satisfies (c), consider any circuit $C$ in $G$, let $l$ be its last edge, and suppose the broken circuit $C - \{l\}$ is in the unoriented part of $A_k(D)$. Since it is not in the unoriented part of $D$, and since the only difference between $D$ and $A_k(D)$ is at the $k$th edge $e$, we conclude that $e \in C - \{l\}$. Being the last edge in $C$, $l$ must be later than $e$ and must therefore be present, in exactly one of its orientations, in $A_k(D)$. But then $A_k(D)$ contains a cycle of length $\geq 3$, namely, one of the two cyclic orientations of $C \subseteq G$. This contradicts the fact, observed above, that $A_k(D)$ is acyclic. So $A_k(D)$ satisfies condition (c) and therefore belongs to $\mathcal{D}_k$. ∎

**Lemma 2.** $A_k$ is one-to-one.

*Proof.* Suppose $D_1$ and $D_2$ are two distinct elements of $\mathcal{D}_{k-1}$ that are sent by $A_k$ to the same element $D$ of $\mathcal{D}_k$. Since $A_k$ affects only the $k$th edge $e$, $D_1$ and $D_2$ must differ only in the orientation they give $e$. So we may assume that $e$ is oriented normally in $D_1$ and abnormally in $D_2$. Acting on $D_2$, the algorithm deletes the abnormally oriented $e$, so $e$ is absent from $D$, which means that the algorithm also deletes the normally oriented $e$ in $D_1$. By definition of the algorithm, this can happen only if unorienting $e$ in $D_1$ would introduce a cycle $C$ of length $\geq 3$. $C$ must contain the abnormal orientation of $e$, for otherwise it would be a cycle in $D_1$, contrary to $D_1 \in \mathcal{D}_{k-1}$. But then $C$, having length $\geq 3$, cannot also contain the normal orientation of $e$, so it is a cycle in $D_2$, contrary to $D_2 \in \mathcal{D}_{k-1}$. ∎

**Lemma 3.** $A_k$ maps $\mathcal{D}_{k-1}$ onto $\mathcal{D}_k$.

*Proof.* Let an arbitrary $D' \in \mathcal{D}_k$ be given. We shall find a $D \in \mathcal{D}_{k-1}$ such that $A_k(D) = D'$. Let $e$ be the $k$th edge of $G$, the one that is deleted or unoriented by $A_k$.

In order for condition (a) to be satisfied by $D$ and for the algorithm to transform $D$ into $D'$, it is necessary that $D$ be the same as $D'$ except at the edge $e$. It is also necessary that exactly one of the orientations of $e$ be present in $D$, so our job is to choose the appropriate orientation. Whichever orientation we choose, conditions (a) and (c) will be satisfied by $D$. Our choice need only ensure that (b) holds and that the algorithm transforms $D$ into $D'$.

If $e$ is present in $D'$, then we have no choice but to orient it normally in $D$, for our algorithm always deletes abnormally oriented edges. The $D$ so obtained satisfies (b) because it is formed by deleting an arc from $D'$ (the abnormal orientation of $e$). When the $k$th step of our algorithm is applied to $D$ it un-orients $e$ and thus produces $D'$, because $D'$ is acyclic.

There remains the case that $e$ is absent from $D'$. If the digraph obtained by adjoining $e$ with its abnormal orientation is acyclic, then let this digraph be $D$. It is clearly in $\mathcal{D}_{k-1}$, and the $k$th step of our algorithm converts it to $D'$ because the abnormally oriented $e$ is deleted.

Finally, suppose $e$ is absent from $D'$ and adding it with abnormal orientation destroys acyclicity. Then we have no choice but to add it with normal orientation to produce $D$. Because the abnormal orientation destroyed acyclicity, the algorithm will delete this normally oriented $e$, producing $D'$ as desired.

To check condition (b), suppose toward a contradiction that $D$ is not acyclic. Thus, adding $e$ to $D'$, with either orientation, has produced a cycle. If $u$ and $v$ are the endpoints of $e$, then $D'$ must contain directed paths from $u$ to $v$ and from $v$ to $u$. These paths together constitute a closed (directed) walk $W$ in $D'$. We consider two cases, obtaining a contradiction in each. First, suppose $W$ contains an arc that is later than $e$ and is therefore in the oriented part of $D'$, i.e., its opposite is not in $D'$. Then the smallest closed sub-walk of $W$ containing this arc is a cycle (by minimality) of length $\geq 3$ (as the opposite arc is absent), which is impossible as $D' \in \mathcal{D}_k$. There remains only the possibility that $W$ consists entirely of arcs earlier than $e$, hence in the unoriented part of $D'$. But $W$ contains a path from $u$ to $v$, and that path is a broken circuit of $G$ because all the edges in it are earlier than $e$. This again contradicts $D' \in \mathcal{D}_k$, so the proof is complete. ∎

*Remarks.* The problem of finding bijective proofs of these theorems was proposed by Herbert Wilf.

The chromatic polynomial of a graph is expressible in terms of the chromatic polynomials of the graphs obtained by deleting and contracting an edge (see [1, p. 145])

$$f(G, \lambda) = f(G - \{e\}, \lambda) - f(G/e, \lambda).$$

This formula and Whitney's theorem imply

$$|\mathcal{A}_i(G)| = |\mathcal{A}_i(G - \{e\})| + |\mathcal{A}_{i-1}(G/e)|.$$

The problem originally posed by Wilf was to prove this formula bijectively for some edge $e$, i.e., to find a bijection between $\mathcal{A}_i(G)$ and the disjoint union of $\mathcal{A}_i(G - \{e\})$ and $\mathcal{A}_{i-1}(G/e)$. Taking $e$ to be the first edge of $G$, we can exhibit such a bijection as

$$A \rightarrow \begin{cases} A \in \mathcal{A}_i(G - \{e\}), & \text{if } e \notin A \\ A - \{e\} \in \mathcal{A}_i(G/e), & \text{if } e \in A. \end{cases}$$

This construction was found before those in our proofs of Whitney's and Stanley's theorems and served as a step in the discovery of these proofs.

## References

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