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Galerkin Approximations for Weakly Nonlinear Second Order Evolution Equations

By

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1. Introduction

The main purpose of this paper is to prove the following theorem.

1.1 Theorem. *Suppose*

a) $\mathscr{H}$ *is a complex Hilbert space with inner product* $(\cdot,\cdot)$ *and the corresponding norm* $\|\cdot\|$

b) $\varphi_n \in \mathscr{H}$ *for* $n \geq 1$; *let* $\mathscr{V}_n$ *denotes the collection of all linear combinations of* $\varphi_1, \dots, \varphi_n$ *and assume that* $\bigcup_{i=1}^{\infty} \mathscr{V}_i$ *is dense in* $\mathscr{H}$

c) $S: \mathscr{D}(S) \equiv \bigcup_{i=1}^{\infty} \mathscr{V}_i \to \mathscr{H}$ *is a linear operator such that for some* $r \in \boldsymbol{R}$ *and some* $\gamma \in (0, \infty)$ *we have that*

$$\gamma |\mathrm{Im}((Sx, x))| \leq \mathrm{Re}((Sx, x)) + r\|x\|^2 \qquad \text{for all } x \in \mathscr{D}(S)$$

d) *there exists* $b \in [0, \infty)$ *such that for all* x, y *in* $\mathscr{D}(S)$ *we have*

$$|\mathrm{Re}((Sx, y) - (x, Sy))| \leq b(\mathrm{Re}((Sx, x)) + (1 + r)\|x\|^2 + \|y\|^2)$$

e) $T \in (0, \infty)$, $F: [0, T] \times \mathscr{H} \to \mathscr{H}$ *is continuous and for some* $L \in [0, \infty)$ *we have that*

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\| \qquad \text{for } t \in [0, T],\ x, y \in \mathscr{H}.$$

Choose any $x_0 \in \mathscr{H}$, $y_0 \in \mathscr{H}$. *Let* S_F *denotes the Friedrichs extension of* S.

Then, for each $n \geq 1$ *there exists a unique* $v_n \in C^2([0, T], \mathscr{V}_n)$ *such that for all* $z \in \mathscr{V}_n$ *we have*

$$(v_n''(t), z) + (Sv_n(t), z) = (F(t, v_n(t)), z) \qquad \text{for } t \in [0, T],$$

$$(v_n'(0), z) = (y_0, z),$$

$$(v_n(0), z) = (x_0, z).$$

Moreover, there exists $v \in C([0, T], \mathscr{H})$ *such that*

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} \|v(t) - v_n(t)\| = 0.$$

Furthermore, $\mathscr{D}(S_F) = \mathscr{D}(S_F^*)$ *and this* v *is the unique element of* $C([0, T], \mathscr{H})$ *that has the following properties*:

$$(v, z) \in C^2([0, T], \boldsymbol{C}) \qquad \textit{for } z \in \mathscr{D}(S_F^*),$$

$$\frac{d^2}{dt^2}(v(t), z) + (v(t), S_F^* z) = (F(t, v(t)), z) \qquad \textit{for } t \in [0, T],\ z \in \mathscr{D}(S_F^*),$$

$$v(0) = x_0,\ \frac{d}{dt}(v, z)|_{t=0} = (y_0, z) \qquad \textit{for } z \in \mathscr{D}(S_F^*).$$

Observe that S is defined only on finite linear combinations of the basis functions φ_n and that c) and d) need to be verified only on $\mathscr{D}(S)$, the domain of S. This and the fact that the only requirement on the initial data x_0, y_0 is that they belong to $\mathscr{H}$ makes the verification of hypotheses elementary for a very large class of problems.

a), b) and c) imply that S has a m-sectorial extension S_F-called the Friedrichs extension of S [10]. When S is elliptic operator the condition c) is, in effect, the Gårding's inequality [12, 16, 18]. Condition d) has been used before in connection with the wave equation, see [19 p. 420, 427–430], [15] and it actually implies the bound in c).

1.2 *Example*, Suppose that Ω is an arbitrary nonempty open set in $\boldsymbol{R}^n$, $n \geq 1$. Let $\mathscr{H} = L^2(\Omega)$. Basis functions φ_k can be chosen so that the assumption b) is satisfied and that $\mathscr{D}(S) \subset W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ (a simple construction of $\varphi_k \in C_0^2(\Omega)$ can be done as in [2]). Assume

$a_{ij} \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$ are real vlued for $1 \leq i, j \leq n$; for some $\delta > 0$

$$\text{(1)} \quad \begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta \sum_{i=1}^n \xi_i^2 && \text{for } (\xi_1, \ldots, \xi_n) \in \boldsymbol{R}^n,\ x \in \Omega; \\ &b_i \in C^1(\Omega) \cap W^{1,\infty}(\Omega) && \text{for } 1 \leq i \leq n;\ c \in L^\infty(\Omega) \end{aligned}$$

and define

$$Su = -\sum_{i=1}^n \sum_{j=1}^n D_i(a_{ij} D_j u) + \sum_{i=1}^n b_i D_i u + cu \in \mathscr{H} \qquad \text{for } u \in \mathscr{D}(S).$$

Observe that it is not required that $a_{ij} = a_{ji}$; S_F may have no eigenvalues.

It will be shown that assumptions c) and d) of Theorem 1.1 are satisfied. In concrete applications (known nice φ_k) no references to the Sobolev spaces are needed to apply the following argument.

Integration by parts gives for $u, v \in \mathscr{D}(S)$

$$\text{(2)} \qquad (Su, v) = \sum_{i=1}^n \sum_{j=1}^n \int_\Omega a_{ij} D_j u D_i \bar{v} + \sum_{i=1}^n \int_\Omega \bar{v} b_i D_i u + \int_\Omega cu\bar{v}$$

Hölder's inequalities and (1) imply that for all $u \in \mathscr{D}(S)$ we have

$$\mathrm{Re}((Su, u)) \geq \delta[u]^2 - \mu_1[u]\|u\| - \mu_2\|u\|^2 \tag{3}$$

$$|\mathrm{Im}((Su, u))| \leq \mu_3[u]^2 + \mu_1[u]\|u\| + \mu_2\|u\|^2 \tag{4}$$

where μ_1, μ_2, μ_3 are some constants and $[u]$ is given by

$$[u] = \left(\sum_{i=1}^{n} \int_{\Omega} |D_i u|^2\right)^{1/2}.$$

(3), (4) and a little of algebra imply c).

(2) implies that for $u, v \in \mathscr{D}(S)$

$$(u, Sv) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\Omega} a_{ij} D_j \bar{v} D_i u + \sum_{i=1}^{n} \int_{\Omega} u \bar{b}_i D_i \bar{v} + \int_{\Omega} \bar{c} u \bar{v}. \tag{5}$$

Two integrations by parts give that for $u, v \in \mathscr{D}(S)$

$$\int_{\Omega} a_{ij} D_j \bar{v} D_i u = \int_{\Omega} a_{ij} D_j u D_i \bar{v} + \int_{\Omega} \bar{v} D_i a_{ij} D_j u - \int_{\Omega} \bar{v} D_j a_{ij} D_i u;$$

this, (2) and (5)-after another integration by parts, give

$$(Su, v) - (u, Sv) = \int_{\Omega} \bar{v}\Big(\sum_{i=1}^{n} (b_i + \bar{b}_i + \sum_{j=1}^{n} D_j(a_{ij} - a_{ji}))\, D_i u + (c - \bar{c} + \sum_{i=1}^{n} D_i \bar{b}_i) u\Big)$$

and therefore

$$|(Su, v) - (u, Sv)| \leq \|v\|(\mu_4[u] + \mu_5\|u\|)$$

for some constants μ_4, μ_5; this, (3) and a little of algebra imply d). □

The assumptions of the theorem can be easily verified also when S is very singular and degenerate "elliptic" operator with no restrictions on the boundary of the domain [2, 15]. Theorem 1.1 obviously applies also to systems.

A result similar to Theorem 1.1 has been obtained in [15]. In [15] the conditions on F are weaker, however, x_0 is required to belong to the completion of $\mathscr{D}(S)$ under the norm

$$|x|^2 = \mathrm{Re}((Sx, x)) + (1 + r)\|x\|^2$$

and the procedure for calculation of $v_n(0)$ is more involved. Results of [15] are needed for proof of Theorem 1.1-which can be found at the end of the paper. It appears that the proofs in [15] can not be adapted to yield the results of this paper directly. A stronger version of Theorem 1.1 is given in Theorem 2.10 which can be used also to prove convergence of finite element approximations. Various other interesting convergence results are known

[5, 7, 8]-under different sets of assumptions. Use of the Friedrichs extension in connection with convergence of Galerkin approximations seems to have first appeared in [14].

2. Sectorial Form Approach

Hypotheses of this section begin with H#: and they are in effect in the text that follows.

H1: $\mathscr{H}$ is a complex Hilbert space with inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$.

H2: $\mathscr{V}$ is a dense subspace of $\mathscr{H}$, moreover, $\mathscr{V}$ is a Hilbert space with inner product $[\cdot,\cdot]$ and norm $|\cdot|$. There exists $M_1 \in (0, \infty)$ such that

$$\|x\| \leq M_1|x| \qquad \text{for all } x \in \mathscr{V}.$$

H3: $\mathscr{F}: \mathscr{V} \times \mathscr{V} \to \boldsymbol{C}$ is a sesquilinear form such that for some $M_2, M_3 \in (0, \infty)$

$$|\mathscr{F}(x, y)| \leq M_2|x||y| \qquad \text{for } x, y \in \mathscr{V},$$

$$\operatorname{Re}(\mathscr{F}(x, x)) \geq M_3|x|^2 \qquad \text{for } x \in \mathscr{V}.$$

Proof of the following representation theorem (or its equivalent) can be found in many places [10, 12, 14, 18, ...].

2.1 Theorem. *There exists a closed densely defined linear operator A in $\mathscr{H}$ with the following additional properties*:

1) *if $\lambda \in \boldsymbol{R}$ and $\lambda < M_3M_1^{-2}$ then λ is in the resolvent set of A and*

$$\|(A - \lambda)^{-1}\| \leq 1/(M_3M_1^{-2} - \lambda)$$

2) *$\mathscr{D}(A) \subset \mathscr{V}$, $\mathscr{D}(A)$ is dense (in $|\cdot|$ norm) in $\mathscr{V}$*

3) *$\mathscr{F}(x, y) = (Ax, y)$ for all $x \in \mathscr{D}(A)$, $y \in \mathscr{V}$*

4) *$x \in \mathscr{D}(A)$ iff $x \in \mathscr{V}$ and for some $r \in [0, \infty)$ we have that $|\mathscr{F}(x, y)| \leq r\|y\|$ for all $y \in \mathscr{V}$*

5) *$\operatorname{Re}((A^\alpha x, x)) \geq (M_3M_1^{-2})^\alpha\|x\|^2$ for $\alpha \in [0, 1]$, $x \in \mathscr{D}(A^\alpha)$*

6) *if $\mathscr{F}(x, y) = \overline{\mathscr{F}(y, x)}$ for x, y in $\mathscr{V}$ then $A = A^*$, $\mathscr{V} = \mathscr{D}(A^{1/2})$ and $\mathscr{F}(x, y) = (A^{1/2}x, A^{1/2}y)$ for all x, y in $\mathscr{V}$.*

Let $\mathscr{F}^*(x, y) = \overline{\mathscr{F}(y, x)}$ for x, y in $\mathscr{V}$ and note that the operator given by Theorem 2.1 when $\mathscr{F}$ is replaced by $\mathscr{F}^*$ equals A^*, the adjoint of A [10]. Let A_R denote the linear operator given by Theorem 2.1 when applied to the form $\mathscr{F}_R(x, y) = (\mathscr{F}(x, y) + \overline{\mathscr{F}(y, x)})/2$. Let us abbreviate $G = A_R^{1/2}$ and observe that $\mathscr{D}(G) = \mathscr{V}$, G is selfadjoint, $(Gx, x) \geq M_3^{1/2}M_1^{-1}\|x\|^2$ for $x \in \mathscr{V}$,

$\|G^{-1}\| \leq M_1 M_3^{-1/2}$,

$$M_3|x|^2 \leq \|Gx\|^2 \leq M_2|x|^2 \qquad \text{for } x \in \mathscr{V}.$$

Proof of the following theorem can be found in [10 p. 337].

2.2 Theorem. *There exists a bounded linear operator B on $\mathscr{H}$ such that*

$$\mathscr{F}(x, y) = ((1 + iB)Gx, Gy) \qquad \textit{for } x, y \in \mathscr{V},$$

moreover, B is selfadjoint and $A = G(1 + iB)G$.

One has to make additional assumptions when studying the second order evolution equations. The following assumption is, in effect, used in [19 p. 427–430] where the wave equation is studied. See also [8, 11, 15].

H4: Assume that for some $b \in [0, \infty)$ we have that

$$|\mathrm{Re}(\mathscr{F}(x, y) - \overline{\mathscr{F}(y, x)})| \leq b(|x|^2 + \|y\|^2)$$

for all x, y in a dense (in $|\cdot|$ norm) subset of $\mathscr{V}$.

It is not hard to see that H4 implies (and is implied by)

$$|\mathscr{F}(x, y) - \overline{\mathscr{F}(y, x)}| \leq 2b|x|\,\|y\| \qquad \text{for } x, y \text{ in } \mathscr{V}.$$

1) and 2) of the lemma below are proven in [11]. For a stronger result see [13].

2.3 Lemma.

1) $\mathscr{D}(A^\alpha) = \mathscr{D}(A_R^\alpha) = \mathscr{D}(A^{*\alpha})$ *for all* $\alpha \in [0, 1]$

2) $A_R = (1/2)(A + A^*)$

3) $\|BGx\| \leq 2c\|x\|$ *for all* $x \in \mathscr{V}$, *where*, $2c \equiv bM_3^{-1/2}$

4) *if* $\lambda \in \boldsymbol{R}$, $|\lambda| > c$ *then* $\|(G + iBG + \lambda^2 G^{-1})^{-1}\| \leq (2|\lambda| - 2c)^{-1}$.

Proof. If $x \in \mathscr{D}(A)$ then $x \in \mathscr{V}$ and for all $y \in \mathscr{V}$ we have

$$|\mathscr{F}^*(x, y)| = |\overline{\mathscr{F}(y, x)} - \mathscr{F}(x, y) + (Ax, y)| \leq (2b|x| + \|Ax\|)\|y\|$$

therefore $x \in \mathscr{D}(A^*)$ by 4) of Theorem 2.1 and thus $\mathscr{D}(A) \subset \mathscr{D}(A^*)$. Similarly we obtain that $\mathscr{D}(A^*) \subset \mathscr{D}(A)$, hence, $\mathscr{D}(A^*) = \mathscr{D}(A)$.

It can be easily seen, by a similar argument, that $\mathscr{D}(A_R) = \mathscr{D}(A)$. From 3) of Theorem 2.1 we obtain 2).

If $x \in \mathscr{D}(A)$, $y \in \mathscr{V}$ then

$$|(Ax - A^*x, y)| = |\mathscr{F}(x, y) - \overline{\mathscr{F}(y, x)}| \leq 2b|x| \|y\|$$

and hence for $x \in \mathscr{D}(A)$

$$\|(A - A^*)x\| \leq 2b|x| \leq 2bM_3^{-1/2} \operatorname{Re}((Ax, x))^{1/2} \leq 2bM_3^{-1/2} \|Ax\|^{1/2} \|x\|^{1/2}$$
$$\|(A - A_R)x\| = \|(A - A^*)x\|/2 \leq bM_3^{-1/2} \|Ax\|^{1/2} \|x\|^{1/2}$$

and this implies 1), see [9 p. 28].

Theorem 2.2 and H4 imply that for $x, y \in \mathscr{V}$ we have

$$|2i(Gx, BGy)| = |\mathscr{F}(x, y) - \overline{\mathscr{F}(y, x)}| \leq 2b|x| \|y\| \leq 2bM_3^{-1/2} \|Gx\| \|y\|$$

and this implies 3).

To show 4) let $P = G + iBG + \lambda^2 G^{-1}$, $\mathscr{D}(P) = \mathscr{V}$. Observe that for $x \in \mathscr{V}$

$$0 \leq \|G^{1/2}x - |\lambda| G^{-1/2}x\|^2 = ((G + \lambda^2 G^{-1})x, x) - 2|\lambda| \|x\|^2;$$

this and 3) imply

$$\operatorname{Re}((Px, x)) \geq 2(|\lambda| - c)\|x\|^2. \tag{1}$$

Since

$$\|(iBG + \lambda^2 G^{-1})(G + \mu)^{-1}\| \leq (2c + \lambda^2 \|G^{-1}\|)(\mu + M_3^{1/2} M_1^{-1})^{-1}$$
$$\text{for } \mu > -M_3^{1/2} M_1^{-1}$$

we have that

$$(P + \mu)^{-1} = (G + \mu)^{-1}(1 + (iBG + \lambda^2 G^{-1})(G + \mu)^{-1})^{-1}$$

for μ large enough; this and (1) imply 4). □

Define $X = \mathscr{H} \times \mathscr{H}$. X is a Hilbert space with inner product

$$(\{x, y\}, \{z, w\}) = (x, z) + (y, w).$$

Define $H: \mathscr{D}(H) \to X$ by $\mathscr{D}(H) = \{\{x, y\} | x \in \mathscr{V},\ y \in \mathscr{V}\}$,

$$H\{x, y\} = \{-Gy, (1 + iB)Gx\} \qquad \text{for } \{x, y\} \in \mathscr{D}(H).$$

For different approaches to second order problems see [4, 5, 6, 7, 8, 15, 18, 19].

2.4 Theorem. *H is the generator of a strongly continuous group e^{Ht}, $t \in \boldsymbol{R}$, on X. Moreover, $\|e^{Ht}\| \leq e^{c|t|}$ for $t \in \boldsymbol{R}$ (c is an in Lemma 2.3).*

Proof. If $u = \{x, y\} \in \mathscr{D}(H)$ then

$$\operatorname{Re}((Hu, u)) = \operatorname{Re}((iBGx, y))$$

and by 3) of Lemma 2.3

$$\text{(1)} \qquad |\mathrm{Re}((Hu, u))| \leq 2c\|x\| \|y\| \leq c\|u\|^2.$$

If $\{w, v\} \in X$, $\lambda \in \boldsymbol{R}$, $|\lambda| > c$ then 4) of Lemma 2.3 enables us to define

$$x = (G + iBG + \lambda^2 G^{-1})^{-1}(v - \lambda G^{-1}w), \ y = -G^{-1}(\lambda x + w);$$

it can be easily verified that $(H - \lambda)\{x, y\} = \{w, v\}$. This and (1) imply that

$$\|(H - \lambda)^{-1}\| \leq (|\lambda| - c)^{-1} \qquad \text{for } \lambda \in \boldsymbol{R},\ |\lambda| > c. \quad \square$$

H5: Suppose $T \in (0, \infty)$, $F: [0, T] \times \mathscr{H} \to \mathscr{H}$ is continuous and that for some $L \in [0, \infty)$ we have that

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\| \qquad \text{for } t \in [0, T],\ x, y \in \mathscr{H}.$$

2.5 Theorem. *For each $x_0 \in \mathscr{H}$, $y_0 \in \mathscr{H}$ there exists a unique $\{x, y\} \in C([0, T], X)$ such that*

$$\{x(t), y(t)\} = e^{-Ht}\{x_0, G^{-1}y_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}F(s, x(s))\}\, ds \qquad \text{for } t \in [0, T].$$

Moreover, if $u_0 \in \mathscr{H}$, $v_0 \in \mathscr{H}$ and $\{u, v\} \in C([0, T], X)$ is such that

$$\{u(t), v(t)\} = e^{-Ht}\{u_0, G^{-1}v_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}F(s, u(s))\}\, ds \qquad \text{for } t \in [0, T]$$

then

$$\|\{x(t), y(t)\} - \{u(t), v(t)\}\| \leq K\|\{x_0, G^{-1}y_0\} - \{u_0, G^{-1}v_0\}\| \qquad \text{for } t \in [0, T]$$

where $K = \exp(LTM_1 M_3^{-1/2} + cT) \geq \exp(\|G^{-1}\|LT + cT)$.

Proof. This follows immediately from the fixed point theorem and the Gronwall's inequality. For details see [16 p. 184]. $\square$

If $x \in C([0, T], \mathscr{H})$ is as in the above theorem then x is said to be the *mild solution* of

$$x''(t) + Ax(t) = F(t, x(t)), \qquad x(0) = x_0,\ x'(0) = y_0.$$

In the following 3 theorems a relation between mild and weak solutions is examined. See also [1].

2.6 Theorem. *If* $x_0 \in \mathscr{V}$, $y_0 \in \mathscr{H}$ *and* $\{x, y\} \in C([0, T], X)$ *is such that*

$$\{x(t), y(t)\} = e^{-Ht}\{x_0, G^{-1}y_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}F(s, x(s))\}\, ds \qquad \text{for } t \in [0, T]$$

then $x \in C([0, T], \mathscr{V}) \cap C^1([0, T], \mathscr{H})$, $x(0) = x_0$, $x'(0) = y_0$,

$$(x', z) \in C^1([0, T], \mathbf{C}) \qquad \text{for } z \in \mathscr{V},$$

$$\frac{d}{dt}(x'(t), z) + \mathscr{F}(x(t), z) = (F(t, x(t)), z) \qquad \text{for } t \in [0, T],\ z \in \mathscr{V},$$

$$\int_0^t x(s)ds \in \mathscr{D}(A) \qquad \text{for } t \in [0, T],$$

$$x'(t) - y_0 + A\int_0^t x(s)ds = \int_0^t F(s, x(s))ds \qquad \text{for } t \in [0, T].$$

Proof. Let $f(t) = \{0, G^{-1}F(t, x(t))\}$. Since f, $Hf \in C([0, T], X)$, $\{x(0), y(0)\} \in \mathscr{D}(H)$ we have that [10, 16] $\{x, y\} \in C^1([0, T], X)$, $H\{x, y\} \in C([0, T], X)$ and

$$\{x'(t), y'(t)\} + H\{x(t), y(t)\} = f(t) = \{0, G^{-1}F(t, x(t))\} \qquad \text{for } t \in [0, T].$$

Since $\|(1 + iB)^{-1}\| \leq 1$ this implies that $x, y \in C([0, T], \mathscr{V}) \cap C^1([0, T], \mathscr{H})$ and that for $t \in [0, T]$ we have

$$x'(t) = Gy(t)$$
$$y'(t) + (1 + iB)Gx(t) = G^{-1}F(t, x(t)).$$

Therefore $x'(0) = y_0$ and for $t \in [0, T]$, $z \in \mathscr{V}$ we have

$$(y'(t), Gz) + ((1 + iB)Gx(t), Gz) = (G^{-1}F(t, x(t)), Gz),$$

$$\frac{d}{dt}(x'(t), z) = \frac{d}{dt}(y(t), Gz) = (F(t, x(t)), z) - \mathscr{F}(x(t), z),$$

$$(x'(t), z) - (x'(0), z) = \int_0^t ((F(s, x(s)), z) - \mathscr{F}(x(s), z))\, ds,$$

$$(x'(t), z) - (x'(0), z) - \left(\int_0^t F(s, x(s))ds, z\right) = -\mathscr{F}\left(\int_0^t x(s)\, ds, z\right);$$

by 4) and 3) of Theorem 2.1 we are done. □

2.7 Theorem. *If* $x \in C([0, T], \mathscr{V}) \cap C^1([0, T], \mathscr{H})$ *and*

$$\frac{d}{dt}(x'(t), z) + \mathscr{F}(x(t), z) = (F(t, x(t)), z) \qquad \textit{fot } t \in [0, T],\ z \in \mathscr{V},$$

then, $\{x, G^{-1}x'\} \in C([0, T], X)$ *and for all* $t \in [0, T]$ *we have*

$$\{x(t), G^{-1}x'(t)\} = e^{-Ht}\{x(0), G^{-1}x'(0)\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}F(s, x(s))\}\, ds.$$

Proof. Let $y = G^{-1}x'$ and note that for $z \in \mathscr{V}$, $t \in [0, T]$ we have

$$\frac{d}{dt}(y(t), Gz) + ((1 + iB)Gx(t), Gz) = (G^{-1}F(t, x(t)), Gz),$$

$$(y(t), Gz) - (y(0), Gz) = \int_0^t (G^{-1}F(s, x(s)) - (1 + iB)Gx(s), Gz)\, ds,$$

$$y(t) - y(0) = \int_0^t (G^{-1}F(s, x(s)) - (1 + iB)Gx(s))\, ds,$$

therefore $y \in C([0, T], \mathscr{V}) \cap C^1([0, T], \mathscr{H})$ and

$$y'(t) + (1 + iB)Gx(t) = G^{-1}F(t, x(t)),$$
$$\{x'(t), y'(t)\} + H\{x(t), y(t)\} = \{0, G^{-1}F(t, x(t))\}$$

and this implies the assertions of the theorem. □

2.8 Theorem. *If* $x_0 \in \mathscr{H}$, $y_0 \in \mathscr{H}$ *and if* $x \in C([0, T], \mathscr{H})$ *is the mild solution of*

$$x''(t) + Ax(t) = F(t, x(t)), \qquad x(0) = x_0,\ x'(0) = y_0,$$

then this x *is the unique element of* $C([0, T], \mathscr{H})$ *with the following properties*:

$$(x, z) \in C^2([0, T], \mathbf{C}) \qquad \textit{for } z \in \mathscr{D}(A^*),$$

$$\frac{d^2}{dt^2}(x(t), z) + (x(t), A^*z) = (F(t, x(t)), z) \qquad \textit{for } t \in [0, T],\ z \in \mathscr{D}(A^*),$$

$$x(0) = x_0,\ \frac{d}{dt}(x, z)|_{t=0} = (y_0, z) \qquad \textit{for } z \in \mathscr{D}(A^*).$$

Proof. For $n \geq 1$ let $w_n \in \mathscr{V}$ be such that $\lim_{n\to\infty} \|w_n - x_0\| = 0$. Let u_n be the mild solution of

$$u_n''(t) + Au_n(t) = F(t, u_n(t)),\ u_n(0) = w_n,\ u_n'(0) = y_0.$$

Theorem 2.6 implies that for $t \in [0, T]$, $z \in \mathscr{D}(A^*)$ we have

$$(u_n(t), z) - (w_n, z) - t(y_0, z) = \int_0^t (t-s)\,((F(s, u_n(s)), z) - \mathscr{F}(u_n(s), z))\,ds$$
$$= \int_0^t (t-s)\,((F(s, u_n(s)), z) - (u_n(s), A^*z))\,ds$$

and since $\|u_n(t) - x(t)\| \leq K\|w_n - x_0\|$, by Theorem 2.5, we have

$$(x(t), z) - (x_0, z) - t(y_0, z) = \int_0^t (t-s)\,((F(s, x(s)), z) - (x(s), A^*z))\,ds$$

and this implies that x has the properties stated.

To show uniqueness suppose that $y \in C([0, T], \mathscr{H})$ also has the properties stated. Define $g(t) = F(t, y(t))$ and note that for $t \in [0, T]$, $z \in \mathscr{D}(A^*)$ we have

$$(y(t), z) - (x_0, z) - t(y_0, z) = \int_0^t (t-s)\,((g(s), z) - (y(s), A^*z))\,ds$$
$$\left(y(t) - x_0 - ty_0 - \int_0^t (t-s)g(s)\,ds, z\right) = -\left(\int_0^t (t-s)y(s)\,ds, A^*z\right)$$

and therefore

$$y(t) - x_0 - ty_0 + A\int_0^t (t-s)y(s)\,ds = \int_0^t (t-s)g(s)\,ds \qquad \text{for } t \in [0, T].$$

This implies that if $v = A^{-1}y$ then $v \in C^2([0, T], \mathscr{H})$ and

$$v''(t) + Av(t) = A^{-1}g(t) \qquad \text{for } t \in [0, T],$$
$$v(0) = A^{-1}x_0,\ v'(0) = A^{-1}y_0.$$

Let $h = G^{-1}v'$ and note that Theorem 2.2 implies

$$\{v'(t), h'(t)\} + H\{v(t), h(t)\} = \{0, G^{-1}A^{-1}g(t)\} \qquad \text{for } t \in [0, T]$$
$$\{v, h\} \in C^1([0, T], X),\ H\{v, h\} \in C([0, T], X)$$

and therefore

$$\{v(t), h(t)\} = e^{-Ht}\{A^{-1}x_0, G^{-1}A^{-1}y_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}A^{-1}g(s)\}\,ds. \tag{1}$$

Using $A^{-1} = G^{-1}(1 + iB)^{-1}G^{-1}$ gives (for any x_0, y_0 in $\mathscr{H}$)

$$H\{A^{-1}x_0, G^{-1}A^{-1}y_0\} = \{-A^{-1}y_0, G^{-1}x_0\},$$
$$H\{-A^{-1}y_0, G^{-1}x_0\} = -\{x_0, G^{-1}y_0\}.$$

Thus, the right hand side of (1) is in $\mathscr{D}(H^2)$ and

$$-H^2\{v(t), h(t)\} = e^{-Ht}\{x_0, G^{-1}y_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}g(s)\}\, ds$$

and an evaluation of $H^2\{v, h\}$ gives

$$\{y(t), (1+iB)Gv'(t)\} = e^{-Ht}\{x_0, G^{-1}y_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}F(s, y(s))\}\, ds$$

and by Theorem 2.5 we have that $y = x$. $\square$

The weakest condition that ensures convergence of the interior approximations of $A^{-1}f$ is given by the following [3, 5]

H6: Let $\mathscr{V}_1, \mathscr{V}_2, \ldots$ be finite dimensional subspaces of $\mathscr{V}$ such that

$$\lim_{n\to\infty} \inf_{z\in\mathscr{V}_n} |y - z| = 0$$

for all y in a dense (in $|\cdot|$ norm) subset of $\mathscr{V}$.

We shall denote by P_n the orthogonal (in $\mathscr{H}$) projection of $\mathscr{H}$ onto $\mathscr{V}_n$ and by P'_n the orthogonal (in $\mathscr{V}$) projection of $\mathscr{V}$ onto $\mathscr{V}_n$. observe that H6 implies

$$\lim_{n\to\infty} |P'_n x - x| = 0 \qquad \text{for all } x\in\mathscr{V}.$$

2.9 Theorem. *Choose any $w_0\in\mathscr{V}$, $y_0\in\mathscr{H}$. Then, for each $n \geq 1$ there exists a unique $u_n \in C^2([0, T], \mathscr{V}_n)$ such that for all $z\in\mathscr{V}_n$ we have*

$$(u''_n(t), z) + \mathscr{F}(u_n(t), z) = (F(t, u_n(t)), z) \qquad \text{for } t\in[0, T],$$
$$(u'_n(0), z) = (y_0, z),$$
$$[u_n(0), z] = [w_0, z].$$

Moreover, there exists $u\in C([0, T], \mathscr{V})\cap C^1([0, T], \mathscr{H})$ such that

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} (|u(t) - u_n(t)| + \|u'(t) - u'_n(t)\|) = 0,$$

$$u(0) = w_0,\ u'(0) = y_0,$$

furthermore, u is the mild solution of

$$u''(t) + Au(t) = F(t, u(t)),\ u(0) = w_0,\ u'(0) = y_0.$$

This result has been proved in [15] (under weaker conditions on F). In [15] a different definition of the mild solution is used, however, in the proof of the following theorem it will be shown that the "furthermore" part of Theorem

2.9 remains valid under the present definition of the mild solution. Observe that we have to have $w_0 \in \mathscr{V}$ and that $u_n(0)$ is determined by $u_n(0) = P'_n w_0$. In order to prove Theorem 1.1 we need to have $w_0 \in \mathscr{H}$ and $u_n(0) = P_n w_0$. This is achieved in the following theorem. See [5, 7, 8] for various other interesting convergence results-under different sets of assumptions.

2.10 Theorem. *Choose any* $x_0 \in \mathscr{H}$, $y_0 \in \mathscr{H}$. *Then, for each* $n \geq 1$ *there exists a unique* $v_n \in C^2([0, T], \mathscr{V}_n)$ *such that for all* $z \in \mathscr{V}_n$ *we have*

$$(v''_n(t), z) + \mathscr{F}(v_n(t), z) = (F(t, v_n(t)), z) \qquad \text{for } t \in [0, T],$$
$$(v'_n(0), z) = (y_0, z),$$
$$(v_n(0), z) = (x_0, z).$$

Moreover, there exists $v \in C([0, T], \mathscr{H})$ *such that*

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} \| v(t) - v_n(t) \| = 0,$$

furthermore, v *is mild solution of*

$$v''(t) + Av(t) = F(t, v(t)), \; v(0) = x_0, \; v'(0) = y_0.$$

Remark. All assumptions are contained in H1, H2, H3, H4, H5, H6.

Proof. Observe that assumptions H1, H2, H3, H4, H5 remain valid (with the same constants!) if both $\mathscr{V}$ and $\mathscr{H}$ are replaced by $\mathscr{V}_n$ and if F is replaced by $P_n F$ (restricted to $[0, T] \times \mathscr{V}_n$). $X_n = \mathscr{V}_n \times \mathscr{V}_n$. G_n, $H_n, \ldots$ are obtained as G, $H, \ldots$; we shall also use the obvious modifications of preceding theorems.

Choose any $w_0 \in \mathscr{V}$ and let u_n, u be as in the Theorem 2.9.

Note that $u_n(0) = P'_n w_n$, $u'_n(0) = P_n y_0$ and Theorem 2.7 implies

$$\{u_n(t), G_n^{-1} u'_n(t)\} = e^{-H_n t}\{P'_n w_0, G_n^{-1} P_n y_0\} + \int_0^t e^{-H_n(t-s)} \{0, G_n^{-1} P_n F(s, u_n(s))\} \, ds.$$

If $z \in \mathscr{V}$, $t \in [0, T]$ then by Theorem 2.9

$$(u'_n(t), P'_n z) - (u'_n(0), P'_n z) = \int_0^t ((F(s, u_n(s)), P'_n z) - \mathscr{F}(u_n(s), P'_n z)) \, ds,$$

the convergence result of Theorem 2.9 and H6 imply

$$(u'(t), z) - (u'(0), z) = \int_0^t ((F(s, u(s)), z) - \mathscr{F}(u(s), z)) \, ds,$$
$$\frac{d}{dt}(u'(t), z) + \mathscr{F}(u(t), z) = (F(t, u(t)), z)$$

and by Theorem 2.7 we have that $\{u, G^{-1}u'\} \in C([0, T], X)$ and for all $t \in [0, T]$

$$\{u(t), G^{-1}u'(t)\} = e^{-Ht}\{w_0, G^{-1}y_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}F(s, u(s))\}\, ds,$$

thus, Theorem 2.5 implies that the "furthermore" part of Theorem 2.9 remains valid under the present definition of the mild solution.

Existence and uniqueness of v_n is obvious (ODE's-or see Theorem 2.8). Theorem 2.7 implies that for $t \in [0, T]$

$$\{v_n(t), G_n^{-1}v_n'(t)\} = e^{-H_nt}\{P_nx_0, G_n^{-1}P_ny_0\} + \int_0^t e^{-H_n(t-s)}\{0, G_n^{-1}P_nF(s, v_n(s))\}\, ds.$$

By Theorem 2.5 there exists $\{v, y\} \in C([0, T], X)$ such that for $t \in [0, T]$

$$\{v(t), y(t)\} = e^{-Ht}\{x_0, G^{-1}y_0\} + \int_0^t e^{-H(t-s)}\{0, G^{-1}F(s, v(s))\}\, ds.$$

Theorem 2.5 implies that for $t \in [0, T]$

$$\| v_n(t) - u_n(t) \| \leq K \| P_nx_0 - P_n'w_0 \|,$$
$$\| v(t) - u(t) \| \leq K \| x_0 - w_0 \|$$

and hence

$$\begin{aligned}\| v_n(t) - v(t) \| &\leq \| v_n(t) - u_n(t) \| + \| u_n(t) - u(t) \| + \| u(t) - v(t) \| \\ &\leq K \| P_nx_0 - P_n'w_0 \| + K \| x_0 - w_0 \| + \| u_n(t) - u(t) \|\end{aligned}$$

and since $\| P_nx_0 - P_n'w_0 \| \leq \| x_0 - w_0 \| + 2M_1 |P_n'w_0 - w_0|$ we have

$$\| v_n(t) - v(t) \| \leq 2K \| x_0 - w_0 \| + 2KM_1 |P_n'w_0 - w_0| + \| u_n(t) - u(t) \|.$$

Since $w_0 \in \mathscr{V}$ is arbitrary and $\mathscr{V}$ is dense in $\mathscr{H}$ we see that H6 and Theorem 2.9 imply the assertions of the theorem. □

3. Proof of Theorem 1.1

It is well known [10, 14, 17] that a), b), c) imply that there exist a subspace $\mathscr{V}$ of $\mathscr{H}$, an inner product $[\cdot,\cdot]$ on $\mathscr{V}$ and a sesquilinear form $\mathscr{F}$ on $\mathscr{V}$ such that the assumptions H1, H2, H3 of Section 2 are satisfied and that also the following holds

i) $\mathscr{D}(S)$ is dense (in $|\cdot|$ norm) subspace of $\mathscr{V}$
ii) $[x, y] = ((Sx, y) + (x, Sy))/2 + (1 + r)(x, y)$ for x, y in $\mathscr{D}(S)$
iii) $\mathscr{F}(x, y) = ((S + r + 1)x, y)$ for x, y in $\mathscr{D}(S)$.

Observe that d), i), ii) imply H4 and that b), i) imply H6. The operator A given by Theorem 2.1 is an extension of $S + r + 1$ and $A - r - 1 \equiv S_F$ is called the Friedrichs extension of S [10]. $\mathscr{D}(S_F) = \mathscr{D}(S_F^*)$ by Lemma 2.3.

Let $F_s(t, x) = (1 + r)x + F(t, x)$ for $t \in [0, T]$, $x \in \mathscr{H}$. So, H5 is satisfied-with F_s in place of F. Theorems 2.8, 2.10 imply Theorem 1.1.

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