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A CONSTRUCTIVE RESPONSE TO 'WHERE MATHEMATICS COMES FROM'

ABSTRACT. Lakoff and Nuñez's book *Where mathematics comes from: How the embodied mind brings mathematics into being* (2000) provided many mathematics education researchers with a novel, and perhaps startling perspective on mathematical thinking. However, as evidenced by reviewers' criticisms (Gold, 2001; Goldin, 2001; Madden, 2001), their perspective – though liberating for many, with its humanistic emphases – remains controversial. Nonetheless, we believe this perspective deserves further constructive response. In this paper, we propose that several of the book's flaws can be addressed through a more rigorous establishment of conceptual distinctions as well as a more appropriate set of methodological approaches.

In the past decade, several mathematics education researchers have emphasised the embodied nature of mathematical understanding, working toward displacing the more prevalent, conventional views found in both psychology and philosophy and studying implications for mathematics learning. These researchers have argued that sensory-motor action plays a crucial role in mathematical activity. A major struggle has been to explain how abstract, formal mathematical ideas can emerge from concrete sensory-motor experiences. This struggle has more recently found promising solutions in another area of inquiry in mathematics education research – the role of metaphor in mathematical meaning.

Though many may have seen literary pursuit as the proper domain for metaphorical expression, Pimm (1987) claimed that "metaphor is as central to the expression of mathematical meaning, as it is to the expression of meaning in everyday language" (pp. 11–12). He was joined by others, such as Presmeg (1992), in emphasising the ways in which metaphor can help in developing and understanding abstract mathematical concepts. Drawing on Lakoff and Johnson's theory, Sfard (1994) pointed to the types of metaphors whose source is in bodily experience, thereby suggesting that metaphor can play a role in translating bodily experiences into the less concrete realm of mathematical ideas. She also emphasised the constitutive role of metaphor, that is, its role in creating the universe of abstract ideas.

In their recent publication, Lakoff and Nuñez (2000) attempt to systematically explain how this metaphoric process works in mathematics. Based on findings in related cognitive science research, they take as given three assumptions: that everything humans know must be derived in some way

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from sensory-motor experiences; that essential aspects of our thinking are not accessible to awareness; and, that humans understand through conceptual metaphor. (Although these are stated as assumptions, we believe they represent fundamental oversimplifications, as the paper will make clear.) They then explain 'where mathematics come from,' at least many areas of mathematics, and a methodology for tracing the metaphorical transformations from bodily experience to abstract ideas.

Lakoff and Nuñez's book *Where Mathematics Comes From: How the embodied mind brings mathematics into being* (WMCF) has had a significant impact on the mathematics education community. Their claims relate to, and in some cases support, many active, important areas of research. While many researchers embraced these claims, several of which had already been advanced in the literature, the criticisms of Madden (2001) and Goldin (2001) suggest that major modifications need to be made in order for their theory to account for the experiences of mathematicians and mathematical learners.

Given the scope and importance of WMCF, we feel that a response to the claims of Lakoff and Nuñez would be useful to the mathematics education community. However, although the ideas in this paper arose initially upon critical reflection of WMCF, we aim to provide more than a critique or review. Rather, we would like to build on some of the important, humanistic claims put forth by Lakoff and Nuñez by bringing needed distinctions to some of their basic terms, and in particular, to the very word *mathematics*. By fleshing out these distinctions, we will attempt to bring clarity to multiple and sometimes vague ways in which notions such as abstract, embodied, and intuition are used both in WMCF and elsewhere in the mathematics education literature. We will also argue that Lakoff and Nuñez's methodology of "mathematical idea analysis" is insufficient to adequately describe the nature of mathematical concepts; for this, further empirical research is necessary, especially research that can probe the very idiosyncratic nature of students' individual conceptions.

1. WHERE DOES *WHICH* 'MATHEMATICS' COME FROM?

In this section, we work out the ways in which the term 'mathematics' may be understood in a question about 'where mathematics comes from.' This is necessary because the different senses of 'mathematics,' i.e., the different aspects of mathematics, may each function differently or at least distinctively in cognition. And further, the ways in which these different aspects function in different individuals (and different groups of individuals) may empirically introduce further differences still.

1.1. *Conceptual vs. ideational mathematics*

WMCF does not differentiate the term ‘mathematics,’ nor indicate whether metaphor might function differently depending on whether one is *learning*, *doing,* or *using* mathematics. It is evident from the historical development of complex numbers that the metaphor used to explain an idea is not (always) the same as the one used to perform abstraction. We thus begin with the interpretation of ‘mathematics’ implicit in WMCF, which we call *conceptual mathematics* (CM): this is mathematics as a subject-matter or discipline. The discipline of mathematics in its core purposes is a public activity, an ongoing game in progress, whose rules are continuously negotiated as shared (even if not perfectly shared) meanings among the participants. According to Devlin’s (1994) characterisation, the core purpose of mathematics is the pursuit of patterns: the patterns of number and counting are the subject matter of number theory, while geometry studies patterns of shapes. Devlin also identifies those patterns of reasoning that underlie mathematical logic and those patterns of motion that are the subject matter of calculus. Patterns of position and closeness are the study of topology and probability theory attends to patterns of chance. As Schiralli points out “In each case the mathematician’s central purpose is the pursuit, representation, exploration, and manipulation of patterns and pattern possibilities” (in preparation).

A mathematical concept, therefore, is a publicly accessible tool – with a history and a future – involved in pursuing, representing, exploring, and manipulating patterns and pattern possibilities. This tool may continue to have utility in its present form, may be improved in future, or even supplanted by newer tools as yet unrepresented. The significance of a mathematical concept lies in the way it connects with related concepts – with the logical patterns of the connective rules functioning as the medium within which mathematical inquiry publicly proceeds.

Next, and this is very important: these CM concepts are not necessarily the same as the mathematical ideas that individual mathematicians (experienced or novice) may form of them. CM concepts are public representations; they exist outside in a public space of shared meanings. As such they are best kept distinct from the internal representations that given people will form of them. How an individual represents these concepts to herself is what we will call *ideational mathematics* (IM) and will probably be influenced by many experiential and genetic factors.

We illustrate the difference between IM and CM using the concept of derivative. WMCF proposes that the concept of derivative depends on a limit metaphor, where a tangent line is conceptualised metaphorically as the limit of a sequence of secant lines, with the secant lines becoming progressively smaller but always having a real length. The limit metaphor, in turn, is based on the conceptual metaphor that continuous processes without end are infinite iterative processes, that is, the process of jumping over and over and over – a continuous process – is iterative in that it will have an intermediate endpoint and an intermediate result. These metaphors give rise to a logical definition of the derivative that can be expressed in symbolic form in mathematical textbooks. However, as the mathematician Thurston (1995) points out, "practising mathematicians understand the derivative in multiple ways" (p. 30). A 'microscopic' way of conceiving the derivative (the limit of what you get by looking at it under a microscope of higher and higher power) might ultimately become reconciled with an 'approximation' way (the best linear approximation to the function near a point). Thurston provides six more distinct ways of representing the concept of the derivative, and he emphasises that individual mathematicians may use, apply, or mean each one in different contexts, and that each carries a personal tone or flavour.

2. THE ABSTRACT NATURE OF MATHEMATICAL THINKING

A chief complaint against WMCF, particularly among mathematicians, was its perceived inability to account for the distinctive kind of abstractness that characterises mathematical thinking. Lakoff and Nuñez claim that abstract concepts are always rooted, through some combination of linking and grounding metaphors, to sensory-motor experiences. They claim that we cannot think of the derivative, for example, without conceptualising it in terms of something more concrete. We might imagine that the microscopic and approximation concepts of the derivative are each understood, via metaphor, in terms of more concrete concepts – through *different* sets of source and target domains. If so, then how do mathematicians develop their abstract concept of the derivative, which is 'detached' from those particular meanings, and can thus be applied in a given situation? We propose that WMCF is right about the sensory-motor basis of abstract concepts, but that their reduction of abstract concepts to more concrete ones through metaphor fails to explain the fundamental processes involved in acts of abstraction.

The very phrases 'abstract thought' and 'abstract concept' are misleading. The expression that needs to be analysed is 'thinking abstractly.' If someone is asked to think about quadrilaterals, a vague image of a prototypical quadrilateral may be 'seen' in her mind's eye. But as soon as she tries to focus it, to attend to it carefully, she finds she is considering a specific kind of quadrilateral, or indeed a specific quadrilateral that she has actually had some experience of, or even a suite of images of quadrilaterals of all kinds and all descriptions. (Are there other non-imagic ways of thinking about quadrilaterals a-contextually?)

But because she can access an initial vague image, we may be tricked into thinking that her mind actually possesses an abstract idea of a quadrilateral as an active unit of thought. The original request to 'think about a quadrilateral,' however, is really quite idle; in order to get a clearer picture what thinking abstractly about quadrilaterals involves, there needs to be a problem.

When she productively thinks about quadrilaterals, when she has a real problem concerning quadrilaterals, we believe she is thinking primarily (but not exclusively) linguistically. She is working on a problem *about* some aspect of quadrilaterals and other things, and so her thought is enmeshed in a propositional net in which the concept of quadrilateral (in its public, generic, 'abstract' sense) is at work. The reason we say 'primarily' is that we believe she also may be aware of images of quadrilaterals as she continues to think about her problem and these images may be subliminally influencing the bearing and trend of her thought in ways of which she may *not* be aware. They may, for instance, be stirring up embodied recollections of (relevant) past experiences involving quadrilaterals that 'push' her linguistic thought in one or another direction – suppose she has recently been creating tesselations using parallelogram-shaped tiles. Or they may simply be provoked by those embodied recollections that the whole problem situation is triggering.

When she finally articulates her thinking, either sub-vocally or verbally or in writing, she is generating publicly contestable assertions whose meaning is in the realm of public, shared representations, namely words and concepts. In that realm, quadrilateral is an abstract concept – a four-sided convex polygon – which can be defined and illustrated using examples and counter-examples.

Importantly here, Wittgenstein (1953) shows that there need not be a common set of necessary and sufficient conditions respecting all legitimate uses of a word. And as words 'map' concepts in the patterns of their uses, there need not be a single 'abstractable' entity (image or form) for a word/concept at work at all in much productive abstract thought. To think abstractly is to think with concepts. And thinking with concepts is not the same as thinking about the ideas or images those concepts may occasion in us.

By focusing on the act of thinking with concepts, rather than on abstract concepts, we can explain how mathematicians, as Poincaré noted, excel at giving the same name to different things. Every definition listed by Thurston has the same name – a derivative – but when he thinks *with* the concept of the derivative in a problem-solving context, various kinds of triggers prompt him to ignore some aspects of his concept of the derivative while highlighting others. Of interest then, are the triggering mechanisms. If you are presented with an equation in symbolic form, you might think of the derivative symbolically, triggering some of the families of derivatives you know: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, etc. If you are reading a graph, and are having trouble understanding the motion represented by the curve, you might think of the derivative as a rate: the instantaneous speed of $f(t)$ when t is time. If your problem is to explain how both these concepts somehow *mean* the same thing, then do you have to locate the fundamental concept of the derivative concept – some metaphoric essence? No! You might reconcile the two by reasoning that the symbolic definition is a method of calculating the rate definition, or you might imagine a particular example that unites the two.

3. CONCEPT VERSUS CONCEPTION

The differences between IM and CM help explain why certain mathematicians (see Dubinsky, 1999; Gold, 2001; Madden, 2001) found some of the metaphors used in WMCF either foreign or forced. (Lakoff and Nuñez would counter, following their cognitive unconscious assumption, that mathematicians do not typically realise they are thinking metaphorically; for mathematicians, this is emotionally unsatisfying: the assumption redefines expert understanding as something both different from and inaccessible to understanding by experts). Lakoff and Nuñez provided metaphorical pathways to the *concepts* of CM, through grounding and linking metaphors, but these are not necessarily the same pathways that an individual will follow in creating his or her *conceptions* in IM. Lakoff and Nuñez do not distinguish IM from CM. In fact, they imply that ideational and conceptual mathematics will be isomorphic by claiming that the metaphors on which mathematics is based are not at all arbitrary. That is, grounding metaphors are forced on us by our physical nature, and metaphorical mappings, blends and special cases have a stable, precise structure (see p. 375).

In contrast, we propose that at least two diversifying factors come into play during the evolution of ideational mathematics which make WMCF's implication untenable. First, as mentioned above, metaphor is not the only meaning-making strategy used in mathematics. Second, metaphorical mappings can also be made to and from the ongoing experiences (including non-mathematical ones) of the mathematician. These are what WMCF calls *extraneous* metaphors (p. 53). These are metaphors, sometimes idiosyncratic, that individual mathematicians or students may construct in their learning experiences, which are a powerful part of sense making – or *conception* forming – in those experiences. Such metaphors may provide links of a different kind than the ones designated *linking metaphors* – but they are nonetheless powerful. WMCF may have used some of these 'extraneous' metaphors themselves.

Studies of student learners also provide many surprising examples of extraneous metaphors. For example, Presmeg (1992) reports on Allison's own construction of a 'water level' metaphor – the *x*-axis is the water level on which a ship sails – for working with trigonometric ratios. This is how Allison understands and solves trigonometric problems: the water level is part of her conception of the *x*-axis. In fact, extraneous metaphors can frequently be observed in use during the course of problem solving: Alex, a grade eight student, was trying to figure out the relationship between the number of sides in a polygon and the sum of its interior angles. He had found 180° for the triangle and 360° for the quadrilateral, and from there had conjectured that the sum of the interior angles of a pentagon would be 540°. One of the present authors, Nathalie Sinclair was surprised since most students she had worked with initially conjectured 720°. He explained: "I know it's going to be 540° because I'm a skateboarder." Alex's conjecture was based on the skateboarder's interest in multiples of half-turns, and did not explain why each new side should increase the sum of the angles by 180°. He subsequently used his skateboarding connection to reason that a half-turn corresponded to adding a new triangle to the existing polygon, and hence an extra 180°. The physical analogy provided Alex with an ambiguous (and potentially deceitful) impetus with which to make sense of the problem; and will he not think with skateboards next time he encounters sums of angles?

4. Where does ideational mathematics come from?

How is the mathematics learner involved in CM or IM? Someone learning mathematics is striving to learn CM, and is involved in a process of initiation into the activities of doing mathematics and becoming progressively more proficient in the use of its rules and tools or concepts. It is in the individual's thinking about mathematical rules and concepts where CM and IM are interactively engaged, but how individuals make sense of these CM rules and concepts may well vary considerably in style as well as in substance.

A further complication arises in transition from IM to CM: the student does not come to this process – does not come to learn or catch on to CM – without some kind of untutored mathematical awareness (UMA) or proficiency. In fact depending on aptitude and environment(s) more or less powerful 'mathematical' ideas will have been formed prior to any exposure to formal mathematics (CM). The street children of Brazil that Saxe (1991) studied have developed a very powerful and functional proficiency with numbers. The young child who spends hours playing with LEGO pieces may develop a different type of proficiency, one related to the way three dimensional shapes compare in terms of size and commensurability. The purpose of a mathematical education is in part to bring these ideas (UMA/IM) into consistency (not necessarily perfect alignment) with whatever corresponding concepts exist in CM.

This untutored mathematical awareness is an aspect of mathematics but its contours and content will also no doubt vary considerably from person to person and there will no doubt be differences across cultures. The linkages between UMA and CM will be 'worked out as ideas' by an individual using a variety of cognitive heuristic devices, like metaphor, encapsulation, conceptual categorisation, analogy, and others. As mentioned above, there is abundant room here for quite individualistic and idiosyncratic cognitive strategies to be employed. Again: this is the aspect of mathematics – an individual's thinking about mathematical concepts and rules – that we are calling *ideational mathematics* (IM).

Particular individuals, gifted mathematicians for example, can even instigate new directions in CM through the manipulation of their UMA/IM ideas to push CM concepts into new territory, and even to propose new CM concepts entirely. The case of Ramanujan is illustrative. This is the area of 'mathematical discovery' and the mathematician's fluency in CM (the shared language of public representations) is what enables these new concepts to be meaningfully presented to, and critiqued by, other mathematicians. As CM is the public language within which the individual mathematician's ideas may be communicated, cognitive strategies like metaphor and so forth in turn become public strategies of communication, assisting others to grasp the point of the new elements of CM proposed. What will function as a successful metaphor in this public space will also presumably vary among the individuals and groups concerned. A successful metaphor for an individual in IM may not be successful in a given attempt at public CM communication.

4.1. *Embodied mathematics*

In the context of the above distinctions, *embodied mathematics* (EM) may be understood as the embodiment – and embodied apprehension – of ordering principles and categories the human organism has/acquires (properly described, we think there are indeed elements of both) beneath the level of conscious awareness but which variously is involved in all the other senses just identified. For example, Bryant and Squire (2001) show how children’s spatial schemas act as ordering principles for judgements about ‘more and less:’ There are more counters in a row that takes up more horizontal space. The bias toward vertical symmetry is another example. Children regularly use vertical symmetry as an ordering principle (in, say, reading faces and perceiving their own bodies), and thus identify vertical symmetry in geometric shapes quite naturally, but often have difficulty in perceiving or, indeed, accepting other symmetric orientations. A final example of EM is the one-to-one principle (that every item in a display should be tagged with one and only one unique tag); Gelman and Meck (1983) have found that preschoolers unconsciously use this principle, before being able to verbalise it explicitly.

EM is thus the underlying source or ‘embodied cognitive infrastructure’ of UMA *and* CM *and* IM. Some forms of UMA may well be highly idiosyncratic and even misleading depending on the individual and contextual circumstances. UMA *proficiency*, therefore, will involve those instances of UMA that are readily representable coherently in terms of CM concepts.

The ‘has/acquires’ angle might also add some value to the Poincaré (1913) position on intuition. In contrast with logic, Poincaré saw intuition, physical analogies, and images derived from experience as providing the necessary insight, purpose, and direction in mathematics; intuition is responsible for motivating and organising mathematical activity – for ‘seeing’ the idea in an argument. An intuitive framework (more Kantian naturalism than Platonic idealism) might be a given, but later ‘intuitions’ might be the result of the ideational being (the thinker) tapping into the embodied ordering principles and categories that the visceral being (the organism) has been subliminally and experientially processing. The thinker’s access to visceral intuition can be mediated by socially achieved conceptual systems that provide tools that guide the formation of new ideas and thereby improve thinking about mathematics and doing mathematics. In fact, Pratt (1998) has described a computer-based probability learning environment that succeeds in achieving this. These concepts and new ideas can participate in the thinker’s tapping of the embodied mathematical knowledge already possessed, making his understanding and representations of mathematics more productive.

5. BEHIND/BEYOND METAPHORS

By focusing almost exclusively on metaphor, WMCF undermines, or at least fails to explain, the acts of understanding needed to move toward CM.

We do not deny the power and pervasiveness of metaphor; rather, we seek to understand the processes by which metaphors are constructed by learners. These processes may have much in common with the triggering mechanisms discussed above. In fact, the basic human acts of understanding identified by Sierpinska (1994), based partly on the empirical work of Vygotsky, might provide some insight into these processes. The four acts of understanding – identification, discrimination, generalisation and synthesis – operate from a very early age; however, both the objects of the acts and the rules governing their use, become more sophisticated with age and experience.

Identification is the process of singling something out from a background. While the toddler *notices* the colourful earring dangling from his mother's ear, and the young child *notices* the vibrant red M&M in the jar, we expect the primary student in the mathematics classroom to notice the equi-angularity of an equilateral triangle, not its colour. Discrimination is the process of identifying differences. The 'terrible two' year old child is learning to discriminate between her mother's desires and her own. In the classroom, we may want students to discriminate between even and odd numbers – to see them as two different types of numbers. Generalisation occurs when a given situation is thought of as a particular case of another situation. Eating breakfast on Saturday morning may be a particular case of having a family meal. Sierpinska notes that while identification may be a more fundamental *notion*, generalisation may be a more fundamental process/operation. There are many forms of generalisation, but a common one students encounter in the classroom is to see a square as a particular case of a rectangle. The fourth act of understanding, synthesis, is the process of searching for a common link, a unifying principle. Reconciling the various definitions of the derivative is certainly a sophisticated example of synthesis. But young children also engage in synthesising; for example, when children refer to their family, they have found a unifying principle that connects parents, siblings, and maybe even aunts, uncles, grand-parents and pets.

A student who can think with the concept quadrilateral has engaged in at least three acts of understanding. The student has noticed the four-sidedness of the quadrilateral, has distinguished the four-sided polygons from the five-sided ones, and has generalised that all the finite, different examples of quadrilateral so far encountered are particular cases of some general shape. When the student is told that quadrilaterals have two distinct diagonals that cross each other, some synthesising will likely be needed. The remarkable thing is not the acts themselves, for they are commonplace in the lives of children; rather, the student has identified, discriminated, and generalised according to *mathematical* principles. The student has learned that in mathematics, there are certain properties and behaviours that are significant. Colour is usually not one of them.

Clearly, these acts of understanding cannot be subsumed under WMCF's construct of "conceptual metaphor;" otherwise we lose cognitively important distinctions in our quest to understand how students come to know mathematics.

6. The Limitations of 'Mathematical Idea Analysis'

The set of metaphors offered in WMCF, for explaining 'where mathematics comes from,' derive from a single methodology of 'mathematical idea analysis.' This is a linguistics-based technique – based almost entirely on standard utterances in textbooks and curricula – of revealing the metaphorical sources of mathematical ideas. Mathematical idea analysis assumes that linguistic expressions can be taken at face value, independent of the many different possible interpretations. Consequently, as Dubinsky (1999) points out, the structure of metaphors identified for each idea – which is unjustifiably put forth as the 'real' way that idea is understood – often seems like a logical creation in servitude of a consistent methodology.

We have attempted to offer a tentative analysis of the different ways 'mathematics' might be understood in a variety of cases that illustrate the issues embedded in any question concerning 'where mathematics comes from.' In developing this analysis we have used some philosophical techniques arising in conceptual or logical analysis (see, e.g., Wilson, 1963). We have identified quite a few important respects in which properly approaching a question concerning where mathematics comes from would require empirical investigation – not just mathematical idea analysis – into the individual and cultural differences affecting learning, doing, or using mathematics. In acknowledging the way in which empirical investigation might thus productively inform questions of concept or meaning, we view the approach we are taking as consistent with contemporary interest among educational researchers in mixed methodology. In calling our analysis 'tentative,' we acknowledge the fact that it is too early to offer a definitive position on where mathematics comes from.

Our own inclination at this point is to view the sensory-motor experiences that form the core of Lakoff and Nuñez's account as more complex and context-dependent than Lakoff and Nuñez take them to be. We therefore invite empirical investigation into the ways in which these sensory-motor experiences might variously be structured by those common neurophysiological predispositions human beings genetically possess *and* might variously be mediated by environmental factors including those developing cultural and symbolic systems into which specific human beings and groups of human beings are variously and progressively initiated. Any adequate account of where mathematics comes from, we propose, would need to account for the ways in which a-cultural cognitive predispositions interact with various cultural and symbolic variables to produce mathematical cognitive structures.

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