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011

# ON LIMIT STATES OF A LINEARIZED BOLTZMANN EQUATION*

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**Abstract.** With methods of mean ergodic theory a very simple criterion for existence of limit states of linearized Boltzmann equation is proven.

**Key words.** linearized Boltzmann equation, ergodic theory

**AMS(MOS) subject classifications.** 45A05, 47A35

A linearized Boltzmann equation can under some conditions ([4], see also [1], [8]) be written in the form

$$\frac{du}{dt}+u=Bu$$

where $B$ is typically a Markov operator on some $L^1$. The solution of this equation is $e^{-t}e^{Bt}u_0$ and one would like to know how the solution behaves as $t\to\infty$. In this paper this equation is considered in an arbitrary Banach space $X$ and $B$ is assumed to be a bounded linear operator on $X$ such that $\sup_{n\geqq 0}\|B^n\|<\infty$. Let $N=\{x\in X \mid Bx=x\}$ and $R=\{x\in X \mid x=y-By$ for some $y\in X\}$.

THEOREM 1. *If $x\in X$, $x_0\in X$, then the following statements are equivalent*:

(a) *There exist integers $n_1<n_2<\cdots$ such that $\lim_{i\to\infty} y((1/n_i)\sum_{k=0}^{n_i-1}B^kx)=y(x_0)$ for all $y\in X^*$.*

(b) *There exist $t_1, t_2,\cdots$ in $(0,\infty)$ such that $\lim_{i\to\infty} t_i=\infty$ and $\lim_{i\to\infty} y(e^{-t_i}e^{Bt_i}x)=y(x_0)$ for all $y\in X^*$.*

(c) *$x_0\in N$ and $x-x_0\in\bar{R}$.*

(d) $\lim_{n\to\infty}\|(1/n)\sum_{k=0}^{n-1}B^kx-x_0\|=0$.

(e) $\lim_{t\to\infty}\|e^{-t}e^{Bt}x-x_0\|=0$.

This theorem implies that all standard mean ergodic theorems (e.g. (2)) are applicable in the study of limits of $e^{-t}e^{Bt}x$! A simple and quite powerful criterion for existence of the limit is given by the following.

THEOREM 2. *If $x\in X$ and if the set $\{B^nx \mid n\geqq 0\}$ is weakly sequentially compact, then there exists $x_0\in N$ such that*

$$\lim_{t\to\infty}\|e^{-t}e^{Bt}x-x_0\|=0.$$

*Proof.* Let $C=\{B^nx \mid n\geqq 0\}$ and let $C_1$ be the convex hull of $C$. Since $(1/n)\sum_{k=0}^{n-1}B^kx\in C_1$ for $n\geqq 1$ and since $C_1$ is weakly sequentially compact [2, Krein-Šmulian Theorem] there exists $x_0\in X$ such that part (a) of Theorem 1 is satisfied.

There are many ways of showing that the set $\{B^nx \mid n\geqq 0\}$ is weakly sequentially compact [2]. In [3], [5] conditions like strong (weak) constrictiveness (and some other conditions) are required; the following observation shows that these conditions are much more restrictive than those of Theorem 2. If $x\in X$, then the set $\{B^nx \mid n\geqq 0\}$ is weakly sequentially compact if and only if there exists a weakly compact set $F$ (for this $x$!) such that $\lim_{n\to\infty}\operatorname{dist}(B^nx, F)=0$.

If $B$ is positive quasi-compact operator then even convergence rates can be estimated.

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THEOREM 3. *Suppose that $X$ is real Banach space and that $X^+$ is a closed subset of $X$ with the following properties*:

(1) *If* $x \in X^+$, $y \in X^+$, $\alpha \in [0, \infty)$ *then* $x + y \in X^+$ *and* $\alpha x \in X^+$;

(2) *There exists* $M_0 \in (0, \infty)$ *such that for each* $x \in X$ *there exist* $x_+ \in X^+$ *and* $x_- \in X^+$ *which satisfy*

$$x = x_+ - x_-, \quad \|x_+\| \leqq M_0 \|x\|, \quad \|x_-\| \leqq M_0 \|x\|$$

*and if* $x = y_+ - y_-$ *for some* $y_+ \in X^+$, $y_- \in X^+$, *then* $y_+ - x_+ \in X^+$.

(3) *If* $x \in X^+$, $y \in X^+$, *then* $\|x\| \leqq \|x + y\|$.

*Suppose also that $T$ is bounded linear operator on $X$ such that*

(4) $TX^+ \subset X^+$.

(5) $\lim_{n\to\infty} (1/n) y(T^n x) = 0$ *for all* $x \in X$ *and all* $y \in X^*$.

(6) $\|T^m - K\| < 1$ *for some integer $m$ and some compact linear operator $K$.*

*Then there exist* $a \in (0, \infty)$, $b \in (0, \infty)$ *such that for every* $x \in X$ *there exists* $x_0 \in X$ *for which*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x - x_0 \right\| \leqq \frac{b}{n} \|x\|, \qquad \|e^{-t} e^{Tt} x - x_0\| \leqq b \|x\| \, e^{-at}$$

*whenever* $n \geqq 1$, $t > 0$.

Proof of Theorem 3 can be found in [7] and follows from the fact that $T^n x$ is actually asymptotically periodic for every $x \in X$ (see also [6]). Operators considered in [5] in connection with the Boltzmann equation satisfy above assumptions.

*Proof of Theorem* 1. Equivalence of statements (a), (c), (d) is well known [2]. It is obvious that (e) implies (b) and (c); it is enough to prove that (b) implies (c) and that (c) implies (e).

Let $M = \sup_{n \geqq 0} \|B^n\|$, $A = I - B$ and

$$F(t) = e^{-t} + e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left| \frac{t}{n+1} - 1 \right|.$$

Identities

$$e^{-At} = e^{-t} e^{Bt} = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} B^n,$$

$$Ae^{-At} = e^{-t} + e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \frac{t}{n+1} - 1 \right) B^{n+1}$$

imply that for $t > 0$

$$\|e^{-At}\| \leqq M, \qquad \|Ae^{-At}\| \leqq MF(t).$$

A nasty but straightforward exercise gives that $\lim_{t\to\infty} F(t) = 0$.

Assume (b). Then for all $y \in X^*$

$$0 = \lim_{i\to\infty} y(Ae^{-At_i} x) = \lim_{i\to\infty} (A^* y)(e^{-At_i} x) = (A^* y)(x_0) = y(Ax_0);$$

hence, $Ax_0 = 0$. If $x - x_0 \notin \bar{R}$, then there exists $y_0 \in X^*$ such that $A^* y_0 = 0$ and $y_0(x - x_0) = 1$, which leads to the contradiction

$$1 = y_0(x - x_0) = (e^{-A^* t_i} y_0)(x - x_0) = y_0(e^{-At_i} x - x_0) = 0$$

and therefore $x - x_0 \in \bar{R}$ and (c) is true.

Assume (c). Pick $\varepsilon > 0$ and let $z \in X$ be such that $\|x - x_0 - Az\| < \varepsilon/(1+M)$; hence, for $t > 0$

$$\|e^{-At}x - x_0\| \leqq \|e^{-At}(x - x_0 - Az)\| + \|Ae^{-At}z\|$$
$$\leqq M\varepsilon/(1+M) + MF(t)\|z\|$$

and this implies (e).

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